# Ramond-Ramond *S*-matrix elements from the *T*-dual Ward identity

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Recently it has been speculated that the Ward identities associated with the string dualities and the gauge symmetries can be used as guiding principles to find all components of the scattering amplitude of $n$ supergravitons from a given component of the S matrix. In this paper, we apply the Ward identities associated with the T duality and the gauge symmetries on the disk-level S-matrix element of one Ramond-Ramond (RR) $(p-3)$ form, one Neveu-Schwarz-Neveu-Schwarz (NSNS), and one Neveu-Schwarz (NS) state to find the corresponding S-matrix elements of the RR $(p-1)$ form, $(p+1)$ form, or the RR $(p+3)$ form on the world volume of a $D_p$-brane. Moreover, we apply these Ward identities on the S-matrix element of one RR $(p-3)$ form and two NSNS states to find the corresponding S-matrix elements of the RR $(p-1)$ form, $(p+1)$ form, $(p+3)$ form, or the RR $(p+5)$ form.

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## I. INTRODUCTION

Higher derivative couplings in superstring theory can be captured from $\alpha'$ expansion of the corresponding S-matrix elements [1,2] and from exploring the dualities of the superstring theory [3–16]. The dualities can be implemented either on-shell or off-shell. At the on-shell level, they appear in the S-matrix elements as S-dual and T-dual Ward identities [17–21]. These identities establish connections between different elements of the scattering amplitude of $n$ supergravitons. By calculating one element explicitly in the world sheet conformal field theory, all other elements of the S matrix may be found by the Ward identities. At the off-shell level, on the other hand, the dualities appear as symmetries of the effective action. By calculating the couplings of one specific component of the supergraviton at order $\alpha'^n$ from the corresponding S-matrix element, the couplings of all other components at this order may be found by the dualities [22–28].

The effective actions of $D_p$-branes in superstring theory at leading order of $\alpha'$ are given by the Dirac-Born-Infeld (DBI) and the Chern-Simons (CS) actions, which are invariant under off-shell T duality. The first higher derivative correction to these actions is at order $\alpha'^2$. The curvature squared corrections to the DBI action has been found in [29] from the $\alpha'$ expansion of the disk-level S-matrix element of two gravitons [30]. At the $\alpha'^2$ order, the S-matrix element of two massless closed-string states in the superstring theory has only contact terms; e.g., for two gravitons, they are the curvature-squared couplings in the momentum space [29]. The T-dual and S-dual Ward identities then dictate that the curvature couplings must be invariant under linear T duality and S duality. The consistency of the curvature couplings with the linear T duality and S duality has been used in [22,25] to find the on-shell couplings of two supergravitons on the world volume of $D_p$-brane at order $\alpha'^2$.

The curvature corrections to the CS action, on the other hand, has been first found in [31–33] by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action. These corrections also starts at order $\alpha'^2$, which is the curvature-squared times the RR potential, i.e., $C^{(p-3)} \wedge R \wedge R$. These couplings have been confirmed in [34–36] by the $\alpha'$ expansion of the disk-level S-matrix element of two gravitons and one RR vertex operator. At the $\alpha'^2$ order, the S-matrix element has only contact terms, which are the coupling $C^{(p-3)} \wedge R \wedge R$ in the momentum space. However, all other S-matrix elements of three massless closed strings have both contact terms and massless poles. As a result, the T-dual Ward identity does not indicate that the curvature couplings must be invariant under the linear T duality. On the other hand, it has been observed in [37] that the CS action at order $\alpha'^2$ has also couplings between one NSNS and one RR $n$ form where $n = p-1,\ p+1,\ p+3$. We expect that the combination of these couplings and the curvature-squared couplings should be extendible to the off-shell nonlinear T duality after including many other couplings at order $\alpha'^2$. Some of these $D_p$-brane couplings involving the RR $(p-3)$ form have been found in [38,39] from the $\alpha'$ expansion of the corresponding S-matrix elements.

