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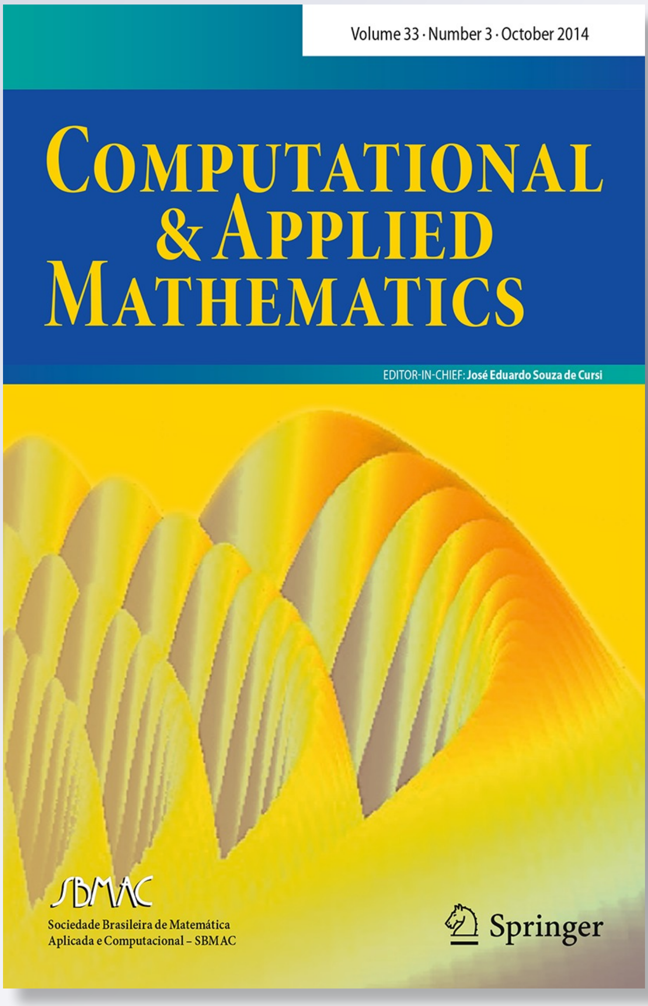
Computational and Applied Mathematics

ISSN 0101-8205
Volume 33
Number 3

Comp. Appl. Math. (2014) 33:687-715
DOI 10.1007/s40314-013-0089-4



Springer

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DOI 10.1007/s40314-013-0089-4

Optimal control of time-varying linear delay systems based on the Bezier curves

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Received: 29 April 2013 / Revised: 2 July 2013 / Accepted: 13 October 2013 /
Published online: 8 November 2013

Abstract In this paper, time-delay control systems with quadratic performance are solved by applying the least square method on the Bezier control points. The approximation process is done in two steps. First, the time interval is divided into $2k$ subintervals, then in each subinterval the trajectory and control functions are approximated by the Bezier curves. We have chosen the Bezier curves as piecewise polynomials of degree n and determined the Bezier curves on any subinterval by $n+1$ control points. By considering a least square optimization problem, the control points can be found, then the Bezier curves that approximate the action of control and trajectory can be computed as well. Some numerical examples are given to verify the efficiency of the proposed method.

Keywords Optimal control problem · Dynamic systems · The Bezier control points · Optimal control of time-delay systems · Time-delay systems · The Bezier curve method

Mathematics Subject Classification (2000) 49N10

1 Introduction

Many physical systems are best modeled using time-delay dynamical systems as follows:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = F(t, \mathbf{x}(t), \mathbf{u}(t), x_1(t-\tau_1), \ldots, \quad x_p(t-\tau_p),$$
$$u_1(t-\eta_1), \ldots, u_m(t-\eta_m)), \quad t \in [t_0, t_f]$$

Communicated by Marko Rojas-Medar.

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$$\mathbf{x}(t) = \boldsymbol{\phi}(t), \quad t \le t_0,$$
$$\mathbf{u}(t) = \boldsymbol{\psi}(t), \quad t \le t_0,$$

where the state $\mathbf{x}(t)$ is a p vector function; $\mathbf{u}(t)$ is a m vector control function, τ_is and η_js $(i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, m)$ are non-negative constant time delays, the vector functions $\boldsymbol{\phi}(t)$ and $\boldsymbol{\psi}(t)$ are defined appropriately and are given (see Eller and Aggarwal 1969; Gollman et al. 2009; Krasovskii 1963; Loxton 2010; Wu et al. 2006).

In some systems of this type, it is desirable to select the optimal pair $(\mathbf{x}^*(.), \mathbf{u}^*(.))$ to minimize a performance criterion modeled by a cost function of the form

$$I = G(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(t, \mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t.$$

Wu et al. (2006) developed a computational method for solving an optimal control problem which is governed by a switched dynamical system with time delay. Kharatishidi (1961) approached this problem by extending the Pontryagin's maximum principle to time-delay systems. The actual solution involves a two-point boundary value problem in which advances and delays are presented. In addition, this solution does not yield a feedback controller. Optimal time control of delay systems has been considered by Oguztoreli (1963) who obtained several results concerning bang–bang controls which are parallel to those of LaSalle (1960) for non-delay systems. For a time-invariant system with an infinite upper limit in the performance measure, Krasovskii (1962) developed the forms of the controller and the performance measure. Ross (1968) obtained a set of differential equations for the unknowns in the forms of Krasovskii. However, Ross's results are not applicable to time-varying systems with a finite limit in the performance measure.

Basin and Perez (2007) presented an optimal regulator for a linear system with multiple states and input delays and a quadratic criterion. The optimal regulator equations were obtained by reducing the original problem to the linear–quadratic regulator design for a system without delays (see Basin and Perez 2007; Basin and Rodriguez-Gonzalez 2006).

This paper aims at minimizing quadratic cost functional over solutions of time-delay system of the following form:

$$\begin{aligned}
\min\ I &= \frac{1}{2}\mathbf{x}^T(t_f)H(t_f)\mathbf{x}(t_f) + \int_{t_0}^{t_f} (\mathbf{x}^T(t)P(t)\mathbf{x}(t) + \mathbf{u}^T(t)Q(t)\mathbf{u}(t))\, \mathrm{d}t \\
& s.t. \\
\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= A_1(t)\mathbf{x}(t) + A_2(t)(x_1(t-\tau_1)\ldots x_p(t-\tau_p))^T \\
&\quad + B_1(t)\mathbf{u}(t) + B_2(t)(u_1(t-\eta_1)\ldots u_m(t-\eta_m))^T \\
&\quad + F(t),\ t \in [t_0, t_f], \\
\mathbf{x}(t) &= \boldsymbol{\phi}(t),\ t \le t_0, \\
\mathbf{u}(t) &= \boldsymbol{\psi}(t),\ t \le t_0,
\end{aligned} \tag{1}$$

where $\mathbf{x}(t) = (x_1(t)\ldots x_p(t))^T \in \mathbb{R}^p$, $\mathbf{u}(t) = (u_1(t)\ldots u_m(t))^T \in \mathbb{R}^m$ are, respectively, state and control functions, while $\boldsymbol{\phi}(t) = (\phi_1(t)\ldots\phi_p(t))^T$, and $\boldsymbol{\psi}(t) = (\psi_1(t)\ldots\psi_m(t))^T$, are known as vectors functions and τ_is and η_js ($i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, m$) are non-negative constant time delays. We assume the matrices $H(t) = [h_{ij}(t)]_{p\times p}$ and $P(t) = [p_{ij}(t)]_{p\times p}$ are positive semi-definite, $Q(t) = [q_{ij}(t)]_{m\times m}$ is positive definite, $A_1(t) = [a^1_{ij}(t)]_{p\times p}$, $A_2(t) = [a^2_{ij}(t)]_{p\times p}$, $B_1(t) = [b^1_{ij}(t)]_{p\times m}$, $B_2(t) = [b^2_{ij}(t)]_{p\times m}$ are matrix functions and $F(t) = (f_1(t)\ldots f_p(t))^T$ is a vector function, which their elements are assumed to be polynomials defined on $[t_0, t_f]$. We need to impose continuity on $\mathbf{x}(t)$ and its first derivative whose constraints are described in Sect. 2.

Piecewise polynomials are often used to represent the approximate solution in the numerical solution of differential equations (see Winkel 2001; Zheng et al. 2004; Heinkenschloss 2005; Juddu 2002). Splines are usually defined as piecewise polynomials of degree n that these functions and their $n-1$ derivatives should be equal to the approximated function at the joining nodes. These conditions dramatically increase the computations. While in using Bezier curves as approximating functions, one needs to consider Bezier curves as C^1 functions in defined interval, these considerations reduce the computations. B-splines, due to numerical stability and arbitrary order of accuracy, have become popular tools for solving differential equations (where Bezier form is a special case of B-splines). There are many papers and books that deal with the Bezier curves or surface techniques. Harada and Nakamae (1982), Nürnberger and Zeilfelder (2000) used the Bezier control points in approximating data and functions. Zheng et al. (2004) proposed the use of control points of the Bernstein–Bezier form for solving differential equations numerically, and also Evrenosoglu and Somali (2008) used this approach for solving singular perturbed two-point boundary value problems. The Bezier curves are used for solving partial differential equations as well. Wave and heat equations are solved in Bezier form (see Beltran and Monterde 2004; Cholewa et al. 2002; Lang 0000; Layton and Van de Panne 2002), Bezier curves are used for solving dynamical systems (see Gachpazan 2011); also the Bezier control point method is used for solving delay differential equation (see Ghomanjani and Farahi 2012). Some other applications of the Bezier functions and control points are found in Chu et al. (2008), Farin (1988), and Shi and Sun (2000) that are used in computer-aided geometric design and image compression. The use of the Bezier curves for the optimal control of time-varying linear delay system is a novel idea. Although the method is very easy to use and straightforward, the obtained results are satisfactory (see the numerical results).

We suggest a technique similar to that used in Zheng et al. (2004), and Evrenosoglu and Somali (2008) for solving quadratic optimal control problems for time- delay systems. In Sect. 2, least square method is discussed. Convergence analysis is stated in Sect. 3. In Sect. 4, some numerical examples are solved which show the efficiency and reliability of the method. Finally, Sect. 5 gives a conclusion briefly.

