International Journal of Applied Mathematical Research, 3 (2) (2014) 116-121
©Science Publishing Corporation
www.sciencepubco.com/index.php/IJAMR
doi: 10.14419/ijamr.v3i2.1595
Research Paper

The numerical solution of the singularly perturbed differential-difference equations based on the Meshless method

Jafar Saberi-Nadjafi *, Fahimeh Akhavan Ghassabzade

Department of Applied Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
**Corresponding author E-mail: najafi141@gmail.com*

Abstract

In this paper, we describe a meshless approach to solve singularly perturbed differential- difference equations of the second order with boundary layer at one end of the interval. In the numerical treatment for such type of problems, first we approximate the terms containing negative and positive shifts which converts it to a singularly perturbed differential equation. Next, a numerical scheme based on the moving least squares (MLS) method is used for solving singularly perturbed differential equation. The MLS methodology is a meshless method, because it does not need any background mesh or cell structures. The proposed scheme is simple and efficient to approximate the unknown function. Several examples are presented to demonstrate the efficiency and validity of the numerical scheme presented in the paper.

Keywords: *Differential-Difference Equation; Singular Perturbations; Boundary Layer; Meshless Method*.

1. Introduction

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest order derivative term is multiplied by a positive small parameter and involving at least one delay or advance term. Such problems arise frequently in the study of human pupil light reflex [1], control theory [2], mathematical biology [3], study of bitable devices [4], etc. The mathematical modeling of the determination of the expected time for the generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general boundary value problem for singularly perturbed differential- difference equation with small shifts.

There exist several numerical studies for approximating the solution of singularly perturbed differential- difference equations. For example Kadalbajoo and Sharma [5-8], Kadalbajoo and Ramesh [9], Amiraliyeva and Erdogan [10], Amiraliyeva and Amiraliyev [11], Rao and Chakravarthy[12] developed robust numerical schemes for dealing with singularly perturbed differential- difference equations.

In recent years, much interest of scientists and engineeres has been paid on meshless based methods, particularly moving least squares (MLS) method [13-17].

In this paper, we employ a numerical method based on the MLS method to approximate the unknown function for solution of singularly perturbed differential-difference equations. The main idea of this method is to approximate the unknown field function by a linear.

Combination of shape function built without having recourse to mesh the domain. Instead, nodes are scattered in the domain and a certain weight functions with a local support is associated with each of these nodes.

The rest of this paper is organized as follow: the outline of MLS method is discussed in section 2. In section3, the proposed method is employed on singularly perturbed differential- difference equations. In section4, numerical results for some problems are obtained. Finally in section 5 this report ends with conclusion and some offers for future researches.

2. The moving least squares approximation

The moving least squares (MLS) approximation was devised by mathematicians in data fitting and surface construction [18-19]. Since the numerical approximations of MLS are based on the scattered set of nodes, there have many meshless methods based on the MLS for the numerical solution of differential equations in recent years.

Consider an unknown scalar function of a field variable $u(x)$ in the domain Ω. The MLS approximation $u_h(x)$ of $u(x)$ can be defined as follows

$$u_h(x) = \sum_{j=1}^{m} p_j(x).a_j(x) = p^T(x).a(x), \forall x \in \Omega \tag{1}$$

Where $p(x)$ a complete monomial basis function of the spatial coordinates is, m is the number of basic functions. For a 1D example, the linear basis is $p^T(x) = [1\ x]$ and the quadratic basis is $p^T(x) = [1\ x\ x^2]$. In equation (1), $a(x)$ is a vector of coefficients given by

$$a^T(x) = \{a_1(x) \quad a_2(x) \cdots \quad a_m(x)\} \tag{2}$$

The coefficients $a(x)$ can be obtained by minimizing the following weighted discrete L_2 norm

$$J = \sum_{i=1}^{n} w_i(x)(p^T(x_i).a(x) - u_i)^2, \tag{3}$$

where n is number of nodes in the support domain of x for which the weight function $w_i(x) \neq 0$ and u_i is the nodal parameter of u at $x = x_i$. The stationary of J in Eq. (3) with respect to $a(x)$ leads to the following linear relations

$$A(x).a(x) = B(x).U_S\ , \tag{4}$$

where U_S the vector that collects the node parameters of the field is functions for all the nodes in the support domain,

$$U_S = \{u_1 \quad u_2 \cdots \quad u_n\}^T \tag{5}$$

A(x) and B(x) are defined by

$$A(x) = P^T W P = \sum_{i=1}^{n} w_i(x) P(x_i) P^T(x_i) \tag{6}$$

$$B(x) = P^T W = [w_1(x)P(x_1) \quad w_2(x)P(x_2) \cdots \quad w_n(x)P(x_n)] \tag{7}$$

