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Burnside condition on some intersection subgroups

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Abstract

In this paper, first we present some preliminaries about graphs, core graphs, and combinatorial algebraic topology. Using these tools, and specially using immersions and covering maps, we establish our main theorem. Indeed, we can prove the Burnside condition for the intersection of those subgroups of free groups satisfying the Burnside condition.

1 Introduction

All our conceptions come from [1], [2] and [3]. A graph X consists of two sets E and V (*edges* and *vertices*), with three functions $^{-1} : E \longrightarrow E$ and $s, t : E \longrightarrow V$ such that $(e^{-1})^{-1} = e$, $e^{-1} \neq e$, $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. We say that the edge $e \in E$ has *initial vertex* $s(e)$ and *terminal vertex* $t(e)$. The edge e^{-1} is the *reverse* of e.

A *map of graphs* $f : X \longrightarrow Y$ is a function which maps edges to edges and vertices to vertices. Also we have $f(e^{-1}) = f(e)^{-1}$, $f(s(e)) = s(f(e))$ and $f(t(e)) = t(f(e))$.

A *path* p in X of length $n = |p|$, with initial vertex u and terminal vertex v, is an n-tuple of edges of X of the form $p = e_1 ... e_n$ such that for $i = 1, ..., n-1$, we have $t(e_i) = s(e_{i+1})$ and $s(e_1) = u$ and $t(e_n) = v$. For $n = 0$, given any vertex v, there

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is a unique path Λ_v of length 0 whose initial and terminal vertices coincide and are equal to v. A path p is called a *circuit* if its initial and terminal vertices coincide.

If p and q are paths in X and the terminal vertex of p equals the initial vertex of q, they may be *concatenated* to form a path pq with $|pq| = |p| + |q|$, whose initial vertex is that of p and whose terminal vertex is that of q.

A *round-trip* is a path of the form ee^{-1}. A *reduced path* is a path in X containing no round-trip. An *elementary reduction* is insertion or deletion a round-trip in a path. Two paths p and q are *homotopic* (written $p \sim q$) iff there is a finite sequence of elementary reductions taking one path to the other. Homotopic paths must have the same start and terminal vertices and also, homotopy is an equivalence relation on the set of paths in X. Moreover, any path in X is homotopic to a unique reduced path in X.

Let v be a fix vertex in X, $\pi_1(X, v)$ is defined to be the set of all homotopy classes of closed paths with initial and terminal vertex v. Then $\pi_1(X, v)$ together with the product $[p][q] := [pq]$ forms a group with identity $[\Lambda_v]$ and inverse element $[w]^{-1} = [w^{-1}]$.

For a fix vertex v in X, the *star* of v in X is defined as follows:

$$St(v, X) = \{e \in E : s(e) = v\}.$$

A map $f : X \longrightarrow Y$ yields, for each vertex $v \in X$, a function $f_v : St(v, X) \longrightarrow St(f(v), Y)$. If for each vertex $v \in X$, f_v is injective, we call f an *immersion*. If each f_v is bijective, we call f a *covering*.

The theory of coverings of graphs is almost completely analogous to the topological theory of coverings. Immersions have some of the properties of coverings. One of them which is more important, and we also need it more, is the following one:

"For a given finite set of elements $\{\alpha_1, ..., \alpha_n\} \subseteq \pi_1(X, u)$, there is a connected graph Y and an immersion $f : Y \longrightarrow X$ such that $f_*(\pi_1(Y)) = S$, in which S is the subgroup of $\pi_1(X, u)$ generated by $\{\alpha_1, ..., \alpha_n\}$".

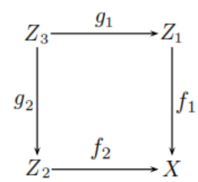
If G is a group, a *G-graph* X is a graph with an action of G on the left on X by maps of graphs, such that for all $g \in G$ and every edge e, $ge \neq e^{-1}$. In this case, the quotient graph X/G, and the quotient map of graphs $X \to X/G$ can be defined. It is easy to see that, in general $X \to X/G$ is locally surjective.