The couplings involving the RR $(p-3)$ form reveal that it is a hard task to find all other couplings involving the RR $n$ form where $n = p-1,\ p+1,\ p+3$, from the off-shell nonlinear T-duality requirement. Partial results for such couplings, however, have been found in [23,24]. We are interested in finding such couplings from the $\alpha'$ expansion of the corresponding S-matrix elements. We are going to benefit from the on-shell linear T-duality requirement to find the S-matrix elements from the S-matrix elements of

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the RR $(p-3)$ form, which have been calculated explicitly in [38,39]. The implicit assumption in the T-duality transformation that fields must be independent of the Killing coordinate causes in some cases that the T-dual Ward identity not to be able to capture the new S-matrix elements in all details. However, the Ward identity corresponding to the gauge symmetries can be used to fix this problem [21].

The disk-level scattering amplitude of one massless RR $(p-3)$ form, one NSNS state, and one open-string NS state has been calculated in [20,39]. The RR potential in this amplitude carries either one or zero transverse indices. Accordingly, it can be split into two parts. The amplitude corresponding to the first part has one integral representing the closed- and open-string channels. The amplitude corresponding to the second part has three integrals that satisfy one constraint equation. The T-dual Ward identity connects the amplitude of the RR $(p-3)$ form to the amplitudes of the RR $(p-1)$ form, $(p+1)$ form, and the RR $(p+3)$ form which furnish a T-dual multiplet. The T-dual multiplet corresponding to the first part has been found in [21]. It has the following structure:

$$A_1(C_i^{(p-3)}) \to A_2(C_{ij}^{(p-1)}) \to A_3(C_{ijk}^{(p+1)}) \to A_4(C_{ijkl}^{(p+3)}), \tag{1}$$

where the number in the label of $A$ refers to the number of transverse indices of the RR potential. All other indices of the RR potential contract with the world volume form. The components $A_2$, $A_3$, $A_4$ carry the same integral that the first component $A_1$ carries. In this case, the T-dual Ward identity captures the new S-matrix elements in full detail. This is confirmed by the fact that each S-matrix element satisfies the Ward identities corresponding to the NSNS and NS gauge transformations [21]. The multiplet does not satisfy the Ward identity corresponding to the RR gauge transformation because it contains only the first part of the RR $(p-3)$ form, which has one transverse index.

In this paper, we find, among other things, the T-dual multiplets corresponding to the second part, which has the RR $(p-3)$ form with no transverse index. The result has the following structure:

$$\begin{array}{ccccccc} A_0(C^{(p-3)}) & \to & A_1(C_i^{(p-1)}) & \to & A_2(C_{ij}^{(p+1)}) & \to & A_3(C_{ijk}^{(p+3)}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A_1'(C_i^{(p-1)}) & \to & A_2'(C_{ij}^{(p+1)}) & \to & A_3'(C_{ijk}^{(p+3)}) \end{array} \tag{2}$$

where the horizontal arrows show the linear T-duality transformation and the vertical arrows show the NSNS or the NS gauge transformations. In this case, the amplitudes in the first line, which are connected by the linear T-duality transformation, do not satisfy the Ward identity associated with the NSNS or the NS gauge transformations. The amplitudes in the second line, which are connected by the T duality, are added to make the whole amplitude to be invariant under the NSNS and the NS gauge transformations. All the above components carry the same three integrals that the first component carries.

One may expect the sum of the multiplets (1) and (2) to satisfy the Ward identity corresponding to the RR gauge transformation. This is the case for the first components, which are calculated explicitly. However, as we show, the other components that are calculated through the Ward identity corresponding to the T duality and the NSNS gauge transformations do not satisfy this condition. This indicates that there should be another T-dual multiplet whose first component is $A_0(C^{(p-1)})$. This component, which should be invariant under the RR Ward identity, is connected to the $C^{(p-1)}$ amplitudes in (2) by the Ward identity corresponding to the NSNS and the NS gauge transformations. Imposing this condition, we are able to find this component. The amplitude has two new integrals that satisfy one new constraint equation. We also find the other amplitudes that are connected to it by the Ward identity. The multiplet has the following structure:

$$\begin{array}{ccc} A_0(C^{(p-1)}) & \to & A_1(C_i^{(p+1)}) \\ & & \downarrow \\ & & A_1'(C_i^{(p+1)}) \end{array} \tag{3}$$

The other components have the same integrals that the first component has. The sum of the multiplets (1), (2), and (3) satisfies the Ward identity corresponding to all of the gauge symmetries and the T duality.