Where matrices P and W are defined as

$$p = \begin{bmatrix} p^T(x_1) \\ p^T(x_2) \\ \vdots \\ p^T(x_n) \end{bmatrix}_{n\times(m+1)}, W = \begin{bmatrix} w_1(x) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n(x) \end{bmatrix}_{n\times n} \tag{8}$$

Solving Eq. (4) for $a(x)$, we have

$$a(x) = A^{-1}(x).B(x).U_S \tag{9}$$

Substituting (9) into Eq. (1), we get

$$u^h(x) = \sum_{i=1}^{n} \phi_i(x).u_i = \Phi^T(x).U_S \tag{10}$$

Where

$$\Phi^T(x) = p^T(x).A^{-1}(x).B(x) \tag{11}$$

Or

$$\varphi_j(x) = \sum_{i=1}^{m} p_i(x)(A^{-1}(x).B(x))_{ij} = P^T(x)(A^{-1}B)_j \tag{12}$$

$\phi_j(x)$ are called the shape functions of MLS approximation, corresponding to the nodal points x_j.

The Gaussian weight function is applied in the present paper is as follows

$$w_j(x) = \begin{cases} \dfrac{\exp[-(d_j/\alpha)^2] - \exp[-(h_j/\alpha)^2]}{1 - \exp[-(h_j/\alpha)^2]} & , 0 \le d_j \le h_j \\ 0 & , d_j > h_j \end{cases}, \tag{13}$$

where $d_j = \|x - x_j\|$, α is a constant controlling the shape of the weight function w_j and h_j which is the size of the support.

3. Description of the problem

We consider a model problem for singularly perturbed differential- difference equation of the form

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x-\delta) + r(x)u(x) + s(x)u(x+\gamma) = f(x) \tag{14}$$

Where $x \in (0,1)$ and $0 < \varepsilon \ll 1$, subject to the interval and the following boundary conditions, respectively

$$\begin{aligned} u(x) &= \varphi(x), \quad -\delta \le x \le 0 \\ u(1) &= \phi(x), \quad 1 \le x \le 1+\gamma, \end{aligned} \tag{15}$$

where $0 < \delta = o(\varepsilon)$ and $0 < \gamma = o(\varepsilon)$, are the delay and advance parameters, respectively and $p(x)$, $q(x), r(x), s(x), \varphi(x), \phi(x)$ and $f(x)$ are sufficiently smooth functions. The solution of (14) and (15) exhibits, layer at the left end of the interval if $p(x) - \delta q(x) + \gamma s(x) > 0$ and layer at the right end of the interval, if $p(x) - \delta q(x) + \gamma s(x) < 0$. If $p(x) = 0$, then one may have oscillatory solution or two layers.

4. Numerical scheme

In this section, we consider the numerical treatment for model problem (14) and (15). The first step in this direction is the use of Taylor approximation for the terms containing delay and advance in the problem (14) and (15). Taking Taylor series expansions of the terms $u(x-\delta)$ and $u(x+\gamma)$ in equation (14), we have

$$u(x-\delta) \approx u(x) - \delta u'(x), \tag{16}$$

and

$$u(x+\gamma) \approx u(x) + \gamma u'(x). \tag{17}$$

Using (16)-(17) in (14)-(15), we obtain

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) \approx f(x),$$
$$u(0) \approx \phi(0),$$
$$u(1) \approx \varphi(1),$$

where $a(x) = p(x) - \delta q(x) + \gamma\, s(x)$ and $b(x) = q(x) + r(x) + s(x)$. Since the delay and advance are sufficiently small the following new problem becomes

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \tag{18}$$
$$u(0) = \phi(0), \tag{19}$$
$$u(1) = \varphi(1). \tag{20}$$

This provides a good approximation to the solution u of the problem (14)-(15).

