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# NILPOTENT PRODUCTS OF CYCLIC GROUPS AND CLASSIFICATION OF $p$-GROUPS

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Abstract. In 1940, P. Hall defined an equivalence relation, called isoclinism, and applied it for classifying all $p$-groups of order at most $p^5$ $(p > 3)$ into ten families $\Phi_1, \Phi_2, \ldots$ and $\Phi_{10}$. This paper intends to characterize all the families which have the nilpotent products of cyclic groups in themselves, and then determine the exact structures of these products.

## 1. Introduction

In 1960, Struik [5] conjectured that "every finite nilpotent group is isoclinic to a nilpotent product of cyclic groups", but unfortunately it was not true in general. She [5] proved that there exists a group of order $p^4$ $(p \geq 3)$, which is not isoclinic to any nilpotent product of cyclic groups. But it remained to find nilpotent groups which are isoclinic to nilpotent products of cyclic groups. In this paper, we determine all

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$p$-groups of order at most $p^5$ $(p > 3)$ with this property. To this end, we characterize all isoclinism families which contain a nilpotent product of cyclic groups, among $\Phi_1, \Phi_2, \ldots$ and $\Phi_{10}$. We also give the exact structures of all nilpotent products of cyclic groups in these families. The following important equivalence relation was defined by P. Hall[2] for classifying finite $p$-groups.

**Definition 1.1.** Two groups $G$ and $H$ are isoclinic (or skew isomorphic), if there exist isomorphisms

$$\alpha : \tfrac{G}{Z(G)} \to \tfrac{H}{Z(H)} \quad \text{and} \quad \beta : G' \to H'$$

such that for all $g_1, g_2 \in G$, $\beta[g_1, g_2] = [h_1, h_2]$, where $h_i Z(H) = \alpha(g_i Z(G))$, for $i = 1, 2$. In this case, we write $G \sim H$ and say the pair $(\alpha, \beta)$ is an isoclinism between $G$ and $H$.

The concept of basic commutator is defined by M. Hall [1]. He also showed that these commutators play an important role in the category of free groups. The number of basic commutators of weight $n$ on $d$ generators is obtained by the following theorem.

**Theorem 1.2.** *( Witt Formula) The number of basic commutators of weight $n$ on $d$ generators is given by the formula*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m},$$

*where $\mu(m)$ is the Möbius function defined by*

$$\mu(m) = \begin{cases} 1 & \text{if } \ m = 1, \\ 0 & \text{if } \ m = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } \ m = p_1 \ldots p_s, \end{cases}$$

*in which the $p_i$, $1 \leq i \leq k$, are the distinct primes dividing $m$.*

Although Golovin defined the nilpotent product of groups in general, in this paper we need the following special case of the definition.

**Definition 1.3.** Let $A_1, A_2, \ldots, A_k$ be cyclic groups. Then $n$-nilpotent product of $A_1, A_2, \ldots, A_k$, denoted by $A_1 \overset{n}{*} A_2 \overset{n}{*} \ldots \overset{n}{*} A_k$, is defined to be the group $\frac{F}{\gamma_{n+1}(F)}$, where $F$ is the free product of $A_1, A_2, \ldots, A_k$.

It is clear that the nilpotency class of a $k$-nilpotent product of cyclic groups is $k$. The following theorems, which are frequently used hereafter, give the structures of the center and the terms of lower central series of some $k$-nilpotent products of cyclic groups.

**Theorem 1.4.** [4, Theorem 4.4] *Let $k$ be a positive integer and $p$ a prime number such that $k \leq p$. Let $C_1, \ldots, C_r$ be cyclic $p$-groups generated by $x_1, \ldots, x_r$, respectively. Let $p^{\alpha_i}$ be the order of $x_i$, and assume that $1 \leq \alpha_1 \leq \ldots \leq \alpha_r$. If $G$ is the $k$-nilpotent product of $C_i$'s , $G = C_1 \overset{k}{*} C_2 \overset{k}{*} \ldots \overset{k}{*} C_r$, then $Z(G) =< x_r^{p^{\alpha_r - 1}}, \gamma_k(G) >$.*