2 Least square method

Consider the optimal control of time-varying linear system (1) with delays in state and control and with quadratic performance. Divide the interval $[t_0, t_f]$ into a set of grid points such that

$$t_i = t_0 + ih,\ i = 0,\ 1,\ \ldots,\ 2k,$$

where $h = \frac{t_f - t_0}{2k}$ and k is a positive integer. Let $S_j = [t_{j-1}, t_j]$ for $j = 1,\ 2,\ \ldots,\ 2k$. Then, for $t \in S_j$, the optimal control problem (1) can be decomposed into the following suboptimal control problems:

$$
\begin{aligned}
&\min\ I_j = C_j + \int\limits_{t_{j-1}}^{t_j} (\mathbf{x}_j^T(t)P(t)\mathbf{x}_j(t) + \mathbf{u}_j(t)^T(t)Q(t)\mathbf{u}_j(t))\mathrm{d}t \\
&\quad s.t. \\
&\frac{\mathrm{d}\mathbf{x}_j(t)}{\mathrm{d}t} = A_1(t)\mathbf{x}_j(t) + A_2(t)\left(x_1^{-k_1^1+j}(t-\tau_1)\ldots x_p^{-k_1^p+j}(t-\tau_p)\right)^T \\
&\qquad + B_1(t)\mathbf{u}_j(t) + B_2(t)\left(u_1^{-k_2^1+j}(t-\eta_1)\ldots u_m^{-k_2^m+j}(t-\eta_m)\right)^T \\
&\qquad + F(t),\ t \in S_j,\ j = 1, 2, \ldots, 2k, \\
&\mathbf{x}_j(\theta) = \boldsymbol{\phi}(\theta), \quad \theta \le t_0, \\
&\mathbf{u}_j(\theta) = \boldsymbol{\psi}(\theta), \quad \theta \le t_0,
\end{aligned} \tag{2}
$$

where $\mathbf{x}_j(t) = (x_1^j(t)\ldots x_p^j(t))^T$ and $\mathbf{u}_j(t) = (u_1^j(t)\ldots u_m^j(t))^T$ are, respectively, vectors of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ which are considered in $t \in S_j$. We should mention that the initial conditions in case $j = 1$ for sub-problem (2) are the same as the initial conditions for the problem (1). The initial conditions in the case $j = 2$ for sub-problem (2) are obtained from the sub-problem (2) when $j = 1$. The initial conditions in case $j = 3$ for sub-problem (2) are obtained from the sub-problem (2) when $j = 1$, and 2, and so on. It is notable that the problem (2) is in fact the problem (1) when it is induced on the limited interval S_j. We mention that $x_i^{-k_1^i+j}(t-\tau_i)$, $1 \le i \le p$, is the ith component of $(x_1^{-k_1^1+j}(t-\tau_1)\ldots x_p^{-k_1^p+j}(t-\tau_p))^T$, where $(t-\tau_i) \in [t_{-k_1^i+j-1}, t_{-k_1^i+j}]$ and $u_i^{-k_2^i+j}(t-\eta_i)$, $1 \le i \le m$, has the same definition as well. Also

$$
C_j = \begin{cases} \frac{1}{2}\mathbf{x}_{2k}^T(t_f)H(t_f)\mathbf{x}_{2k}(t_f) & j = 2k \\ 0 & j \ne 2k \end{cases}, \tag{3}
$$

$$
k_1^i = \begin{cases} \frac{\tau_i}{h} & \frac{\tau_i}{h} \in \mathbb{N} \\ ([\frac{\tau_i}{h}]+1) & \frac{\tau_i}{h} \notin \mathbb{N} \end{cases}, \quad 1 \le i \le p, \tag{4}
$$

$$
k_2^i = \begin{cases} \frac{\eta_i}{h} & \frac{\eta_i}{h} \in \mathbb{N} \\ ([\frac{\eta_i}{h}]+1) & \frac{\eta_i}{h} \notin \mathbb{N} \end{cases}, \quad 1 \le i \le m, \tag{5}
$$

where $[\frac{\tau_i}{h}]$ and $[\frac{\eta_i}{h}]$ denote the integer part of $\frac{\tau_i}{h}$ and $\frac{\eta_i}{h}$, respectively.

Let $\mathbf{x}(t) = \sum_{j=1}^{2k} \chi_j^1(t)\mathbf{x}_j(t)$ and $\mathbf{u}(t) = \sum_{j=1}^{2k} \chi_j^2(t)\mathbf{u}_j(t)$, where $\chi_j^1(t)$ and $\chi_j^2(t)$ ar, respectively, the characteristic function of $\mathbf{x}_j(t)$ and $\mathbf{u}_j(t)$ for $t \in [t_{j-1}, t_j]$. It is trivial that $[t_0, t_f] = \bigcup_{j=1}^{2k} S_j$.

Our strategy is to use Bezier curves to approximate the solutions $\mathbf{x}_j(t)$ and $\mathbf{u}_j(t)$ by $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$, respectively, which are given below. Individual Bezier curves that are defined over the subintervals are joined together to form the Bezier spline curves. For $j = 1, 2, \ldots, 2k$, define the Bezier polynomials $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$ of degree n that approximate, respectively, the actions of $\mathbf{x}_j(t)$ and $\mathbf{u}_j(t)$ over the interval $[t_{j-1}, t_j]$ as follows:

$$\mathbf{v}_j(t) = \sum_{r=0}^{n} \mathbf{a}_r^j B_{r,n}\left(\frac{t-t_{j-1}}{h}\right),$$
$$\mathbf{w}_j(t) = \sum_{r=0}^{n} \mathbf{b}_r^j B_{r,n}\left(\frac{t-t_{j-1}}{h}\right), \tag{6}$$

where

$$B_{r,n}\left(\frac{t-t_{j-1}}{h}\right) = \begin{matrix} n \\ r \end{matrix} \frac{1}{h^n}(t_j - t)^{n-r}(t-t_{j-1})^r$$

is the Bernstein polynomial of degree n over the interval $[t_{j-1}, t_j]$; $\mathbf{a}_r^j$ and $\mathbf{b}_r^j$ are, respectively, p and m ordered vectors of the control points (see Zheng et al. 2004). By substituting (6) in (2), one may define $R_{1,j}(t)$ and $R_{2,j}(t)$ for $t \in [t_{j-1}, t_j]$ as

$$\begin{aligned} R_{1,j}(t) = {} & \frac{\mathrm{d}\mathbf{v}_j(t)}{\mathrm{d}t} - (A_1(t)\mathbf{v}_j(t) \\ & + A_2(t)(v_1^{-k_1^1+j}(t-\tau_1)\ldots v_p^{-k_1^p+j}(t-\tau_p))^T) \\ & -(B_1(t)\mathbf{w}_j(t) \\ & + B_2(t)(w_1^{-k_2^1+j}(t-\eta_1)\ldots w_m^{-k_2^m+j}(t-\eta_m))^T) \\ & -F(t), \\ R_{2,j}(t) = {} & \mathbf{v}_j^T(t)P(t)\mathbf{v}_j(t) + \mathbf{w}_j^T(t)Q(t)\mathbf{w}_j(t). \end{aligned} \tag{7}$$

Let $\mathbf{v}(t) = \sum_{j=1}^{2k} \chi_j^1(t)\mathbf{v}_j(t)$ and $\mathbf{w}(t) = \sum_{j=1}^{2k} \chi_j^2(t)\mathbf{w}_j(t)$, where $\chi_j^1(t)$ and $\chi_j^2(t)$ are, respectively, the characteristic function of $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$ for $t \in [t_{j-1}, t_j]$. Beside the boundary conditions on $\mathbf{v}(t)$, at each node, we need to impose the continuity condition on each successive pair of $\mathbf{v}_j(t)$ to guarantee the smoothness. Since the differential equation is of first order, the continuity of $\mathbf{x}$ (or $\mathbf{v}$) and its first derivative give

$$\mathbf{v}_j^{(s)}(t_j) = \mathbf{v}_{j+1}^{(s)}(t_j), \ s = 0, 1, \ j = 1, 2, \cdots, \quad 2k-1, \tag{8}$$

where $\mathbf{v}_j^{(s)}(t_j)$ is the s-th derivative $\mathbf{v}_j(t)$ with respect to t at $t = t_j$.

Thus, the vector of control points $\mathbf{a}_r^j$ $(r = 0, 1, n-1, n)$ must satisfy (see "Appendix")

$$\begin{aligned} \mathbf{a}_n^j(t_j - t_{j-1})^n &= \mathbf{a}_0^{j+1}(t_{j+1} - t_j)^n, \\ (\mathbf{a}_n^j - \mathbf{a}_{n-1}^j)(t_j - t_{j-1})^{n-1} &= (\mathbf{a}_1^{j+1} - \mathbf{a}_0^{j+1})(t_{j+1} - t_j)^{n-1}. \end{aligned} \tag{9}$$

One may recall that $\mathbf{a}_r^j$ is a p ordered vector. This approach is called the subdivision scheme (or h-refinement in the finite element literature). In Sect. 3, we prove the convergence in the approximation via Bezier curves when n tends to infinity.

Remark 2.1 By considering the C^1 continuity of $\mathbf{w}$, the following constraints will be added to constraints in (9),

$$\begin{aligned} \mathbf{b}_n^j(t_j - t_{j-1})^n &= \mathbf{b}_0^{j+1}(t_{j+1} - t_j)^n, \\ (\mathbf{b}_n^j - \mathbf{b}_{n-1}^j)(t_j - t_{j-1})^{n-1} &= (\mathbf{b}_1^{j+1} - \mathbf{b}_0^{j+1})(t_{j+1} - t_j)^{n-1}, \end{aligned}$$

where the so-called $\mathbf{b}_r^j$ $(r = 0, 1, n-1, n)$ is a m ordered vector.

Now, the residual function can be defined in S_j as follows:

$$R_j = (C_j)^2 + \int_{t_{j-1}}^{t_j} (M\|R_{1,j}(t)\|^2 + (R_{2,j}(t))^2)\mathrm{d}t, \tag{10}$$

where $\|.\|$ is the Euclidean norm (recall that $R_{1,j}(t)$ is a p vector where $t \in S_j$) and M is a sufficiently large penalty parameter. Our aim is to solve the following problem over $S = \bigcup_{j=1}^{2k} S_j$:

$$\begin{aligned} &\min \sum_{j=1}^{2k} R_j \\ &s.t. \quad \mathbf{a}_n^j(t_j - t_{j-1})^n = \mathbf{a}_0^{j+1}(t_{j+1} - t_j)^n, \\ &(\mathbf{a}_n^j - \mathbf{a}_{n-1}^j)(t_j - t_{j-1})^{n-1} = (\mathbf{a}_1^{j+1} - \mathbf{a}_0^{j+1})(t_{j+1} - t_j)^{n-1}, \\ &j = 1, 2, \ldots, 2k-1. \end{aligned} \tag{11}$$

The mathematical programming problem (11) can be solved by many subroutine algorithms. Here, we used Maple 12 to solve this optimization problem.

From the solution of (11), it is obvious that the pair $(\mathbf{v}(.), \mathbf{w}(.))$ approximates well the optimal solution $(\mathbf{x}^*(.), \mathbf{u}^*(.))$.

Remark 2.2 In problem (1), if $\mathbf{x}(t_f)$ be unknown, then we set $C_{2k} = 0$.

Remark 2.3 Now, the main steps of this algorithm discussed in this section are as follows:

Give the problem (1), specify k and n, and define h, t_j's (for $j = 0, 1, \ldots, 2k$), define k_1^i's (for $i = 1, \ldots, p$) from (4) and k_2^i's (for $i = 1, \ldots, m$) from (5).

↓

Define $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$ as (6), then calculate $R_{1,j}(t)$, $R_{2,j}(t)$ and R_j in term of unknown vectors a_r^j and b_r^j (for $r = 0, 1, \ldots, n$, $j = 1, 2, \ldots, 2k$) respectively from (7) and (10).

↓

Solve the mathematical programming problem (11), according to its outputs, identify $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$ which are respectively approximated solutions of $\mathbf{x}_j(t)$ and $\mathbf{u}_j(t)$ for $t \in [t_{j-1}, t_j]$.