The disk-level S-matrix element of one RR $(p-3)$ form and two NSNS states has been calculated in [38–41]. The RR potential in this amplitude carries two, one, or zero transverse indices. Accordingly, it has three parts. The amplitude for the first part has one integral. The amplitude for the second part has five integrals that satisfy two constraint equations, and the amplitude for the third part has 14 integrals that satisfy eight constraint equations. The T-dual Ward identity connects these three parts to the amplitudes of the RR $(p-1)$ form, $(p+1)$ form, $(p+3)$ form, and the RR $(p+5)$ form. The T-dual multiplets corresponding to the first part have been found in [21]. It has the following structure:

$$A_2(C_{ij}^{(p-3)}) \to A_3(C_{ijk}^{(p-1)}) \to A_4(C_{ijkl}^{(p+1)}) \to A_5(C_{ijklm}^{(p+3)}) \to A_6(C_{ijklmn}^{(p+5)}). \tag{4}$$

Each amplitude satisfies the Ward identity corresponding to the NSNS gauge transformation. They all contain one integral. The T-dual multiplet corresponding to the second part, which has been found in [21], has the following structure:

$$\begin{array}{ccccccccc} \mathcal{A}_1(C_i^{(p-3)}) & \to & \mathcal{A}_2(C_{ij}^{(p-1)}) & \to & \mathcal{A}_3(C_{ijk}^{(p+1)}) & \to & \mathcal{A}_4(C_{ijkl}^{(p+3)}) & \to & \mathcal{A}_5(C_{ijklm}^{(p+5)}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{A}_2'(C_{ij}^{(p-1)}) & \to & \mathcal{A}_3'(C_{ijk}^{(p+1)}) & \to & \mathcal{A}_4'(C_{ijkl}^{(p+3)}) & \to & \mathcal{A}_5'(C_{ijklm}^{(p+5)}) \end{array} \quad (5)$$

where the T-dual multiplet in the second line is needed for the NSNS gauge symmetry. Each component contains the same five integrals that the first component has. It has been speculated in [21] that there are three multiplets corresponding to the third part. In this paper, we find these multiplets. We find they have the following structure:

$$\begin{array}{ccccccccc} \mathbf{A}_0(C^{(p-3)}) & \to & \mathbf{A}_1(C_i^{(p-1)}) & \to & \mathbf{A}_2(C_{ij}^{(p+1)}) & \to & \mathbf{A}_3(C_{ijk}^{(p+3)}) & \to & \mathbf{A}_4(C_{ijkl}^{(p+5)}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{A}_1'(C_i^{(p-1)}) & \to & \mathbf{A}_2'(C_{ij}^{(p+1)}) & \to & \mathbf{A}_3'(C_{ijk}^{(p+3)}) & \to & \mathbf{A}_4'(C_{ijkl}^{(p+5)}) \\ & & & & \downarrow & & \downarrow & & \\ & & & & A_2''(C_{ij}^{(p+1)}) & \to & A_3''(C_{ijk}^{(p+3)}) & & \end{array} \quad (6)$$

The multiplets in the second and the third lines are needed for the NSNS gauge symmetry. All of the above components carry the same 14 integrals that the first component carries. Here also one may expect the sum of the multiplets (4), (5), and (6) to satisfy the Ward identity corresponding to the RR gauge transformation. Even though the first components, which are calculated explicitly, satisfy this condition, the other components do not satisfy this condition. This again indicates that there should be another T-dual multiplet like (3) in which the first component is invariant under the RR gauge transformation. In this case, we find that it is hard to find this amplitude from the Ward identities of the NSNS gauge transformations. This component may be calculated explicitly in string theory in which we are not interested in this paper.