It is said that G *acts freely* on X when, whenever v is a vertex of X, $g \in G$, and $gv = v$, then $g = 1$, the identity element of G. In this case, $X \to X/G$ is an immersion, and hence is a covering.

A t*translation* of a map of graphs $f : X \to Y$ is a map $g : X \to X$ which is an isomorphism of graphs and for which $fg = f$. The set of all translations of f forms a group $G(f)$ which acts on X. If f is an immersion, and X is connected, then $G(f)$ acts freely on X.

The *universal cover* $f : \tilde{X} \to X$, of a connected graph X, is a covering with ($\tilde{X}$ is connected and $\pi_1(\tilde{X})$ trivial. In this case, $G(f) \cong \pi_1(X)$ which acts freely, by covering translations, on $\tilde{X}$, and f is isomorphic to the quotient map $\tilde{X} \to \tilde{X}$.

Theorem 1.1. [4] Let



be a pullback diagram of graphs, where f_1 and f_2 are immersions. Let v_1 and v_2 be vertices in Z_1 and Z_2 that $f_1(v_1) = f_2(v_2) = w$. Let v_3 be corresponding vertex in Z_3. Define $f_3 = f_1 g_1 = f_2 g_2 : Z_3 \to X$, and $S_i = f_{i*}(\pi_1(Z_i, v_i))$, for $i = 1, 2, 3$. Then $S_3 = S_1 \cap S_2$.

Theorem 1.2. [4] Let $f : X \longrightarrow Y$ be an immersion of graphs. Suppose that Y has only one vertex and X has only finitely many vertices. Then there exists a graph $\acute{X}$ containing X such that $\acute{X} - X$ consists only of edges, and there exists a map $\acute{f} : \acute{X} \longrightarrow Y$ extending f such that $\acute{f}$ is a covering.

2 Main results

In this section, we deduce our main result. before it, we recall some notes from [4] which are essential in the proof of the main theorem. First, we note to the *core graphs* whose roles are more important.

A *cyclically reduced circuit* in a graph X is a circuit $p = e_1 ... e_n$, which is reduced as a path and for which $e_1 \neq e_n^{-1}$. A graph X is said to be a *core-graph* if X is connected, has at least one edge and every edge belongs to at least one cyclically reduced circuit.

If X is a connected graph with non-trivial fundamental group, an *essential edge* of X is an edge belonging to some cyclically reduced circuit. The *core* of X consists of all essential edges of X and all initial vertices of essential edges.

If X is a connected graph with non-trivial fundamental group and $\acute{X}$ is the core of X, then $\acute{X}$ is a core-graph. If v is a vertex of $\acute{X}$, then the inclusion $\pi_1(\acute{X}, v) \longrightarrow \pi_1(X, v)$ is an isomorphism.

Another notion, we are dealing with, is the *Burnside condition* for subgroups. If S is a subgroup of a group G, we say that $S \subseteq G$ satisfies the *Burnside condition* when, for every $g \in G$, there exists some positive integer n such that $g^n \in S$.

Lemma 2.1. [4] (a) Let $f : X \longrightarrow Y$ be a finite-sheeted covering of connected graphs, v a vertex of X. Then $f_*(\pi_1(X, v)) \subseteq \pi_1(Y, f(v))$ satisfies the Burnside condition.

(b) Let $f : X \longrightarrow Y$ be an immersion of connected graphs. Suppose that Y is a core-graph; v a vertex of X, $f_*(\pi_1(X, v)) \subseteq \pi_1(Y, f(v))$ satisfies the Burnside condition. Then f is a covering.

Finally, using all the above notes, we establish the following theorem, which is our main result in this paper.

Theorem 2.2. Let S_1 and S_2 be finitely generated subgroups of a free group F. Suppose that $S_1 \cap S_2$ satisfies the Burnside condition both in S_1 and S_2. Then $S_1 \cap S_2$ satisfies the Burnside condition in the join $S_1 \vee S_2$, the subgroup generated by $A \cup B$.

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