3 Convergence analysis

In this section without loss of generality, we analyze the convergence of the Bezier curve method applied to linear optimal control problem (1) with time delays in state and control when $p = m = 1$ and the time interval is [0, 1]. So, the following problem is considered:

$$\begin{aligned} \min\ I = \frac{1}{2}x(1)H(1)x(1) + \int_0^1 (x(t)P(t)x(t) \\ + u(t)Q(t)u(t))\,\mathrm{d}t \end{aligned}$$

$$\begin{aligned}
&s.t.\\
&L\left(x(t), u(t), x(t-\tau), u(t-\eta), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)\\
&\quad = \frac{\mathrm{d}x(t)}{\mathrm{d}t} - A_1(t)x(t) - A_2(t)x(t-\tau)\\
&\quad\quad -B_1(t)u(t) - B_2(t)u(t-\eta)\\
&\quad = F(t), \quad t \in [0,1]\\
&x(t) = x_0 = a, \quad t \le 0, \quad x(1) = x_f = b,\\
&u(t) = u_0 = a_1, \quad t \le 0.
\end{aligned} \tag{12}$$

where $x(t) \in R$, $u(t) \in R$, and a, b, a_1 are the given real numbers, and $A_1(t)$, $A_2(t)$, $B_1(t)$, $B_2(t)$ and $F(t)$ are the known polynomials for $t \in [0,1]$, and $H(t)$, $P(t)$ and $Q(t)$ are the given non-negative functions. The constant time delays τ and η are non-negative.

Without loss of generality, we consider the interval $[0,1]$ instead of $[t_0, t_f]$ since the variable t can be changed with the new variable z by $t = (t_f - t_0)z + t_0$ where $z \in [0,1]$.

Lemma 3.1 *For a polynomial in Bezier form*

$$x(t) = \sum_{i=0}^{n_1} a_{i,n_1} B_{i,n_1}(t),$$

we have

$$\frac{\sum_{i=0}^{n_1} a_{i,n_1}^2}{n_1+1} \ge \frac{\sum_{i=0}^{n_1+1} a_{i,n_1+1}^2}{n_1+2} \ge \dots \ge \frac{\sum_{i=0}^{n_1+m_1} a_{i,n_1+m_1}^2}{n_1+m_1+1} \to \int_0^1 x^2(t)\mathrm{d}t, \; m_1 \to +\infty,$$

where a_{i,n_1+m_1} is the Bezier coefficient of $x(t)$ after being degree-elevated to degree n_1+m_1.

Proof See Zheng et al. (2004). □

The convergence of the approximated solution could be done in two ways:

(1) Degree raising the Bezier polynomial approximation
(2) Subdivision of the time interval.

In the following, we prove the convergence in each case, although in numerical examples, we use only subdivision case.

3.1 Degree raising

Theorem 3.2 *If the linear optimal control problem* (12) *with time delays in state and control has a unique C^1 continuous trajectory solution $\bar{x}$, C^0 continuous control solution $\bar{u}$, then the approximate solution obtained by the Bezier curve method converges to the exact solution $(\bar{x}, \bar{u})$ as the degree of the approximated solution tends to infinity.*

Proof Given an arbitrary small positive number $\epsilon > 0$, by the Weierstrass Theorem (see Rudin 1986 and Sohrab 2003), it is obvious that polynomials $Q_{1,N_1}(t)$ of degree N_1 and $Q_{2,N_2}(t)$ of degree N_2 such that $\|\frac{\mathrm{d}^i Q_{1,N_1}(t)}{\mathrm{d}t^i} - \frac{\mathrm{d}^i \bar{x}(t)}{\mathrm{d}t^i}\|_\infty \le \frac{\epsilon}{16}$, $\|\frac{\mathrm{d}^i Q_{1,N_1}(t-\tau)}{\mathrm{d}t^i} - \frac{\mathrm{d}^i \bar{x}(t-\tau)}{\mathrm{d}t^i}\|_\infty \le$

$\frac{\epsilon}{16}$, $i = 0, 1$, $\|Q_{2,N_2}(t) - \bar{u}(t)\|_\infty \le \frac{\epsilon}{16}$, and $\|Q_{2,N_2}(t-\eta) - \bar{u}(t-\eta)\|_\infty \le \frac{\epsilon}{16}$, where $\|.\|_\infty$ stands for the L_∞-norm over [0, 1]. Especially, we have

$$\begin{aligned} \|a - Q_{1,N_1}(0)\|_\infty &\le \frac{\epsilon}{16}, \\ \|b - Q_{1,N_1}(1)\|_\infty &\le \frac{\epsilon}{16}, \\ \|a_1 - Q_{2,N_2}(0)\|_\infty &\le \frac{\epsilon}{16}. \end{aligned} \tag{13}$$

In general, $Q_{1,N_1}(t)$ and $Q_{2,N_2}(t)$ do not satisfy the boundary conditions. After a small perturbation with linear and constant polynomials $\alpha t + \beta$ and γ, respectively, for $Q_{1,N_1}(t)$ and $Q_{2,N_2}(t)$, we can obtain polynomials $P_{1,N_1}(t) = Q_{1,N_1}(t) + (\alpha t + \beta)$ and $P_{2,N_2}(t) = Q_{2,N_2}(t) + \gamma$ such that $P_{1,N_1}(t)$ satisfies the boundary conditions $P_{1,N_1}(0) = a$, $P_{1,N_1}(1) = b$, and $P_{2,N_2}(0) = a_1$. Thus, $Q_{1,N_1}(0) + \beta = a$ and $Q_{1,N_1}(1) + \alpha + \beta = b$. Using (13), one has

$$\begin{aligned} \|a - Q_{1,N_1}(0)\|_\infty &= \|\beta\|_\infty \le \frac{\epsilon}{16}, \\ \|b - Q_{1,N_1}(1)\|_\infty &= \|\alpha + \beta\|_\infty \le \frac{\epsilon}{16}. \end{aligned}$$

Since

$$\begin{aligned} \|\alpha\|_\infty - \|\beta\|_\infty \le \|\alpha + \beta\|_\infty &\le \frac{\epsilon}{16}, \\ \|\alpha\|_\infty &\le \frac{\epsilon}{16} + \|\beta\|_\infty \\ &\le \frac{\epsilon}{16} + \frac{\epsilon}{16} = \frac{\epsilon}{8}. \end{aligned}$$

By the time, $a_1 = P_{2,N_2}(0) = Q_{2,N_2}(0) + \gamma$, so

$$\|a_1 - Q_{2,N_2}(0)\|_\infty = \|\gamma\|_\infty \le \frac{\epsilon}{16}.$$

Now, we have

$$\begin{aligned} \|P_{1,N_1}(t) - \bar{x}(t)\|_\infty &= \|Q_{1,N_1}(t) + \alpha t + \beta - \bar{x}(t)\|_\infty \\ &\le \|Q_{1,N_1}(t) - \bar{x}(t)\|_\infty \\ &\quad + \|\alpha + \beta\|_\infty \le \frac{\epsilon}{8} < \frac{\epsilon}{5}, \\ \left\| \frac{\mathrm{d}P_{1,N_1}(t)}{\mathrm{d}t} - \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} \right\|_\infty &= \left\| \frac{\mathrm{d}Q_{1,N_1}(t)}{\mathrm{d}t} + \alpha - \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} \right\|_\infty \\ &\le \left\| \frac{\mathrm{d}Q_{1,N_1}(t)}{\mathrm{d}t} - \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} \right\|_\infty \\ &\quad + \|\alpha\|_\infty \le \frac{3\epsilon}{16} < \frac{\epsilon}{5}, \\ \|P_{2,N_2}(t) - \bar{u}(t)\|_\infty &= \|Q_{2,N_2}(t) + \gamma - \bar{u}(t)\|_\infty \\ &\le \|Q_{2,N_2}(t) - \bar{u}(t)\|_\infty + \|\gamma\|_\infty \\ &\le \frac{\epsilon}{8} < \frac{\epsilon}{5}, \end{aligned}$$

so,

$$\|P_{1,N_1}(t-\tau)-\bar{x}(t-\tau)\|_\infty < \frac{\epsilon}{5},$$
$$\|P_{2,N_2}(t-\eta)-\bar{u}(t-\eta)\|_\infty < \frac{\epsilon}{5}.$$

Now, let define $LP_N(t) = L(P_{1,N_1}(t), P_{2,N_2}(t), P_{1,N_1}(t-\tau), P_{2,N_2}(t-\eta), \frac{\mathrm{d}P_{1,N_1}(t)}{\mathrm{d}t}) = \frac{\mathrm{d}P_{1,N_1}(t)}{\mathrm{d}t} - A_1(t)P_{1,N_1}(t) - A_2(t)P_{1,N_1}(t-\tau) - B_1(t)P_{2,N_2}(t) - B_2(t)P_{2,N_2}(t-\eta) = F(t)$, for every $t \in [0,1]$. Thus, for $N \geq \max\{N_1, N_2\}$, for the following residual, an upper bound can be found:

$$\begin{aligned}\|LP_N(t)-F(t)\|_\infty &= \|L\Bigg(P_{1,N_1}(t), P_{2,N_2}(t), P_{1,N_1}(t-\tau),\\ &\qquad P_{2,N_2}(t-\eta), \frac{\mathrm{d}P_{1,N_1}(t)}{\mathrm{d}t}\Bigg) - F(t)\|_\infty\\ &\leq \|\frac{\mathrm{d}P_{1,N_1}(t)}{\mathrm{d}t} - \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\|_\infty\\ &\quad +\|A_1(t)\|_\infty\|P_{1,N_1}(t)-\bar{x}(t)\|_\infty\\ &\quad +\|A_2(t)\|_\infty\|P_{1,N_1}(t-\tau)-\bar{x}(t-\tau)\|_\infty\\ &\quad +\|B_1(t)\|_\infty\|P_{2,N_2}(t)-\bar{u}(t)\|_\infty\\ &\quad +\|B_2(t)\|_\infty\|P_{2,N_2}(t-\eta)-\bar{u}(t-\eta)\|_\infty\\ &\leq C_1\left(\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}\right) = C_1\epsilon,\end{aligned}$$

where $C_1 = 1 + \|A_1(t)\|_\infty + \|A_2(t)\|_\infty + \|B_1(t)\|_\infty + \|B_2(t)\|_\infty$ is a constant.
Since the residual $R(P_N) := LP_N(t) - F(t)$ is a polynomial, we can represent it by a Bezier form. Therefore, we have

$$R(P_N) := \sum_{i=0}^{m_1} d_{i,m_1} B_{i,m_1}(t).$$

Then, by Lemma 3.1, there exists an integer $M(\geq N)$ such that when $m_1 > M$, we have

$$\left|\frac{1}{m_1+1}\sum_{i=0}^{m_1} d_{i,m_1}^2 - \int_0^1 (R(P_N))^2\,\mathrm{d}t\right| < \epsilon,$$

which gives

$$\begin{aligned}\frac{1}{m_1+1}\sum_{i=0}^{m_1} d_{i,m_1}^2 &< \epsilon + \int_0^1 (R(P_N))^2\,\mathrm{d}t\\ &\leq \epsilon + C_1^2\epsilon^2.\end{aligned} \tag{14}$$

Suppose $x(t)$ and $u(t)$ are approximated solutions of (12) obtained by the Bezier curve method of degree m_2 $(m_2 \geq m_1 \geq M)$. Let

$$
\begin{aligned}
&R\left(x(t), u(t), x(t-\tau), u(t-\eta), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)\\
&= L\left(x(t), u(t), x(t-\tau), u(t-\eta), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) - F(t)\\
&= \sum_{i=0}^{m_2} c_{i,m_2} B_{i,m_2}(t),\ m_2 \geq m_1 \geq M, \quad t \in [0,1].
\end{aligned}
$$

Define the following norm for difference approximated solution $(x(t), u(t))$ and exact solution $(\bar{x}(t), \bar{u}(t))$:

$$
\begin{aligned}
&\|(x(t), u(t)) - (\bar{x}(t), \bar{u}(t))\|\\
&:= \int_0^1 \sum_{j=0}^{1} \left| \frac{\mathrm{d}^j x(t)}{\mathrm{d}t^j} - \frac{\mathrm{d}^j \bar{x}(t)}{\mathrm{d}t^j} \right|^2 \mathrm{d}t\\
&+ \int_0^1 |u(0) - \bar{u}(0)|\, \mathrm{d}t.
\end{aligned} \tag{15}
$$