The outline of the paper is as follows: We begin with Sec. II, which is a review for the T-dual Ward identity. In Sec. III, using the consistency of the S-matrix element of one RR $(p-3)$ form, one NSNS state, and one open-string NS state that has been calculated in [39,20], with the Ward identity corresponding to the T duality and the gauge symmetries, we find the corresponding S-matrix elements for all other RR potentials. In Sec. IV, we perform the same calculations for the S-matrix element of one RR $(p-3)$ form and two NSNS states, which have been calculated in [38,39]. The amplitudes in this section, however, do not fully satisfy the Ward identity corresponding to the gauge transformation because of our lack of knowledge of the amplitude of the RR field strength $F^{(p)}$ with no transverse index. In Sec. V, we briefly discuss our results.

## II. T-DUAL WARD IDENTITY

It is known that the gauge symmetries of a given theory appear in the S-matrix elements through the corresponding Ward identities. That is, the S-matrix elements of the theory should be invariant under the linearized gauge transformations on the external states and should be invariant under the full nonlinear gauge transformation on the background fields. This idea has been speculated in [17] to hold even for the duality transformations of the theory. In particular, the S-matrix elements should be invariant/covariant under linear T-duality transformation of the external states and under nonlinear T-duality transformation of the background fields.

The full set of nonlinear T-duality transformations for massless RR and NSNS fields has been found in [4,7–10]. The nonlinear T-duality transformations of the RR field $C$ and the antisymmetric field $B$ are such that the expression $\mathcal{C} = e^B C$ transforms linearly under T duality [42]. When the T-duality transformation acts along the Killing coordinate $y$, the massless NSNS fields and $\mathcal{C}$ transform as

$$\begin{aligned} &e^{2\tilde{\phi}} = \frac{e^{2\phi}}{G_{yy}}; \qquad \tilde{G}_{yy} = \frac{1}{G_{yy}} \\ &\tilde{G}_{\mu y} = \frac{B_{\mu y}}{G_{yy}}; \qquad \tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}} \\ &\tilde{B}_{\mu y} = \frac{G_{\mu y}}{G_{yy}}; \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}} \\ &\tilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu y} = \mathcal{C}^{(n-1)}_{\mu\cdots\nu}; \qquad \tilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu} = \mathcal{C}^{(n+1)}_{\mu\cdots\nu y}, \end{aligned} \quad (7)$$

where $\mu, \nu$ denote any coordinate directions other than $y$. In the above transformation, the metric is given in the string frame. If $y$ is identified on a circle of radius $R$, i.e., $y \sim y + 2\pi R$, then after T duality the radius becomes $\tilde{R} = \alpha'/R$. The string coupling is also shifted as $\tilde{g}_s = g_s\sqrt{\alpha'}/R$.

We would like to study the T-dual Ward identity of scattering amplitudes, so we need the above transformations at the linear order. Assuming that the NSNS fields are small, perturbations around the flat space, e.g., $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the above transformations take the following linear form:

$$\tilde{\phi} = \phi - \frac{1}{2}h_{yy}, \qquad \tilde{h}_{yy} = -h_{yy}, \qquad \tilde{h}_{\mu y} = B_{\mu y}, \qquad \tilde{B}_{\mu y} = h_{\mu y}, \qquad \tilde{h}_{\mu\nu} = h_{\mu\nu}, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu}$$
$$\tilde{C}^{(n)}_{\mu\cdots\nu y} = C^{(n-1)}_{\mu\cdots\nu}, \qquad \tilde{C}^{(n)}_{\mu\cdots\nu} = C^{(n+1)}_{\mu\cdots\nu y}. \tag{8}$$

The T-duality transformation of the gauge field on the world volume of D-brane, when it is along the Killing direction, is $\tilde{A}_y = \phi^y$ where $\phi^y$ is the transverse scalar. When the gauge field is not along the Killing direction, it is invariant under the T duality.

The method for finding the couplings that are invariant under the T duality is given in [22]. It can be used to find the T-dual multiplet corresponding to a given scattering amplitude, which satisfies the T-dual Ward identity. Let us review this method here. Suppose we are implementing T duality along a world volume direction $y$ of a $D_p$-brane. We first separate the world volume indices along and orthogonal to $y$ and then apply the T-duality transformations (8). The orthogonal indices are the complete world volume indices of the T-dual $D_{p-1}$-brane. However, $y$ in the T-dual theory, which is a normal bundle index, is not complete. On the other hand, the normal bundle indices of the original theory are not complete in the T-dual $D_{p-1}$-brane. They do not include the $y$ index. In a T-dual multiplet, the index $y$ must be combined with the incomplete normal bundle indices to make them complete. If the scattering amplitudes are not invariant under the T duality, one should then add new amplitudes to them to have the complete indices after the T-duality transformation. In this way, one can find the T-dual multiplet, which satisfies the T-dual Ward identity.