Now, to employ the MLS method, at first N nodal points $\{x_j\}$ are selected on interval [0, 1], where $0 = x_1 < x_2 < \Lambda < x_N = 1$. The distribution of the nodes could be selected regularly or randomly. Next, instead of u, we can replace u_h from Eq. (10), therefore, Eq. (18) becomes

$$\varepsilon u_h''(x) + a(x)u_h'(x) + b(x)u_h(x) = f(x) \tag{21}$$

Or

$$\sum_{i=1}^{N} \left[\varepsilon\varphi_i''(x) + a(x)\varphi_i'(x) + b(x)\varphi_i(x) \right] u_i = f(x) \tag{22}$$

Assume that Eq. (22) holds true at x_j, namely

$$\sum_{i=1}^{N} \varepsilon\phi_i''(x_j) + a(x_j)\phi_i'(x_j) + b(x_j)\phi_i(x_j)]u_i = f(x_j)$$
$$j = 2,3,\ldots,N-1, \tag{23}$$

and

$$\sum_{i=1}^{N} \phi_i(x_j)u_i = \phi(x_j), \quad j = 1$$
$$\sum_{i=1}^{N} \phi_i(x_j)u_i = \varphi(x_j), \quad j = N.$$

Now, if we set A as an N by N matrix defined by the following

$$A_{ij} = \begin{cases} \varepsilon\varphi_i''(x_j) + a(x_j)\varphi_i'(x_j) + b(x_j)\varphi_i(x_j) \\ j = 2,3,\mathrm{L}\ ,N-1 \\ \varphi_i(x_j) \qquad ; j = 1, N \end{cases} \tag{24}$$

And vectors

$$U = [u_1, u_2, \mathrm{L}\ , u_N]^T \tag{25}$$

and

$$F = [f_1, f_2, \ldots, f_N]^T \tag{26}$$

Then we have the following linear system of equations

$$AU = F \tag{27}$$

Finding the values of u_i by solving the linear system (27) with an appropriate procedure such as Gauss elimination method, yields the following approximate solution

$$u(x) \cong u_h(x) = \sum_{i=1}^{N} \varphi_i(x) u_i, \forall x \in [0,1]$$

5. Numerical results

To show the applicability of MLS method, we consider boundary value problem (14)-(15) with constant coefficients. The exact solution of such a problem is given by:

$$u(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{f}{c},$$

where

$$c = q + r + s,$$

$$c_1 = (\beta c - f + e^{m_2}(f - \alpha c)) \Big/ (e^{m_1} - e^{m_2})c,$$

$$c_2 = (f - \beta c + e^{m_1}(-f + \alpha c)) \Big/ (e^{m_1} - e^{m_2})c,$$

$$m_1 = \left\{ -(p - \delta q + \gamma s) + \sqrt{(p - \delta q + \gamma s)^2 + 4\varepsilon c} \right\} \Big/ 2\varepsilon,$$

$$m_2 = \left\{ -(p - \delta q + \gamma s) - \sqrt{(p - \delta q + \gamma s)^2 + 4\varepsilon c} \right\} \Big/ 2\varepsilon,$$

$$\alpha = u(0),$$

$$\beta = u(1).$$

Cubic basis functions and Gaussian weight functions are utilized in the above computation. All routines are written in MATLAB 2007a. In this regard, we have reported the values of absolute error in the following graphs.

Example 1: Consider the following singularly perturbed differential- difference equation with layer at the left end

$$\varepsilon u''(x) + u'(x) + 2u(x - \delta) - 3u(x) = 0,$$

$$u(x) = 1, \qquad -\delta \le x \le 0, \quad u(1) = 1.$$

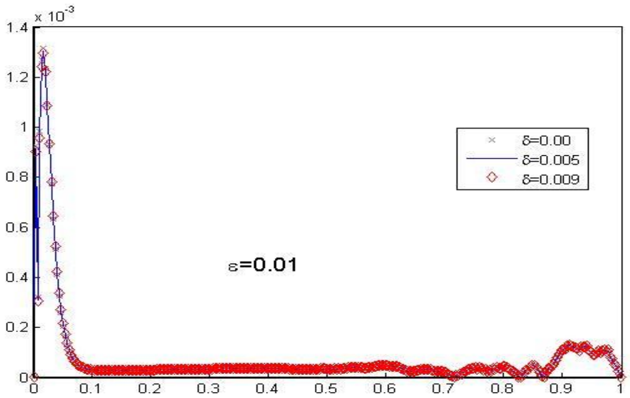

Fig. 1: Graph of the absolute errors for example1 for $\varepsilon = 0.01$ and different values of δ.