By (15) and Lemma 3.1, one can show that

$$
\begin{aligned}
&\|(x(t), u(t)) - (\bar{x}(t), \bar{u}(t))\| \leq C(|x(0) - \bar{x}(0)|\\
&\quad + |x(1) - \bar{x}(1)| + |u(0) - \bar{u}(0)|\\
&\quad + \left\| R\left(\left(x(t), u(t), x(t-\tau), u(t-\eta), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)\right.\right.\\
&\quad \left.\left. - \left(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)\right)\right\|_2^2\\
&= C \int_0^1 \sum_{i=0}^{m_2} (c_{i,m_2} B_{i,m_2}(t))^2\, \mathrm{d}t\\
&\leq \frac{C}{m_2+1} \sum_{i=0}^{m_2} c_{i,m_2}^2.
\end{aligned} \tag{16}
$$

The last inequality in (16) is obtained by Lemma 3.1 where C is a constant positive number. Now, by Lemma 3.1 and (17), one has

$$
\begin{aligned}
\|(x(t), u(t)) - (\bar{x}(t), \bar{u}(t))\| &\leq \frac{C}{m_2+1} \sum_{i=0}^{m_2} c_{i,m_2}^2\\
&\leq \frac{C}{m_2+1} \sum_{i=0}^{m_2} d_{i,m_2}^2\\
&\leq \frac{C}{m_1+1} \sum_{i=0}^{m_1} d_{i,m_1}^2\\
&\leq C(\epsilon + C_1^2 \epsilon^2)\\
&= \epsilon_1,\ m_1 \geq M,
\end{aligned} \tag{17}
$$

where the last inequality in (17) comes from (14).

Thus, from (17) we have

$$\|x(t) - \bar{x}(t)\| \leq \epsilon_1,$$
$$\|u(t) - \bar{u}(t)\| \leq \epsilon_1.$$

Since the infinite norm and the norm defined in (15) are equivalent, there is a $\rho_1 > 0$ where

$$\|x(t) - \bar{x}(t)\|_\infty \leq \rho_1\epsilon_1,$$
$$\|u(t) - \bar{u}(t)\|_\infty \leq \rho_1\epsilon_1.$$

Now, we show that the approximated cost function tends to exact cost function as the degree of Bezier approximation increases. Let

$$I_{\text{exact}} = \frac{1}{2}\bar{x}(1)H(1)\bar{x}(1) + \int_0^1 (\bar{x}(t)P(t)\bar{x}(t) + \bar{u}(t)Q(t)\bar{u}(t))\,\mathrm{d}t,$$
$$I_{\text{approx}} = \frac{1}{2}x(1)H(1)x(1) + \int_0^1 (x(t)P(t)x(t) + u(t)Q(t)u(t))\,\mathrm{d}t,$$

for $t \in [0, 1]$. Now, there are four positive integers $M_i \geq 0, i = 1, \ldots, 4$ such that $\|P(t)\|_\infty \leq M_1$, $\|Q(t)\|_\infty \leq M_2$, $\|\bar{x}(t)\|_\infty \leq M_3$, and $\|\bar{u}(t)\|_\infty \leq M_4$. Since

$$\|x(t)\|_\infty - \|\bar{x}(t)\|_\infty \leq \|\bar{x}(t) - x(t)\|_\infty \leq \rho_1\epsilon_1,$$
$$\|u(t)\|_\infty - \|\bar{u}(t)\|_\infty \leq \|\bar{u}(t) - u(t)\|_\infty \leq \rho_1\epsilon_1,$$

we have

$$\|x(t)\|_\infty \leq \|\bar{x}(t)\|_\infty + \rho_1\epsilon_1 \leq M_3 + \rho_1\epsilon_1,$$
$$\|u(t)\|_\infty \leq \|\bar{u}(t)\|_\infty + \rho_1\epsilon_1 \leq M_4 + \rho_1\epsilon_1,$$

so,

$$\|\bar{x}(t) + x(t)\|_\infty \leq \|\bar{x}(t)\|_\infty + \|x(t)\|_\infty \leq 2M_3 + \rho_1\epsilon_1,$$
$$\|\bar{u}(t) + u(t)\|_\infty \leq \|\bar{u}(t)\|_\infty + \|u(t)\|_\infty \leq 2M_4 + \rho_1\epsilon_1,$$

now, we have

$$\begin{aligned}\|I_{\text{exact}} - I_{\text{approx}}\|_\infty &= \Big\| \int_0^1 \bar{x}(t)P(t)\bar{x}(t) + \bar{u}(t)Q(t)\bar{u}(t) - x(t)P(t)x(t) - u(t)Q(t)u(t)\,\mathrm{d}t \Big\|_\infty \\ &\leq \int_0^1 \|\bar{x}(t)P(t)\bar{x}(t) - x(t)P(t)x(t)\|_\infty \mathrm{d}t \\ &\quad + \int_0^1 \|\bar{u}(t)Q(t)\bar{u}(t) - u(t)Q(t)u(t)\|_\infty \mathrm{d}t\end{aligned}$$

$$\leq \int_0^1 \|P(t)\|_\infty \|\bar{x}^2(t) - x^2(t)\|_\infty \mathrm{d}t$$
$$+ \int_0^1 \|Q(t)\|_\infty \|\bar{u}^2(t) - u^2(t)\|_\infty \mathrm{d}t$$
$$\leq \int_0^1 \|P(t)\|_\infty \|\bar{x}(t) - x(t)\|_\infty \|\bar{x}(t) + x(t)\|_\infty \mathrm{d}t$$
$$+ \int_0^1 \|Q(t)\|_\infty \|\bar{u}(t) - u(t)\|_\infty \|\bar{u}(t) + u(t)\|_\infty \mathrm{d}t$$
$$\leq M_1\rho_1\epsilon_1(\rho_1\epsilon_1 + 2M_3) + M_2\rho_1\epsilon_1(\rho_1\epsilon_1 + 2M_4).$$

This completes the proof. □

3.2 Subdivision

Theorem 3.3 *Let* (x, u) *be the approximated solution of the linear optimal control problem* (12) *obtained by the subdivision scheme of the Bezier curve method. If* (12) *has a unique solution* $(\bar{x}, \bar{u})$ *where* $(\bar{x}, \bar{u})$ *is smooth enough so that the cubic spline* $T(\bar{x}, \bar{u})$ *interpolates to* $(\bar{x}, \bar{u})$ *and converges to* $(\bar{x}, \bar{u})$ *in the order* $O(h^q)$, $(q > 2)$, *where h is the maximal width of all subintervals, then* (x, u) *converges to* $(\bar{x}, \bar{u})$ *as* $h \to 0$.

Proof We first impose a uniform partition $\prod_d = \bigcup_i [t_i, t_{i+1}]$ on the interval $[0, 1]$ as $t_i = id$, where $d = \frac{1}{n_1+1}$.

Let $I_d(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t})$ be the cubic spline over $\prod_d$ which is interpolating to $(\bar{x}, \bar{u})$. Then, for an arbitrarily small positive number $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\Bigg\| L\left(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)$$
$$-L\left(I_{\mathrm{d}}\left(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)\right)\Bigg\|_\infty \leq \epsilon,$$

provided that $d < \delta_1$. Let $R(I_d(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t})) = L(I_d(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t})) - F(t)$ be the residual. For each subinterval $[t_i, t_{i+1}]$, $R(I_d(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}))$ is a polynomial. On each interval $[t_i, t_{i+1}]$, we impose another uniform partition $\prod_{i,h} = \bigcup_j [t_{i,j}, t_{i,j+1}]$ as $t_{i,j} = id + jh$, where $h = \frac{d}{m_1}$, $j = 0, \ldots, m_1$. Express $R(I_d(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}))$ in $[t_{i,j-1}, t_{i,j}]$ as

$$R\left(I_d\left(\bar{x}(t), \bar{u}(t), \bar{x}(t-\tau), \bar{u}(t-\eta), \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)\right)$$
$$= \sum_{p_1=0}^{l} r_{p_1}^{i,j} B_{p_1,l}(t),\ t \in [t_{i,j-1}, t_{i,j}].$$

By Lemma 3 in Zheng et al. (2004), there exists a $\delta_2 > 0$ ($\delta_2 \leq \delta_1$) such that when $h < \delta_2$, we have

$$\left|\sum_{j=1}^{m_1}(t_{i,j}-t_{i,j-1})\sum_{p_1=0}^{l}(r_{p_1}^{i,j})^2 -(l+1)\int_{t_i}^{t_{i+1}} R^2\left(I_d\left(\bar{x}(t),\bar{u}(t),\bar{x}(t-\tau),\bar{u}(t-\eta),\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)\right)\right| \le \frac{\epsilon}{d}.$$

Thus,

$$\left|\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}(t_{i,j}-t_{i,j-1})\sum_{p_1=0}^{l}(r_{p_1}^{i,j})^2 -(l+1)\int_{0}^{1} R^2\left(I_d\left(\bar{x}(t),\bar{u}(t),\bar{x}(t-\tau),\bar{u}(t-\eta),\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)\right)\right| \le \epsilon,$$

or

$$\begin{aligned}&\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}(t_{i,j}-t_{i,j-1})\sum_{p_1=0}^{l}(r_{p_1}^{i,j})^2\\ &\quad< (l+1)\int_0^1 R^2\left(I_d(\bar{x}(t),\bar{u}(t),\bar{x}(t-\tau),\bar{u}(t-\eta),\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)+\epsilon\\ &\quad< (l+1)\epsilon^2+\epsilon.\end{aligned}$$

Now, combining the partitions $\prod_d$ and all $\prod_{i,h}$ gives a denser partition with the length h for each subinterval. Suppose $(x(t), u(t))$ is the approximated solution by the Bezier curve method with respect to this partition, and denote the residual over $[t_{i,j-1}, t_{i,j}]$ by

$$\begin{aligned}&R\left(x(t),u(t),x(t-\tau),u(t-\eta),\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)\\ &\quad= L\left(x(t),u(t),x(t-\tau),u(t-\eta),\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)-F(t)\\ &\quad=\sum_{p_1=0}^{l} c_{p_1}^{i,j}B_{p_1,l}(t).\end{aligned}$$

Define the following norm for difference approximated solution $(x(t), u(t))$ and exact solution $(\bar{x}(t), \bar{u}(t))$:

$$\begin{aligned}&\|(x(t),u(t))-(\bar{x}(t),\bar{u}(t))\|\\ &\quad:=\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}\int_{t_{i,j-1}}^{t_{i,j}}|x(t)-\bar{x}(t)|^2\,\mathrm{d}t\\ &\qquad+\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}\int_{t_{i,j-1}}^{t_{i,j}}\left|\frac{\mathrm{d}x(t)}{\mathrm{d}t}-\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right|^2\mathrm{d}t\\ &\qquad+\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}\int_{t_{i,j-1}}^{t_{i,j}}|u(0)-\bar{u}(0)|\,\mathrm{d}t. \qquad (18)\end{aligned}$$