The linear T-duality transformation of the RR potential (8) reveals that the $D_p$-brane world volume couplings of the RR $n$ form, which have no transverse index, are not related by the T duality to the couplings in which the RR $n$ form have one transverse index. The couplings in which the RR $n$ form have one transverse index are not related by the T duality to the couplings in which the RR $n$ form have two transverse indices and so on. To clarify this, one may first write $n = p + m$. If T duality is implemented along a world volume direction of a $D_p$-brane, then the RR $(p+m)$ form with no transverse index transforms to the RR $(p+m+1)$ form with one transverse index; however, at the same time, the $D_p$-brane transforms to $D_{p-1}$-brane. As a result, the RR $n$ form with no transverse index does not transform to the RR $(n)$ form with one transverse index. It transforms to RR $(n+2)$ form with one transverse index. Therefore, to study the T duality of the world volume amplitudes involving the RR potential, it is convenient to classify the RR potential according to its transverse indices.

## III. TWO CLOSED- AND ONE OPEN-STRING AMPLITUDES

The disk-level S-matrix element of one RR $(p-3)$ form, one NSNS state, and one-open string NS state has been calculated in [20,39]. The amplitude is nonzero only for the case that the NSNS polarization tensor is antisymmetric, the open string is the gauge field, and the RR polarization tensor has one and zero transverse indices. Accordingly, the amplitude has two parts that should be studied under the T-dual Ward identity separately. The first part is[1]

$$A_1(C_i^{(p-3)}) \sim T_p(\varepsilon_1)_i{}^{a_5\cdots a_p}\epsilon_{a_0\cdots a_p} p_3^i p_3^{a_4}(\varepsilon_3^A)^{a_3a_2} p_2^{a_0}\varepsilon_2^{a_1}\mathcal{Q}, \tag{9}$$

where $\mathcal{Q}$ is the integral that represents the open- and closed-string channels. In this amplitude, $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are the polarization of the RR, the gauge field, and the B field, respectively.

Using the totally antisymmetric property of the $D_p$-brane world volume form $\epsilon_{a_0\cdots a_p}$, one can easily rewrite the amplitude in terms of the B-field strength and the gauge field strength. Using the fact that the amplitude should satisfy the Ward identity corresponding to the NSNS and NS gauge symmetries, one realizes that the above coupling is the only possibility. So even without using the string-theory calculation, one can find the above amplitude. The string theory, however, gives information about the integral $\mathcal{Q}$ as well. The explicit form of this integral in terms of the Mandelstam variables has been found in [39]. The T-dual Ward identity then produces the following terms [21]:

$$\begin{aligned}
A_2(C_{ij}^{(p-1)}) &\sim T_p(\varepsilon_1)_{ij}{}^{a_4\cdots a_p}\epsilon_{a_0\cdots a_p} p_3^i p_3^{a_3}[2(\varepsilon_3^S)^{a_2j} p_2^{a_0}\varepsilon_2^{a_1} \\
&\quad + (\varepsilon_3^A)^{a_1a_2} p_2^{a_0}\phi^j]\mathcal{Q} \\
A_3(C_{ijk}^{(p+1)}) &\sim \frac{1}{2}T_p(\varepsilon_1)_{ijk}{}^{a_3\cdots a_p}\epsilon_{a_0\cdots a_p} p_3^i p_3^{a_2}[-2(\varepsilon_3^A)^{jk} p_2^{a_0}\varepsilon_2^{a_1} \\
&\quad + 4(\varepsilon_3^S)^{a_1j} p_2^{a_0}\phi^k]\mathcal{Q} \\
A_4(C_{ijkl}^{(p+3)}) &\sim T_p(\varepsilon_1)_{ijkl}{}^{a_2\cdots a_p}\epsilon_{a_0\cdots a_p} p_3^i p_3^{a_1} p_2^{a_0}(\varepsilon_3^A)^{jk}\phi^l\mathcal{Q}.
\end{aligned} \tag{10}$$