Example 2: Consider the following singularly perturbed differential- difference equation with layer at the left end

$$\varepsilon u''(x) + u'(x) - 2u(x - \delta) - 5u(x) + u(x + \gamma) = 0,$$

$$u(x) = 1, \qquad -\delta \le x \le 0, \quad u(1) = 1, \quad 1 \le x \le 1 + \gamma.$$

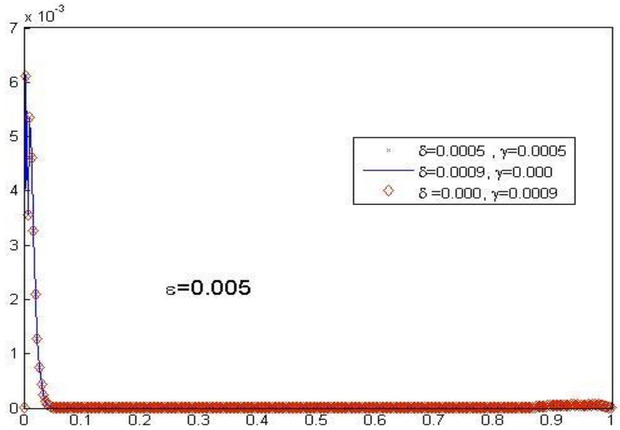

Fig. 2: Graph of the absolute error for example 2 for $\varepsilon = 0.005$ and different values of δ and γ.

Example 3: In this example we consider the singularly perturbed differential- difference equation with layer at the right end

$\varepsilon u''(x) - u'(x) - 2u(x-\delta) + u(x) = 0,$

$u(x) = 1, \qquad -\delta \le x \le 0, \quad u(1) = -1.$

Example 4: In this last example, we consider the following singularly perturbed differential- difference equation

$\varepsilon u''(x) - u'(x) - 2u(x-\delta) + u(x) - 2u(x+\gamma) = 0,$

$u(x) = 1, \qquad -\delta \le x \le 0, \quad u(1) = -1, \qquad 1 \le x \le 1+\gamma$.

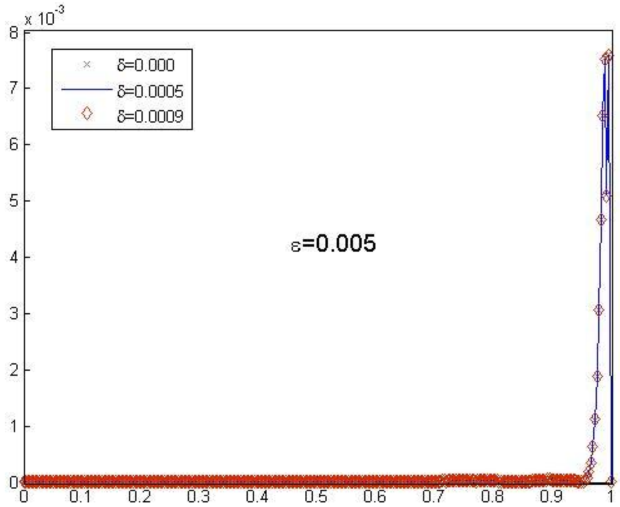

Fig. 3: Graph of the absolute error for example 3 for $\varepsilon = 0.005$ and different values of δ .

6. Conclusion

We have described an efficient and simple numerical scheme based on the moving least squares (MLS) for solving BVPs for singularly perturbed differential- difference equation with small shift. The MLS method is a truly meshless method, which requires no domain discretization for approximation. The method can be easily implemented and its algorithm is simple and efficient to approximate the unknown function. A number of numerical experiments are carried out in support of the predicated theory via tabulating the maximum absolute errors which show the efficiency of the method for solving these types of singularly perturbed equation.