Then, there is a constant C such that

$$\begin{aligned}
&\|(x(t),u(t))-(\bar{x}(t),\bar{u}(t))\| \\
&\leq C\left\|R\left(\left(x(t),u(t),x(t-\tau),u(t-\eta),\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)\right.\right. \\
&\quad\left.\left.-\left(\bar{x}(t),\bar{u}(t),\bar{x}(t-\tau),\bar{u}(t-\eta),\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right)\right)\right\|_2^2 \\
&\leq \frac{C}{l+1}\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}(t_{i,j}-t_{i,j-1})\sum_{p_1=0}^{l}(c_{p_1}^{i,j})^2,
\end{aligned} \tag{19}$$

the last inequality in (19) is obtained by Lemma 3.1. It can be shown that

$$\begin{aligned}
&\|(x(t),u(t))-(\bar{x}(t),\bar{u}(t))\| \\
&\quad\leq \frac{C}{l+1}\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}(t_{i,j}-t_{i,j-1})\sum_{p_1=0}^{l}(c_{p_1}^{i,j})^2 \\
&\quad\leq \frac{C}{l+1}\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}(t_{i,j}-t_{i,j-1})\sum_{p_1=0}^{l}(r_{p_1}^{i,j})^2 \\
&\quad\leq C\left(\epsilon^2+\frac{\epsilon}{l+1}\right)=\epsilon_2.
\end{aligned} \tag{20}$$

Thus, from (20) we have:

$$\begin{aligned}
\|x(t)-\bar{x}(t)\| &\leq \epsilon_2, \\
\|u(t)-\bar{u}(t)\| &\leq \epsilon_2.
\end{aligned}$$

Since the infinite norm and the norm defined in (18) are equivalent, there is a $\rho_2>0$ where

$$\begin{aligned}
\|x(t)-\bar{x}(t)\|_\infty &\leq \rho_2\epsilon_2, \\
\|u(t)-\bar{u}(t)\|_\infty &\leq \rho_2\epsilon_2.
\end{aligned}$$

Now, it can be shown that the approximated cost function converges to exact cost function in the subdivision case. Define

$$\begin{aligned}
I_{\text{exact}} &= \frac{1}{2}\bar{x}(1)H(1)\bar{x}(1)+\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}\int_{t_{i,j-1}}^{t_{i,j}}(\bar{x}(t)P(t)\bar{x}(t) \\
&\quad+\bar{u}(t)Q(t)\bar{u}(t))\,\mathrm{d}t, \\
I_{\text{approx}} &= \frac{1}{2}x(1)H(1)x(1)+\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}\int_{t_{i,j-1}}^{t_{i,j}}(x(t)P(t)x(t) \\
&\quad+u(t)Q(t)u(t))\,\mathrm{d}t,
\end{aligned}$$

for $t\in[t_{i,j-1},t_{i,j}]$. Now, there are four positive integers $M_i\geq 0$, $i=1,\ldots,4$, such that $\|P(t)\|_\infty\leq M_1$, $\|Q(t)\|_\infty\leq M_2$, $\|\bar{x}(t)\|_\infty\leq M_3$, and $\|\bar{u}(t)\|_\infty\leq M_4$. Since

$$\begin{aligned}
\|x(t)\|_\infty-\|\bar{x}(t)\|_\infty &\leq \|\bar{x}(t)-x(t)\|_\infty\leq\rho_2\epsilon_2, \\
\|u(t)\|_\infty-\|\bar{u}(t)\|_\infty &\leq \|\bar{u}(t)-u(t)\|_\infty\leq\rho_2\epsilon_2,
\end{aligned}$$

we have

$$\|x(t)\|_\infty \le \|\bar{x}(t)\|_\infty + \rho_2\epsilon_2 \le M_3 + \rho_2\epsilon_2,$$
$$\|u(t)\|_\infty \le \|\bar{u}(t)\|_\infty + \rho_2\epsilon_2 \le M_4 + \rho_2\epsilon_2,$$

so,

$$\|\bar{x}(t) + x(t)\|_\infty \le \|\bar{x}(t)\|_\infty + \|x(t)\|_\infty \le 2M_3 + \rho_2\epsilon_2,$$
$$\|\bar{u}(t) + u(t)\|_\infty \le \|\bar{u}(t)\|_\infty + \|u(t)\|_\infty \le 2M_4 + \rho_2\epsilon_2,$$

now, we have

$$\begin{aligned}
\|I_{\text{exact}} - I_{\text{approx}}\|_\infty &= \left\| \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \bar{x}(t)P(t)\bar{x}(t) \right. \\
&\quad + \bar{u}(t)Q(t)\bar{u}(t) - x(t)P(t)x(t) \\
&\quad \left. - u(t)Q(t)u(t)\,\mathrm{d}t \right\|_\infty \\
&\le \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \|\bar{x}(t)P(t)\bar{x}(t) - x(t)P(t)x(t)\|_\infty \mathrm{d}t \\
&\quad + \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \|\bar{u}(t)Q(t)\bar{u}(t) - u(t)Q(t)u(t)\|_\infty \mathrm{d}t \\
&\le \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \|P(t)\|_\infty \|\bar{x}^2(t) - x^2(t)\|_\infty \mathrm{d}t \\
&\quad + \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \|Q(t)\|_\infty \|\bar{u}^2(t) - u^2(t)\|_\infty \mathrm{d}t \\
&\le \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \|P(t)\|_\infty \|\bar{x}(t) - x(t)\|_\infty \|\bar{x}(t) + x(t)\|_\infty \mathrm{d}t \\
&\quad + \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} \int_{t_{i,j-1}}^{t_{i,j}} \|Q(t)\|_\infty \|\bar{u}(t) - u(t)\|_\infty \|\bar{u}(t) + u(t)\|_\infty \mathrm{d}t \\
&\le \frac{n_1(n_1+1)}{2}\frac{m_1(m_1+1)}{2} h M_1 \rho_2\epsilon_2(\rho_2\epsilon_2 + 2M_3) \\
&\quad + \frac{n_1(n_1+1)}{2}\frac{m_1(m_1+1)}{2} h M_2 \rho_2\epsilon_2(\rho_2\epsilon_2 + 2M_4).
\end{aligned}$$

By Lemma 3 in Zheng et al. (2004), we conclude that the approximated solution converges to the exact solution in the order $o(h^q)$, $(q > 2)$. This completes the proof. □

4 Numerical examples

While applying the presented method, in Examples 1, 2, 3, 4, 5, and 6 we choose the Bezier curves as piecewise polynomials of degree 3.

Example 1 Consider the optimal control of linear time-delay system (see Palanisamy and Rao 1983),

$$\begin{aligned} \min\ I &= \frac{1}{2}\int_0^2 (x^2(t)+u^2(t))\,\mathrm{d}t,\\ s.t.\quad & \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t-1)+u(t),\\ & x(t)=1,\quad t\le 0,\\ & u(t)=-2,\quad t\le 0. \end{aligned}$$

We need to mention that in Example 1, there is a delay in the state only. Let $k = 3$, then the time interval $[0, 2]$ is divided into 6 subintervals. From (11), the following approximated solutions can be found for the state $x(t)$ and the control $u(t)$,

$$x(t)=\begin{cases} 0.9999999997-1.017369899t+1.95635344t^2-1.049862820t^3, & 0\le t\le \frac{1}{3},\\ 0.8673937557-0.1209966900t+0.1584828t^2-0.1432411700t^3, & \frac{1}{3}\le t\le \frac{2}{3},\\ 1.475965806-2.304655589t+2.601598153t^2-0.9486123500t^3, & \frac{2}{3}\le t\le 1,\\ -0.03981875368+2.321307955t-2.102975263t^2+.64578208t^3, & 1\le t\le \frac{4}{3},\\ 6.672738304-12.03971118t+8.11111341t^2-1.76857117t^3, & \frac{4}{3}\le t\le \frac{5}{3},\\ 8.719375271-14.13399691t+8.413888355t^2-1.63836686t^3, & \frac{5}{3}\le t\le 2, \end{cases}$$

$$u(t)=\begin{cases} -1.999999999+3.497666466t+0.4582933066t^2-8.13100258t^3, & 0\le t\le \frac{1}{3},\\ -3.059435875+12.66068894t-25.91507291t^2+17.1266624t^3, & \frac{1}{3}\le t\le \frac{2}{3},\\ -4.087320158+10.01605060t-11.04293897t^2+4.23800721t^3, & \frac{2}{3}\le t\le 1,\\ -6.969863914+16.84971548t-16.06263745t^2+5.30658457t^3, & 1\le t\le \frac{4}{3},\\ -1.6262781-6.448864322t+9.867931179t^2-3.290216032t^3, & \frac{4}{3}\le t\le \frac{5}{3},\\ 4.929764665-14.10458008t+10.38979280t^2-2.263380566t^3, & \frac{5}{3}\le t\le 2. \end{cases}$$

The graphs of approximated trajectory and control are shown in Fig. 1. The approximated and exact objective functions are, respectively, $I = 1.593587244$ and $I^* = 1.59$ (see Palanisamy and Rao 1983). The computation takes 20 s of CPU time when it is performed by Maple 12 on an AMD Athelon X4 PC with 2 GB of RAM. The QPSolve command solves (11), which involves computing the minimum of a quadratic objective function possibly subject to linear constraints. The QPSolve command uses an iterative active-set method implemented in a built-in library provided by the numerical algorithms group.

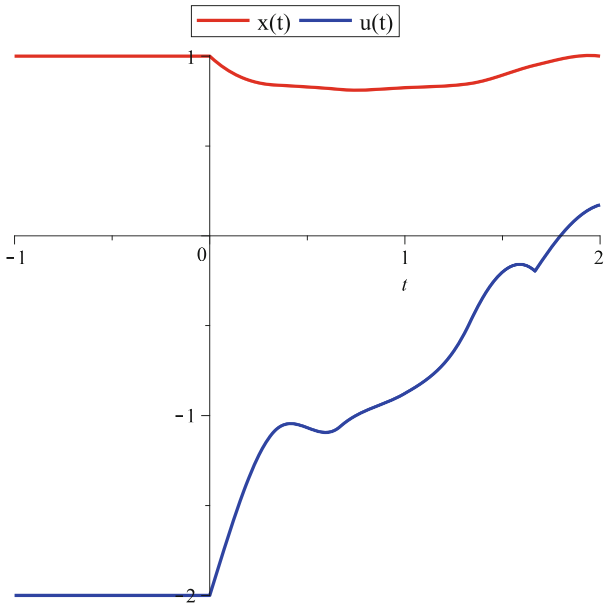

Fig. 1 The graphs of approximated trajectory and control for Example 1

We mention that the value of objective function with the proposed method is more accurate than that with the presented method in Palanisamy and Rao (1983).