On the other hand, the RR Ward identity connects the amplitude (9) to the second part in which the RR $(p-3)$ form has no transverse index. In this section, we are

[1]Our conventions set $\alpha' = 2$ in the string-theory amplitudes. Our index convention is that the Greek letters $(\mu, \nu, \ldots)$ are the indices of the space-time coordinates, the Latin letters $(a, d, c, \ldots)$ are the world-volume indices, and the letters $(i, j, k, \ldots)$ are the normal bundle indices.

going to find the T-dual multiplet corresponding to the second part.

### A. RR $(p-3)$ form with no transverse index

The explicit calculation of the S-matrix element of the RR $(p-3)$ form with no transverse index gives the following result [39,20]:

$$\begin{aligned} A_0 \sim -(F_1)^{a_0a_1}\Big[&\varepsilon_2{}^{a_3}p_3^{a_2}(p_1\cdot D\cdot\varepsilon_3^A)^{a_4}J_1 \\ &+\varepsilon_2{}^{a_3}p_3^{a_2}(p_1\cdot\varepsilon_3^A)^{a_4}J_3+2\varepsilon_2{}^{a_3}p_3^{a_2}(p_2\cdot\varepsilon_3^A)^{a_4}J_2 \\ &-\frac{1}{4}\varepsilon_2{}^{a_2}p_3\cdot V\cdot p_3(\varepsilon_3^A)^{a_3a_4}(J_1+J_3) \\ &+\frac{1}{2}p_3^{a_2}p_3\cdot\varepsilon_2(\varepsilon_3^A)^{a_3a_4}(J_1-2J_2+J_3)\Big] \\ &-(f_1)^{a_0i}\varepsilon_2{}^{a_2}p_3^{a_1}p_{3i}(\epsilon_3^A)^{a_3a_4}(J_1-J_3), \end{aligned} \tag{11}$$

where the RR field strength $(F_1)^{a_0a_1}=p_1^{a_0}\varepsilon_1{}^{a_1}-p_1^{a_1}\varepsilon_1{}^{a_0}$ and the RR factor $(f_1)^{a_0i}=-p_1^i\varepsilon_1{}^{a_0}$. The diagonal matrix $D$ is $D=V-N$ where $V$ is the flat metric of the world volume space and $N$ is the flat metric of the transverse space. There is an overall factor of $T_4\epsilon_{a_0\cdots a_4}$. For simplicity, we have written the amplitude for $p=4$. It can easily be extended to arbitrary $p$ by contracting the extra world volume indices with the RR potential. The closed- and open-string channels appear in the integrals $J_1$, $J_2$, $J_3$. The explicit form of these integrals has been found in [20].

Note that $(f_1)^{a_0i}$ in the last line of (11) is not the RR field strength. In fact, the RR Ward identity connects the amplitude (9) to the last term in (11), so there is the following relation between $\mathcal{Q}$, $J_1$, $J_3$:

$$\mathcal{Q}=J_1-J_3, \tag{12}$$

which can be verified using the explicit form of these integrals. The last term in (11), however, breaks the NS gauge symmetry. The RR gauge-invariant terms in the first two lines are needed to make this term invariant under the NS and the NSNS gauge transformations. These constraints give the following relation between the integrals:

$$\begin{aligned} &2p_1\cdot N\cdot p_3(J_1-J_3)+p_3\cdot V\cdot p_3(J_1+J_3) \\ &\quad+2p_2\cdot p_3(J_1-2J_2+J_3)=0. \end{aligned} \tag{13}$$

Therefore, there are two independent integrals in the amplitude (11).