**Theorem 1.5.** *Let $G = \mathbb{Z}_{r_1} \overset{k}{*} \mathbb{Z}_{r_2} \overset{k}{*} \ldots \overset{k}{*} \mathbb{Z}_{r_t}$, such that $r_{i+1} \mid r_i$ for all $i$, $1 \leq i \leq t$. If $(p, r_1) = 1$ for any prime $p$ less than or equal to $k$, then*
*i) if $k \leq c$, then $\gamma_{c+1}(G) = 1$;*
*ii) if $c < k \leq 2c+1$, then $\gamma_{c+1}(G) = \mathbb{Z}_{r_2}^{(f_2 - f_1)} \oplus \ldots \oplus \mathbb{Z}_{r_t}^{(f_t - f_{t-1})}$, where $f_j = \sum_{i=1}^{k-c} \chi_{c+i}(j)$ for all $j$, $1 \leq j \leq t$, and $\mathbb{Z}_r^{(d)}$ denotes the direct sum of $d$ copies of the cyclic group $\mathbb{Z}_r$.*

## 2. Main results

Throughout this section, we use the notations $\Phi_1, \Phi_2, \ldots$ and $\Phi_{10}$ for isoclinism families, which are introduced by P. Hall [2] , in order to classifying all $p$-groups of order at most $p^5$ $(p > 3)$. Every group $H$ with the property $Z(H) \subseteq H'$ is called a stem group. P. Hall also proved that every isoclinism family includes a stem group and the order of a stem group divides the order of any member of its family. In the following lemma, we try to accumulate some useful statement and information which were sporadically proved by P. Hall [2].

**Lemma 2.1.** *Let $u_i = (z_i, t_i)$ be a pair such that $z_i$ and $t_i$ denote the nilpotency class and the order of derived subgroup in family $\Phi_i$, respectively. Then we have $u_1 = (1,1), u_2 = (2,p), u_3 = (3,p^2), u_4 = (2,p^2), u_5 = (2,p), u_6 = (3,p^3), u_7 = (3,p^2), u_8 = (3,p^2), u_9 = (4,p^3)$ and $u_{10} = (4,p^3)$.*

Now, we are going to determine isoclinism families including a nilpotent product of cyclic groups. Let $p$ be a prime number greater than 3 and $G$ be an $n$-nilpotent product of cyclic groups. If $G$ lies in $\Phi_1, \Phi_2, \ldots$ and $\Phi_{10}$, then $n \leq 4$, by Lemma 2.1. Whereas 1-nilpotent product and direct product of cyclic groups coincide, we have $G \in \Phi_1$ if and only if $n = 1$. In what follows, the results for $n = 2, 3, 4$ are given.

**Theorem 2.2.** *If $i \neq 2$, then there is not any 2-nilpotent product of cyclic $p$-groups in $\Phi_i$.*

The above Theorem shows that any 2-nilpotent product of cyclic $p$-groups of order at most $p^5$ belongs to $\Phi_2$.

**Theorem 2.3.** *If $p$ is an odd prime number and $i \neq 6$, then there is not any 3-nilpotent product of cyclic $p$-groups in $\Phi_i$.*

Now let $G$ be a 4-nilpotent product of cyclic $p$-groups. Theorem 1.5 implies that the order of the third term of lower central series of $G$ is greater than $p^4$ and so is the order of its derived subgroup. Then $G$ does not lie in $\Phi_i$ for all $i, 1 \leq i \leq 10$, by Lemma 2.1.

**Theorem 2.4.** *Let $G$ be a 2-nilpotent product of cyclic $p$-groups. Then $G$ belongs to $\Phi_2$ if and only if $G \cong \mathbb{Z}_{p^\alpha} \overset{2}{*} \mathbb{Z}_p$, for any positive integer $\alpha$.*

P. Hall [2] proved that there exists just one stem group, up to isoclinism, of order $p^5$ such that its center factor is isomorphic to $E_1$. This fact helps us to state the following lemma.

**Theorem 2.5.** *Let $G$ be a 3-nilpotent product of cyclic $p$-groups. Then $G$ belongs to $\Phi_6$ if and only if $G \cong \mathbb{Z}_{p^\alpha} \overset{3}{*} \mathbb{Z}_p$, for any positive integer $\alpha$.*

Our main results of this paper are accumulated in the following proposition.

**Proposition 2.6.** *Let $p$ be a prime number greater than 3 and $G$ be an $n$-nilpotent product of cyclic $p$-groups. Then $G$ does not lie in families $\Phi_i$, for $3 \leq i \leq 10$ and $i \neq 6$. Moreover,*
*i) $G \in \Phi_1$ if and only if $n = 1$,*
*ii) $G \in \Phi_2$ if and only if $n = 2$ and $G \cong \mathbb{Z}_{p^\alpha} \overset{2}{*} \mathbb{Z}_p$,*
*iii) $G \in \Phi_6$ if and only if $n = 3$ and $G \cong \mathbb{Z}_{p^\alpha} \overset{3}{*} \mathbb{Z}_p$.*

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