Example 2 Consider the optimal control of linear time-delay system (see Palanisamy and Rao 1983),

$$\min I = \frac{1}{2}\int_0^1 (x(t)^2 + u(t)^2)\,\mathrm{d}t,$$

$$s.t. \quad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + x\left(t - \frac{1}{3}\right) + u(t) - \frac{1}{2}u\left(t - \frac{2}{3}\right),$$

$$x(t) = 1, \quad t \in \left[-\frac{1}{3}, 0\right],$$

$$u(t) = -1, \quad t \in \left[-\frac{2}{3}, 0\right].$$

Let $k = 3$. From (11), the following approximated solutions can be found for the state $x(t)$ and the control $u(t)$:

$$x(t) = \begin{cases} 1.000000001 - 1.094274275t + 7.009277868t^2 - 21.4398625t^3, & t \in [0, \frac{1}{6}], \\ 1.443587597 - 6.822905439t + 27.84539125t^2 - 36.0407421t^3, & t \in [\frac{1}{6}, \frac{1}{3}], \\ -0.2629450168 + 9.543968302t - 24.2794706t^2 + 19.1083604t^3, & t \in [\frac{1}{3}, \frac{1}{2}], \\ 0.9618527121 - 0.9806318053t + 3.1213571t^2 - 3.3932764t^3, & t \in [\frac{1}{2}, \frac{2}{3}], \\ -.2065981021 + 9.498503471t - 20.42900575t^2 + 12.297735t^3, & t \in [\frac{2}{3}, \frac{5}{6}], \\ 14.94439046 - 51.03072302t + 59.38886727t^2 - 22.5025347t^3, & t \in [\frac{5}{6}, 1] \end{cases}$$

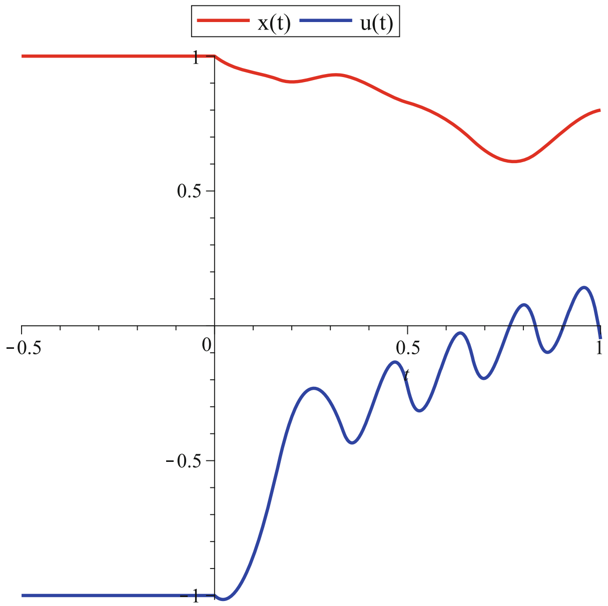

Fig. 2 The graphs of approximated trajectory and control for Example 2

$$u(t) = \begin{cases} -1.000000001 - 1.329818401t + 31.07777526t^2 - 32.0406571t^3, & t \in [0, \frac{1}{6}], \\ -2.835696746 + 22.5624782t - 57.37453536t^2 + 35.06102613t^3, & t \in [\frac{1}{6}, \frac{1}{3}], \\ 28.52820978 - 218.0458504t + 539.4499599t^2 - 436.7629786t^3, & t \in [\frac{1}{3}, \frac{1}{2}], \\ 94.62327190 - 495.4215408t + 855.8119763t^2 - 488.7447465t^3, & t \in [\frac{1}{2}, \frac{2}{3}], \\ 200.2261824 - 809.5696358t + 1085.436615t^2 - 482.7583142t^3, & t \in [\frac{2}{3}, \frac{5}{6}], \\ 426.7110543 - 1414.224688t + 1558.194094t^2 - 570.729872t^3, & t \in [\frac{5}{6}, 1]. \end{cases}$$

The graphs of approximated trajectory and control by this method are shown in Fig. 2. The approximated and exact objective functions are, respectively, $I = 0.4220497643$ and $I^* = 0.4220$. Our approach is more accurate with results in Palanisamy and Rao (1983).

Palanisamy and Rao (1983) presented a computational algorithm via Walsh Functions (WF), and the results by the WF method with $m = 100$ were shown, where m was a number of components in Walsh series.

Walsh series is used for $m = 100$, but the obtained results of the proposed method with $k = 3$ are more accurate than the results of the presented method in Palanisamy and Rao (1983).

Example 3 Consider the previous example while we increase k from 3 to 4. From (11), the following approximated solutions can be found for the state $x(t)$ and the control $u(t)$:

$$x(t)=\begin{cases}1-0.9138164531t+5.529958375t^2-20.0135236t^3, & t\in[0,\frac{1}{8}],\\ 1.177058837-3.914197484t+19.5407582t^2-30.7296608t^3, & t\in[\frac{1}{8},\frac{1}{4}],\\ 1.886094914-10.0860121t+34.88154345t^2-38.7220768t^3, & t\in[\frac{1}{4},\frac{3}{8}],\\ -4.078875011+36.13850158t-84.3965045t^2+63.7585686t^3, & t\in[\frac{3}{8},\frac{1}{2}],\\ -1.692056145+13.76608415t-23.54866115t^2+12.4580007t^3, & t\in[\frac{1}{2},\frac{5}{8}],\\ 7.253866273-29.18890961t+45.20263472t^2-24.2217869t^3, & t\in[\frac{5}{8},\frac{3}{4}],\\ -4.579939593+33.49864909t-58.85055718*t^2+31.1217563t^3, & t\in[\frac{3}{4},\frac{7}{8}],\\ 7.904364831-26.62673081t+29.66079178*t^2-10.1384258t^3, & t\in[\frac{7}{8},1],\end{cases}$$

$$u(t)=\begin{cases}-1.-3.777279531t+55.65743925t^2-47.0312335t^3, & t\in[0,\frac{1}{8}],\\ -2.288335188+17.96402697t-44.84310858t^2+25.1571492t^3, & t\in[\frac{1}{8},\frac{1}{4}],\\ 7.582984429-86.52818596t+317.2712533t^2-383.1893467t^3, & t\in[\frac{1}{4},\frac{3}{8}],\\ 114.3612068-796.4503892t+1825.587592t^2-1381.86206t^3, & t\in[\frac{3}{8},\frac{1}{2}],\\ 196.0023593-1061.199677t+1904.890913t^2-1134.600772t^3, & t\in[\frac{1}{2},\frac{5}{8}],\\ 488.1937132-2151.73991t+3150.590061t^2-1532.752197t^3, & t\in[\frac{5}{8},\frac{3}{4}],\\ 743.3521221-2741.564191t+3362.609963t^2-1371.688832t^3, & t\in[\frac{3}{4},\frac{7}{8}],\\ 723.0849239-2316.117024t+2469.573623t^2-876.5088905t^3, & t\in[\frac{7}{8},1].\end{cases}$$

The graphs of approximated trajectory and control by this method are shown in Fig. 3. The approximated and exact objective functions are, respectively, $I=0.4220385520$ and $I^*=0.4220$ (see Palanisamy and Rao 1983).

Example 4 Consider the linear time-varying system with delays described by (see Wang 2007)

$$\begin{aligned}\min\ I &= \frac{1}{2}\int_0^1\left([\,x_1(t)\ x_2(t)\,]\begin{bmatrix}1 & t\\ t & t^2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\right.\\ &\qquad\left.+(t^2+1)u(t)\right)\mathrm{d}t\\ s.t.\ \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= (t^2+1)x_1\left(t-\frac{1}{2}\right)+x_2\left(t-\frac{1}{2}\right)+u(t)\\ &\quad+(t+1)u\left(t-\frac{1}{4}\right),\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= 2x_2\left(t-\frac{1}{2}\right)+(t+1)u(t)\\ &\quad+(t^2+1)u\left(t-\frac{1}{4}\right),\\ \left[x_1(t)\ x_2(t)\right]^T &= [1\ 1]^T,\ t\in\left[-\frac{1}{2},0\right],\\ u(t) &= 1,\ t\in\left[-\frac{1}{4},0\right],\end{aligned}$$

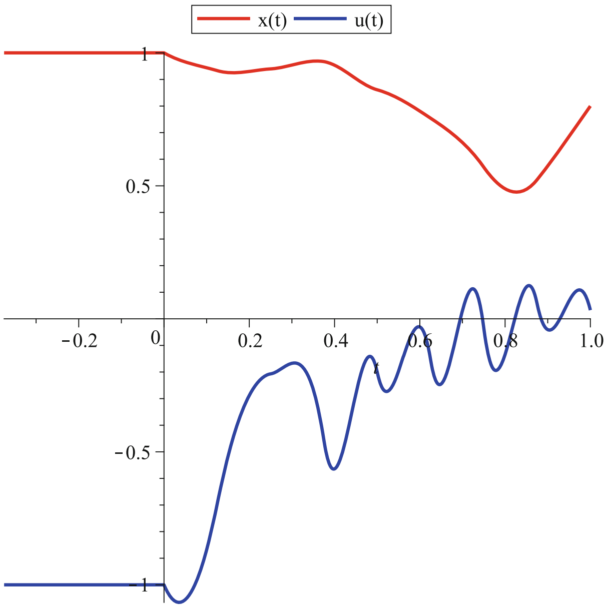

Fig. 3 The graphs of approximated trajectory and control for Example 3

where $\mathbf{x}(t) = [x_1(t)\, x_2(t)]^T$ is two-dimensional state function and $u(t)$ one-dimensional control function. Let $k = 4$. From (11), one can find the following approximated solutions $x_1(t)$, $x_2(t)$ and $u(t)$:

$$x_1(t) = \begin{cases} 1. + 3.499822625t - 2.285880500t^2 - 12.58707780t^3, & t \in [0, \frac{1}{8}], \\ 1.382213734 - 3.433186286t + 35.25722514t^2 - 64.91278450t^3, & t \in [\frac{1}{8}, \frac{1}{4}], \\ 0.7593214115 + 4.243682498t + 3.741106488t^2 - 21.8131018t^3, & t \in [\frac{1}{4}, \frac{3}{8}], \\ -0.1380354631 + 15.57008948t - 37.52278401t^2 + 24.6971462t^3, & t \in [\frac{3}{8}, \frac{1}{2}], \\ 4.889311879 - 14.13800018t + 20.98140654t^2 - 13.6976550t^3, & t \in [\frac{1}{2}, \frac{5}{8}], \\ 6.041661631 - 18.05850758t + 24.67698414t^2 - 14.2941048t^3, & t \in [\frac{5}{8}, \frac{3}{4}], \\ -72.19820146 + 297.8975685t - 400.5932822t^2 + 176.4906795t^3, & t \in [\frac{3}{4}, \frac{7}{8}], \\ -0.6086887156 - 7.597396124t + 17.16691348t^2 - 8.798192522t^3, & t \in [\frac{7}{8}, 1], \end{cases}$$

$$x_2(t) = \begin{cases} 1 + 3.475849531t - 1.85421075t^2 - 17.8394708t^3, & t \in [0, \frac{1}{8}], \\ 1.438831326 - 4.490440233t + 41.35081084t^2 - 78.3187377t^3, & t \in [\frac{1}{8}, \frac{1}{4}], \\ 1.430371686 - 2.102217545t + 22.65109209t^2 - 41.1900087t^3, & t \in [\frac{1}{4}, \frac{3}{8}], \\ -1.461938047 + 25.22794969t - 61.40719215t^2 + 43.4643224t^3, & t \in [\frac{3}{8}, \frac{1}{2}], \\ 0.1718009449 + 10.06871987t - 20.37514082t^2 + 8.96722710t^3, & t \in [\frac{1}{2}, \frac{5}{8}], \\ -2.993469106 + 23.94330086t - 40.4645260t^2 + 18.55626216t^3, & t \in [\frac{5}{8}, \frac{3}{4}], \\ -84.28234975 + 344.754301t - 462.4198296t^2 + 203.5174208t^3, & t \in [\frac{3}{4}, \frac{7}{8}], \\ -4.107469433 + 1.348483676t + 8.35311977t^2 - 5.65671536t^3, & t \in [\frac{7}{8}, 1], \end{cases}$$

