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On countability of homotopy groups

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Abstract

In this talk we intend to generalize some results of Conner and Lamereaux on the countability of $\pi_1(X, x)$. For this, we show that some properties of topological spaces can be transferred from X to the loop space $\Omega^n(X, x)$, for some $x \in X$. Finally, we intend to give some conditions in which the space X is semilocally n-connected.

1 Introduction

Conner and Lamereaux [2] proved that several results concerning the existence of universal covering spaces for separable metric spaces. They defined several homotopy theoretic conditions which are equivalent to the existence of a universal covering space. For instance they proved that every connected, locally path connected separable metric space whose fundamental group is a free group admits a universal covering space. As an application of these results, they proved that a connected, locally path connected subset of the Euclidean plane, E^2, admits a universal covering space if and only if its fundamental group is free, if and only if its fundamental group is countable. In this talk we intend to generalize some of this results on the

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countability of $\pi_1(X, x)$. For this, we show that some properties of topological spaces can be transferred from X to the loop space $\Omega^n(X, x)$, for some $x \in X$. Finally, we intend to give some conditions in which the space X is semilocally n-connected.

2 Main results

In this section, we intend to generalize some results of Conner and Lamereaux [2] on the countability of $\pi_1(X, x)$. For this, we show that some properties of topological spaces can be transferred from X to the loop space $\Omega^n(X, x)$, for some $x \in X$.

Lemma 2.1. Let X be locally (n-1)-connected space. If $\Omega^{(n-1)}(X, x)$ is semilocally simply connected at the constant (n-1)-loop e_x, then X is semilocally n-connected at x, for all $n \geq 2$.

Note that the converse of this fact has been shown by Wada [5, Remark]

Conner and Lamereaux [2] proved that the fundamental group of a path connected, locally connected, separable metric space which admits a universal cover is countable. Using this result, Lemma 2.1, the fact that the separability, metrizability, locally connected properties of space X can be transferred to the loop space $\Omega^n(X, x)$, for some $x \in X$ and applying the group isomorphism $\pi_1(\Omega^{(n-1)}(X, x), e_x) \cong \pi_n(X, x)$ we conclude the following result. The following result is the first result on countability of homotopy groups.

Theorem 2.2. Let X be an (n-1)-connected, locally (n-1)-connected, semilocally n-connected and separable metric space. Then $\pi_n(X, x)$ is countable.

A space X is called n-homotopically Hausdorff at $x \in X$ if for any essential n-loop α based at x, there is an open neighborhood U of x for which α is not homotopic (rel $\dot{I^n}$) to any n-loop lying entirely in U. X is said to be n-homotopically Hausdorff if it is n-homotopically Hausdorff at any $x \in X$ (see [4]).

Consider $\overline{\Omega^n(X, x)}$ as the space of homotopy classes rel $\dot{I^n}$ of n-loops at x in X. If p is an n-loop at x, and U is an open neighborhood of x, then we define $O^n(p, U)$ to be the collection of homotopy classes of n-loops rel $\dot{I^n}$ containing n-loops of the form $p * \alpha$, where α is an n-loop in U at x. It is routine to check that the collection $O^n(p, U)$ is a basis for $\overline{\Omega^n(X, x)}$. In the following, we show that $\overline{\Omega^n(X, x)}$ is Hausdorff if and only if X is n-homotopically Hausdorff at x. (see also [1] for the case $n = 1$.)

The following lemmas are essential to prove of the second result on countability of homotopy groups.

Lemma 2.3. $\overline{\Omega^n(X, x)}$ is Hausdorff if and only if X is n-homotopically Hausdorff at x.

Lemma 2.4. $\overline{\Omega^n(X,x)}$ with the above topology is homeomorphic to $\overline{\Omega(\Omega^{(n-1)}(X,x),e_x)}$, where $\Omega^{(n-1)}(X,x)$ is equipped with the compact-open topology, for all $n \geq 2$.

Lemma 2.5. Let $n \geq 2$. Then a space X is n-homotopically Hausdorff at x if and only if $\Omega^{(n-1)}(X,x)$ is homotopically Hausdorff at e_x, for any $x \in X$.

We shall also need the following well-known result of Dugundji [3].

Theorem 2.6. If X is second countable and Y is locally compact and second countable, then the function space X^Y is second countable. In particular, if X is second countable then $\Omega^n(X,x)$ is also second countable, for all $x \in X$.

Now, the second result on countability of homotopy groups is as follows.

Theorem 2.7. Suppose that X is a second countable, locally (n-1)-connected and n-homotopically Hausdorff space at x which is not semilocally n-connected at this point. Then $\pi_n(X,x)$ is uncountable.

The following corollary is a consequence of Theorems 2.2 and 2.7.

Corollary 2.8. If X is an (n-1)-connected, locally (n-1)-connected, separable metric space, then the following statements are equivalent.
(i) X is semilocally n-connected.
(ii) X is n-homotopically Hausdorff and $\pi_n(X)$ is countable.

One of the main conditions of Theorem 2.2 is assuming that X is semilocally n-connected. In the follow, we intend to give some conditions in which the space X is semilocally n-connected.

Conner and Lamereaux [2] proved that if X is a connected, locally path connected separable metric space with a fundamental group which is a free group then X admits a universal covering space. Using this result, the fact that the separability, metrizability, locally connected properties of space X can be transferred to the loop space $\Omega^n(X,x)$, for some $x \in X$ and applying the group isomorphism $\pi_1(\Omega^{(n-1)}(X,x),e_x) \cong \pi_n(X,x)$ we conclude the following result.

Proposition 2.9. Let X be an (n-1)-connected, locally (n-1)-connected, separable metric space in which $\pi_n(X,x)$ is free. Then X is semilocally n-connected at x.

The following results are the extension of some results of Conner and Lamereaux [2].

Definition 2.10. Let $i : X \to Y$ be an embedding of one path connected space into another. Then we say that X is a π_n-retract of Y if there exists a homomorphism $r : \pi_n(Y) \to \pi_n(X)$ such that the composition $ri_* : \pi_n(X) \to \pi_n(X)$ is an isomorphism. In this case the homomorphism r is called a π_n-retraction for X in Y. Also, X is called a π_n-neighborhood retract in Y if X is a π_n-retract of one of its open neighborhoods in Y.

Definition 2.11. A separable metric space X is called a π_n-absolute neighborhood retract (π_n-ANR) if whenever X is a subspace of a separable metric space Y, then X is a π_n-neighborhood retract in Y .

Lemma 2.12. Let Y be locally (n-1)-connected and semilocally n-connected and X be a π_n-retract of Y. Then X is semilocally n-connected.

Corollary 2.13. Let X be a separable metric space. If X is π_n-ANR, then it is semilocally n-connected.

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