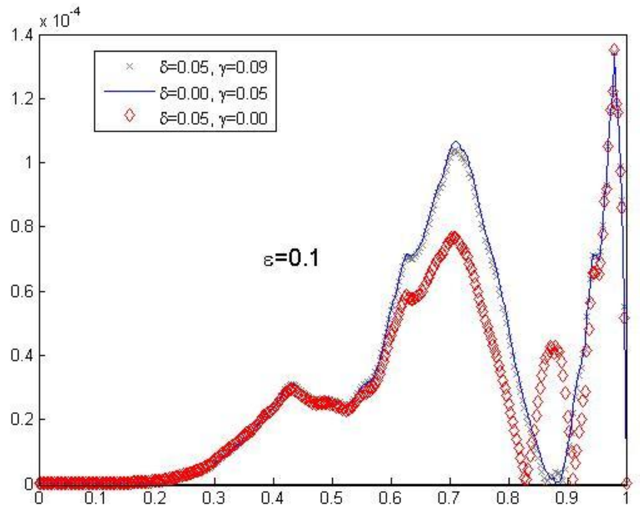

Fig. 4: Graph of the absolute error for example 4 for $\varepsilon = 0.1$ and different values of δ and γ.

References

[1] A Longtin, J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90 (1988), pp. 183-199.

[2] E.L. Elsgolts, Qualitative method in mathematical analysis: Translations of mathematical monographs, vol.12, American mathematical society, Providence, RI, 1964.

[3] J. D. Murray, Mathematical biology I: An introduction, 3rd ed., Springer-verlag, Berlin, 2001.

[4] M.W. Gerstein, H.M. Gibbs, F.A. Hopf, D.L. Kaplan, Bifurcation gap in a hybrid optical system, Physs. Rev. A 26 (1982), pp. 3720-3722.

[5] M. K. Kadalbajoo, K. K. sharma , "Numerical treatment of mathematical model arising from a model of neuronal variability ," J. math. Anal. Appl., 307(2005), pp. 606-627.

[6] M. K. Kadalbajoo, K. K. Sharma, "Numerical Analysis of singularly perturbed delay differential equations with layer behavior," Appl. Math. Compute, 157(2004), pp. 11-28.

[7] M. K. Kadalbajoo, K. K. Sharma, "Numerical Analysis of boundary value problem for singularly perturbed delay differential equations with layer behavior," Appl. Math. Comput, 151(1) (2004), pp. 11-28.

[8] M. K. Kadalbajoo, K. K. Sharma, "A Numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equation ," Appl. Math. Comput., 197(2008), pp. 692-707.

[9] M. K. Kadalbajoo,V.P. Ramesh. Hybrid method for numerical solution of singularly perturbed delay differential equations, Appl. Math. Comput. 187(2007), pp. 797-814.

[10] I. G. Amiraliyeva, F. Erdogan, "Uniform Numerical method for singularly perturbed delay differential equations," comput. Math. Appl. 53(2007), pp. 1251-1259.

[11] I. G. Amiraliyeva, G. M. Amiraliyev, "Uniform difference method for parameterized singularly perturbed delay differential equations," Numer, Algo. 52(2009), pp. 509-521.

[12] Rao, R. N., Chakravarthy, P. P. A fourth order finite difference method for singularly perturbed differential-difference equations, American journal of computational and applied mathematics. 1(2011), pp. 5-10.

[13] S. N. Atluri, T. Zhv, "A new meshless local petrov-Galerkin (MLPG) approach in computational mechanics," Comput. Mech. 22(2) (1998), pp. 117-127.

[14] M. Dehghan D. Mirzaei, "Meshless local petrov-Galerkin (MLPG) method for the unsteady magnetohydrodynamic(MHD) flow through pipe with orbitrary wall conductivity," Appl. Numer. Math. 59(2009), pp. 1043-1058.

[15] D. Mirzaei, M. Dehghan, "A meshless based method for solution of integral equations," Appl. Num. Math., 60(2010), pp. 245-262.

[16] M. Dehghan, R. Salehi, The numerical solution of the non-linear integro-differential equations based on the meshless method, J. Comput. Appl. Math. 236(2012), pp. 2367-2377.

[17] M. Dehghan D. Mirzaei, The meshless local Petrov-Galerkin (MLPG) method for the generalized two-dimensional non-linear Schrodinger equation, Engineering Analysis with Boundary Elements 32(2008), pp. 747-756.

[18] D. Shepard, "A two-dimensional interpolation function for irregularly space points," in: proc. 23rd Nat. conf. ACM. ACMpress, Newyork, 1968, pp. 517-524.

[19] P. Lancaster, K. Salkauskas, Surfaces generated by moving least squares methods, Math. Comput. 37 (1981), pp. 141-158.