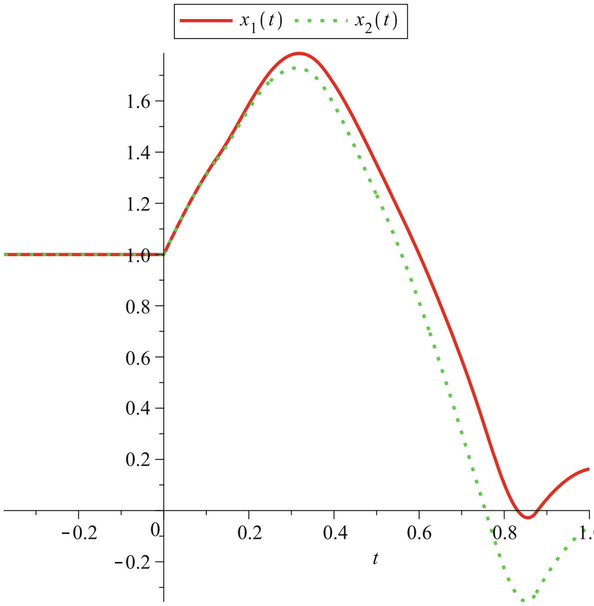

Fig. 4 The graphs of approximated trajectories for Example 4

$$u(t) = \begin{cases} 1 - 30.5639081t + 206.8628318t^2 - 393.8794758t^3, & t \in [0, \frac{1}{8}], \\ -1.683199715 + 19.68237552t - 81.90336092t^2 + 74.2861682t^3, & t \in [\frac{1}{8}, \frac{1}{4}], \\ -0.9864745645 + 14.9784254t - 77.71456712t^2 + 88.2037853t^3, & t \in [\frac{1}{4}, \frac{3}{8}], \\ 4.887497295 - 13.77658341t - 49.66591976t^2 + 106.4995442t^3, & t \in [\frac{3}{8}, \frac{1}{2}], \\ -112.5266494 + 564.7695841t - 954.8808295t^2 + 542.0578672t^3, & t \in [\frac{1}{2}, \frac{5}{8}], \\ -1.939987697 - 24.15985884t + 80.38782599t^2 - 59.67557402t^3, & t \in [\frac{5}{8}, \frac{3}{4}], \\ 339.3535284 - 1258.596941t + 1551.987958t^2 - 636.2463098t^3, & t \in [\frac{3}{4}, \frac{7}{8}], \\ -59.82398778 + 229.5467337t - 285.3591522t^2 + 115.7379162t^3, & t \in [\frac{7}{8}, 1]. \end{cases}$$

The graphs of approximated trajectories and control by this method are shown, respectively, in Figs. 4 and 5. The approximated objective function by this method and the approximated objective function in Wang (2007) are, respectively, $I = 1.536409753$ and $J = 1.562240664$.

According to the result obtained from our method for cost function, this result is more accurate than the obtained cost function of the presented method in Wang (2007).

Example 5 Consider the optimal control of linear time-delay system (see Basin and Rodriguez-Gonzalez 2006),

$$\min\ I = \frac{1}{2}\int_0^{0.25} (x^2(t) + u^2(t))\mathrm{d}t$$

$$s.t.\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) + u(t-0.1) + u(t),$$

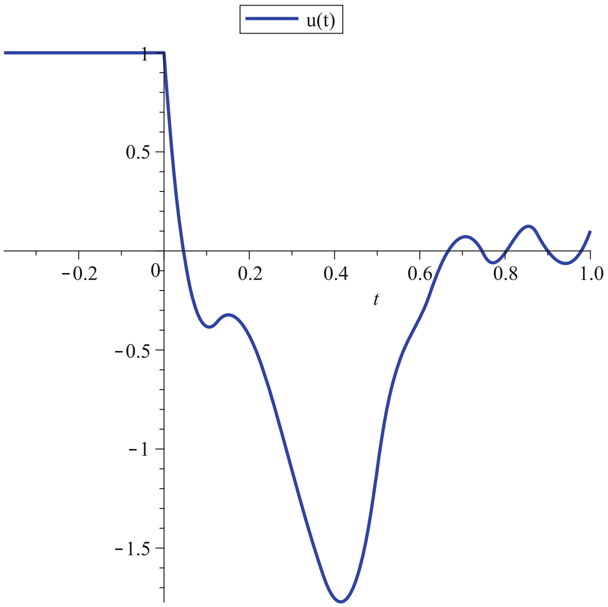

Fig. 5 The graph of approximated control for Example 4

$$\begin{aligned}x(0) &= 1,\\ u(t) &= 0,\ t \in [-0.1, 0].\end{aligned}$$

The optimal control problem is to minimize the state x using the minimum energy of control u.

Let $k = 5$. From (11), one can find the following approximated solutions $x(t)$ and $u(t)$:

$$x(t) = \begin{cases} 1 + 0.99832875t - 4.0936575t^2 + 74.947150t^3, & t \in [0, 0.025],\\ 1.001606585 + 0.7877042125t + 5.0446990t^2 - 56.409270t^3, & t \in [0.025, 0.05],\\ 0.9892882272 + 1.555923762t - 10.9020555t^2 + 53.784860t^3, & t \in [0.05, 0.075],\\ 1.013058989 + 0.5866384125t + 2.2678130t^2 - 5.8415000t^3, & t \in [0.075, 0.1],\\ 1.09110482 - 2.23685085t + 35.3238485t^2 - 132.09876t^3, & t \in [0.1, 0.125],\\ 0.8097325684 + 6.876248412t - 56.4622665t^2 + 163.0144t^3, & t \in [0.125, 0.15],\\ 1.263865389 - 4.494380525t + 34.5950765t^2 - 73.23114t^3, & t \in [0.15, 0.175],\\ 1.191288641 - 1.9891374t + 13.073286t^2 - 18.511320t^3, & t \in [0.175, 0.2],\\ 0.6813145184 + 5.666839712t - 25.2384265t^2 + 45.394580t^3, & t \in [0.2, 0.225],\\ 2.240447094 - 16.53889212t + 79.752815t^2 - 119.48t^3, & t \in [0.225, 0.25],\end{cases}$$

$$u(t) = \begin{cases} -9.922962338t + 302.4537965t^2 - 2031.087914t^3, & t \in [0, 0.025],\\ -0.3037434199 + 16.80712747t - 377.9849724t^2 + 1857.898024t^3, & t \in [0.025, 0.05],\\ 0.8645233559 - 37.71115742t + 400.8262922t^2 - 1257.14752t^3, & t \in [0.05, 0.075],\\ -0.5629148875 + 8.515404269t - 70.58162325t^2 + 193.79326t^3, & t \in [0.075, 0.1],\\ -0.6053531744 + 9.579442912t - 79.13091t^2 + 215.32055t^3, & t \in [0.1, 0.125],\\ 3.867717010 - 81.86852425t + 525.2070892t^2 - 1056.92548t^3, & t \in [0.125, 0.15],\\ -3.389970920 + 51.64438898t - 287.2733633t^2 + 576.129669t^3, & t \in [0.15, 0.175],\\ -0.1062186132 - 20.48085613t + 215.3414569t^2 - 553.556774t^3, & t \in [0.175, 0.2],\\ 4.397795419 - 48.67393673t + 159.4712105t^2 - 132.3802809t^3, & t \in [0.2, 0.225],\\ -21.16959667 + 239.4476588t - 886.5049222t^2 + 1069.711335t^3, & t \in [0.225, 0.25],\end{cases}$$

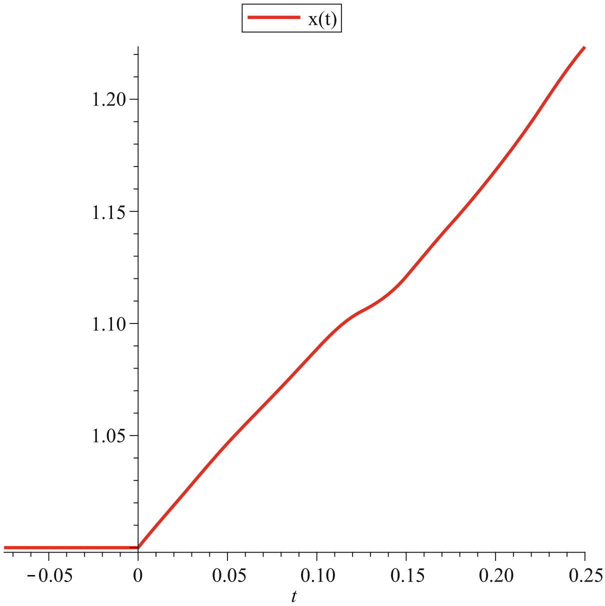

Fig. 6 The graph of approximated trajectory for Example 5

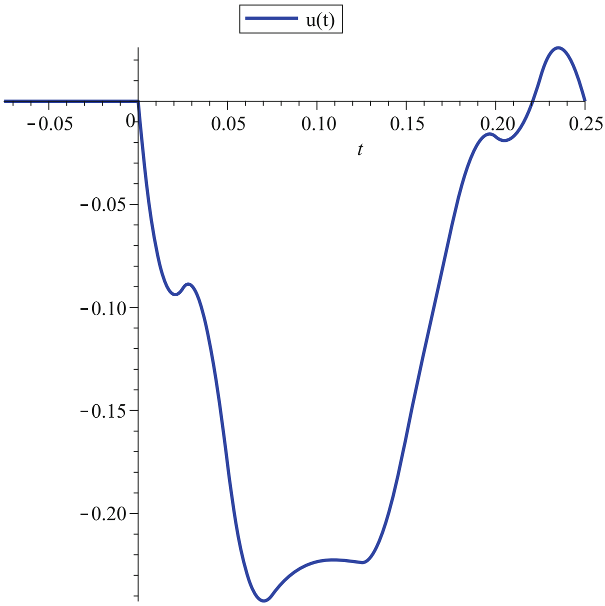

Fig. 7 The graph of approximated control for Example 5

The graphs of approximated trajectory and control by this method are shown in Figs. 6 and 7. The approximated objective function by this method and the approximated objective function in Basin and Rodriguez-Gonzalez (2006) are, respectively, $I = 0.1565866913$ and $J = 0.1563$ (see Basin and Rodriguez-Gonzalez 2006).

Basin and Rodriguez-Gonzalez (2006) used the terminal condition for state, but no terminal condition is used here.

Example 6 Consider the optimal control of linear time-delay system (see Marzban and Razzaghi 2004),

$$\min\ I = \frac{1}{2}\left[10^5 x^2(2) + \int_0^2 u^2(t)\mathrm{d}t\right], \tag{21}$$

$$s.t.\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t-1) + u(t), \tag{22}$$

$$x(t) = 1, \quad t \in [-1, 0]. \tag{23}$$

The problem is to find the optimal control $u(t)$ which minimizes (21) subject to (22) and (23). The exact solution is given by

$$u(t) = \begin{cases} -2.1 + 1.05t, & t \in [0, 1], \\ -1.05, & t \in [1, 2]. \end{cases}$$

$$x(t) = \begin{cases} 1 - t + 0.525t^2, & t \in [0, 1], \\ -0.25 + 1.575t - 1.075t^2 + 0.175t^3, & t \in [1, 2]. \end{cases}$$

Here, we solve this problem by means of the Bezier curves and taking $k = 5$. From (11), one can find the following approximated solutions $x(t)$, and $u(t)$ (see Tables 1 and 2):

$$x(t) = \begin{cases} 1. - 1.051031912t + 0.15046066t^2 + 0.6491195t^3, & t \in [0, 0.1], \\ 1.004156605 - 1.113380987t + 0.462206016t^2 + 0.12954394t^3, & t \in [0.1, 0.2], \\ 1.016185952 - 1.203601091t + 0.687756292t^2 - 0.05841464t^3, & t \in [0.2, 0.3], \\ 1.048067919 - 1.363010963t + 0.953439472t^2 - 0.20601644t^3, & t \in [0.3, 0.4], \\ 1.065591824 - 1.428725643t + 1.035582864t^2 - 0.24024287t^3, & t \in [0.4, 0.5], \\ 3.312711536 - 8.170084775t + 7.776941994t^2 - 2.48736258t^3, & t \in [0.5, 0.6], \\ -4.699542268 + 11.86054973t - 8.915253426t^2 + 2.14935837t^3, & t \in [0.6, 0.7], \\ 3.277064739 - 5.232179562t + 3.293838924t^2 - 0.75756838t^3, & t \in [0.7, 0.8], \\ -1.620292370 + 3.950365053t - 2.445251484t^2 + 0.43807546t^3, & t \in [0.8, 0.9], \\ 1.321572306 - 0.9527427448t + 0.278697296t^2 - 0.0663595t^3, & t \in [0.9, 1], \end{cases}$$

Table 1 Exact and estimated values of $x(t)$ for Example 6

t	Exact $x(t)$	Present $x(t)$
0.0	1.000000	1.0000000000
0.2	0.801000	0.8010050000
0.4	0.644000	0.6410479850
0.6	0.529000	0.5290000003
0.8	0.456000	0.4623799938
1.0	0.425000	0.4322061750
1.2	0.394400	0.4092437380
1.4	0.328200	0.3291700070
1.6	0.234800	0.2348050010
1.8	0.122600	0.1226060000
2.0	0.000000	0.0000000000

Table 2 Exact and estimated values of $u(t)$ for Example 6

t	Exact $u(t)$	Present $u(t)$
0.0	−2.1	−2.100000000
0.2	−1.89	−1.915132832
0.4	−1.68	−1.677935526
0.6	−1.47	−1.469999999
0.8	−1.26	−1.240620577
1.0	−1.05	−1.050000000
1.2	−1.05	−1.050000000
1.4	−1.05	−1.050000002
1.6	−1.05	−1.049999996
1.8	−1.05	−1.050000000
2.0	−1.05	−1.050000100

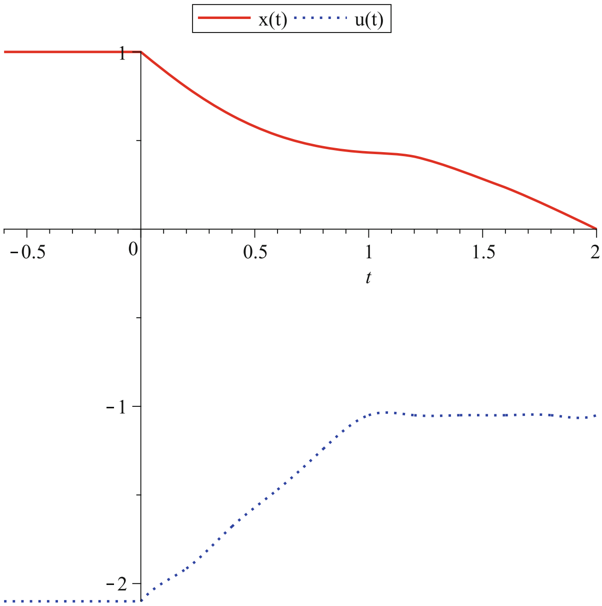

Fig. 8 The graphs of approximated trajectory and control for Example 6

$$u(t)=\begin{cases} -2.100000000+1.477399952t-5.5029091t^2+13.6879428t^3, & t\in[0,0.1],\\ -1.935541603-0.98947596t+6.83147032t^2-6.8693561t^3, & t\in[0.1,0.2],\\ -2.604971822+4.031250676t-5.72034628t^2+3.5904911t^3, & t\in[0.2,0.3],\\ -1.580075748-1.093229756t+2.82045454t^2-1.1543983t^3, & t\in[0.3,0.4],\\ 1.155971116-11.35340554t+15.64567432t^2-6.4982399t^3, & t\in[0.4,0.5],\\ -13.50691046+32.63523922t-28.34297046t^2+8.1646417t^3, & t\in[0.5,0.6],\\ 4.632590256-12.71351265t+9.44765616t^2-2.3327546t^3, & t\in[0.6,0.7],\\ -4.969028736+7.861385024t-5.2486992t^2+1.1663776t^3, & t\in[0.7,0.8],\\ 9.363416448-19.01194989t+11.54713524t^2-2.3327546t^3, & t\in[0.8,0.9],\\ -51.85739897+83.02274236t-45.13880484t^2+8.1646417t^3, & t\in[0.9,1], \end{cases}$$

The graphs of approximated trajectory and control by this method are shown in Fig. 8. The approximated and exact objective function are, respectively, $I = 1.837574909$ and $I^* = 1.8375$ (see Marzban and Razzaghi 2004).

The proposed method is more straight and easier than the hybrid method in Marzban and Razzaghi (2004), although the results of two papers more or less have the same accuracy.

5 Conclusions

Using Bezier curve, we have used least square method for numerical solutions of time-varying linear optimal control problems with time delays in state and control.

The control point structure provides a bound on the residual function. Numerical examples show that the proposed method is efficient and very easy to use. One may extend this method for optimal control problems governed by nonlinear ordinary differential equations (ODEs), but a complicated manipulation seems inevitable.

Although the method is simple, by solving various numerical examples, accuracy in comparison of other methods can be found.

Appendix

In this section, we specify the derivative of Bezier curve.

By (6), we have

$$v_j(t) = \sum_{i=0}^{n} a_i^j B_{i,n}(t),\ t \in [0, 1],$$

where $B_{i,n}(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$.

Now, we have

$$\frac{\mathrm{d}B_{i,n}(t)}{\mathrm{d}t} = n(B_{i-1,n-1}(t) - B_{i,n-1}(t)), \tag{24}$$

where $B_{-1,n-1}(t) = B_{n,n-1}(t) = 0$, and

$$B_{i-1,n-1}(t) = \frac{(n-1)!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i},$$

$$B_{i,n-1}(t) = \frac{(n-1)!}{i!(n-i-1)!} t^{i}(1-t)^{n-i-1}.$$

Using (24), the first derivative $\mathbf{v}_j(t)$ is shown as

$$\begin{aligned} \frac{\mathrm{d}\mathbf{v}_j(t)}{\mathrm{d}t} &= \sum_{i=1}^{n-1} n\mathbf{a}_i^j B_{i-1,n-1}(t) - \sum_{i=0}^{n-1} n\mathbf{a}_i^j B_{i,n-1}(t) \\ &= \sum_{i=0}^{n-1} n\mathbf{a}_{i+1}^j B_{i,n-1}(t) - \sum_{i=0}^{n-1} n\mathbf{a}_i^j B_{i,n-1}(t) \\ &= \sum_{i=0}^{n-1} B_{i,n-1}(t) n\{\mathbf{a}_{i+1}^j - \mathbf{a}_i^j\}. \end{aligned} \tag{25}$$

Now, we specify the procedure of derivation (9) from (8).

By (6), we have

$$\begin{aligned}\mathbf{v}_j(t) &= \genfrac{}{}{0pt}{}{n}{0}\mathbf{a}_0^j\frac{1}{h^n}(t_j-t)^n\\ &\quad+\ldots+\genfrac{}{}{0pt}{}{n}{n}\mathbf{a}_n^j\frac{1}{h^n}(t-t_{j-1})^n, \qquad (26)\\ \mathbf{v}_{j+1}(t) &= \genfrac{}{}{0pt}{}{n}{0}\mathbf{a}_0^{j+1}\frac{1}{h^n}(t_{j+1}-t)^n\\ &\quad+\ldots+\genfrac{}{}{0pt}{}{n}{n}\mathbf{a}_n^{j+1}\frac{1}{h^n}(t-t_j)^n, \qquad (27)\end{aligned}$$

by substituting $t=t_j$ into (26) and (27), one has

$$\mathbf{v}_j(t_j)=\mathbf{a}_n^j\frac{1}{h^n}(t_j-t_{j-1})^n, \qquad (28)$$

$$\mathbf{v}_{j+1}(t_j)=\mathbf{a}_0^{j+1}\frac{1}{h^n}(t_{j+1}-t_j)^n. \qquad (29)$$

To preserve the continuity of Bezier curves at the nodes, one needs to impose the condition $\mathbf{v}_j(t_j)=\mathbf{v}_{j+1}(t_j)$, so from (28) and (29), we have

$$\mathbf{a}_n^j(t_j-t_{j-1})^n=\mathbf{a}_0^{j+1}(t_{j+1}-t_j)^n. \qquad (30)$$

From (25), the first derivatives of $\mathbf{v}_j(t)$ and $\mathbf{v}_{j+1}(t)$ are, respectively:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{v}_j(t)}{\mathrm{d}t} &= \sum_{i=0}^{n-1}B_{i,n-1}(t)n(\mathbf{a}_{i+1}^j-\mathbf{a}_i^j)\\ &= \sum_{i=0}^{n-1}\binom{n-1}{i}(t_j-t)^{n-1-i}(t-t_{j-1})^i\\ &\quad\times\frac{1}{h^n}\{n(\mathbf{a}_{i+1}^j-\mathbf{a}_i^j)\}\\ &= \binom{n-1}{0}\{n(\mathbf{a}_1^j-\mathbf{a}_0^j)\}\frac{1}{h^n}(t_j-t)^{n-1}\\ &\quad+\ldots+\binom{n-1}{n-1}\{n(\mathbf{a}_n^j-\mathbf{a}_{n-1}^j)\}\\ &\quad\times\frac{1}{h^n}(t-t_{j-1})^{n-1}, \qquad (31)\\ \frac{d\mathbf{v}_{j+1}(t)}{dt} &= \sum_{i=0}^{n-1}\binom{n-1}{i}(t_{j+1}-t)^{n-1-i}(t-t_j)^i\\ &\quad\times\frac{1}{h^n}\{n(\mathbf{a}_{i+1}^{j+1}-\mathbf{a}_i^{j+1})\}\\ &= \binom{n-1}{0}\{n(\mathbf{a}_1^{j+1}-\mathbf{a}_0^{j+1})\}\frac{1}{h^n}(t_{j+1}-t)^{n-1}\end{aligned}$$

$$+\ldots+\binom{n-1}{n-1}\{n(\mathbf{a}_n^{j+1}-\mathbf{a}_{n-1}^{j+1})\}$$
$$\times\frac{1}{h^n}(t-t_j)^{n-1}. \tag{32}$$

By substituting $t = t_j$ into (31) and (32), we have

$$\frac{\mathrm{d}\mathbf{v}_j(t_j)}{\mathrm{d}t} = n(\mathbf{a}_n^j - \mathbf{a}_{n-1}^j)\frac{1}{h^n}(t_j - t_{j-1})^{n-1}, \tag{33}$$
$$\frac{\mathrm{d}\mathbf{v}_{j+1}(t_j)}{\mathrm{d}t} = n(\mathbf{a}_1^{j+1} - \mathbf{a}_0^{j+1})\frac{1}{h^n}(t_{j+1} - t_j)^{n-1}, \tag{34}$$

and to preserve the continuity of the first derivative of the Bezier curves at nodes, by equalizing (33) and (34), we have

$$(\mathbf{a}_n^j - \mathbf{a}_{n-1}^j)(t_j - t_{j-1})^{n-1} = (\mathbf{a}_1^{j+1} - \mathbf{a}_0^{j+1})(t_{j+1} - t_j)^{n-1},$$

where it shows the equality (9).

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