One can verify that the terms in the first two lines of (14) are all possible independent contractions between $(F_1)^{a_0a_1}$, the B field, the gauge field, two momenta, and $\epsilon_{a_0a_1a_2a_3a_4}$. One may consider the terms $\varepsilon_2{}^{a_2}p_2\cdot p_3(\varepsilon_3^A)^{a_3a_4}$ or $\varepsilon_2{}^{a_2}p_1\cdot N\cdot p_3(\varepsilon_3^A)^{a_3a_4}$ as well. However, before fixing the integrals, these terms can be absorbed into the fourth term in (11). Therefore, the string theory calculates the coefficients of all independent terms such that the amplitude satisfies various Ward identities. The coefficients have information about the open- and closed-string poles as well. As we see in the next subsections, the T-dual Ward identity, which connects the above amplitude to all other RR potential, does not produce any new integrals.

To apply the T-dual Ward identity on the amplitude (11), it is convenient to rewrite the amplitude in terms of the flat metrics $V$, $N$. Using the relations $D=V-N$ and $\eta=V+N$, one finds

$$\begin{aligned} A_0 \sim (F_1)^{a_0a_1}\Big[&\varepsilon_2{}^{a_3}p_3^{a_2}(p_1\cdot N\cdot\varepsilon_3^A)^{a_4}\mathcal{Q} \\ &+\varepsilon_2{}^{a_3}p_3^{a_2}(p_2\cdot\varepsilon_3^A)^{a_4}\mathcal{Q}_2+\varepsilon_2{}^{a_3}p_3^{a_2}(p_3\cdot V\cdot\varepsilon_3^A)^{a_4}\mathcal{Q}_1 \\ &+\frac{1}{4}\varepsilon_2{}^{a_2}p_3\cdot V\cdot p_3(\varepsilon_3^A)^{a_3a_4}\mathcal{Q}_1 \\ &-\frac{1}{2}p_3^{a_2}p_3\cdot\varepsilon_2(\varepsilon_3^A)^{a_3a_4}\mathcal{Q}_2\Big] \\ &-(f_1)^{a_0i}\varepsilon_2{}^{a_2}p_3^{a_1}p_{3i}(\epsilon_3^A)^{a_3a_4}\mathcal{Q}, \end{aligned} \tag{14}$$

where

$$\mathcal{Q}_1=J_1+J_3;\qquad \mathcal{Q}_2=J_1-2J_2+J_3. \tag{15}$$

The identity (13) then becomes

$$2p_1\cdot N\cdot p_3\mathcal{Q}+p_3\cdot V\cdot p_3\mathcal{Q}_1+2p_2\cdot p_3\mathcal{Q}_2=0. \tag{16}$$

One may write $\mathcal{Q}=p_3\cdot V\cdot p_3p_2\cdot p_3\mathcal{Q}'$, $\mathcal{Q}_1=p_1\cdot N\cdot p_3p_2\cdot p_3\mathcal{Q}_1'$, and $\mathcal{Q}_2=p_1\cdot N\cdot p_3p_3\cdot V\cdot p_3\mathcal{Q}_2'$. Then the above constraint can be solved to write the amplitude (14) in terms of two integrals. However, we prefer to work with the three integrals and the constraint (16).

#### *1. RR $(p-1)$ form with one transverse index*

In this section, we are going to apply the T-dual Ward identity on the amplitude (14). We have reviewed the method for applying the linear T duality on the scattering amplitudes (the T-dual Ward identity) in Sec. II. We refer the interested readers to [21] for more details on how to apply it to the specific cases. Following [21], one finds that the amplitude (14) is covariant under the linear T duality when the isometric index $y$ is carried by the RR potential. However, when the $y$ index is carried by the NS or the NSNS polarizations, it is not invariant. Using the same steps as we have done in [21], one finds that the following amplitude has to be add to the amplitude (14) to make it invariant under the linear T-duality transformations:

$$A_1 \sim (f_1)^{a_0 a_1}{}_i \bigg[ -\frac{1}{2}(\varepsilon_2)^{a_3} p_3\cdot V\cdot p_3(\varepsilon_3^S)^{a_2 i}\mathcal{Q}_1 - p_3^{a_2} p_3\cdot V\cdot \varepsilon_2(\varepsilon_3^S)^{a_3 i}\mathcal{Q}_2 + \frac{1}{4}\phi^i p_3\cdot V\cdot p_3(\varepsilon_3^A)^{a_2 a_3}\mathcal{Q}_1 + (\varepsilon_2)^{a_3} p_3^{a_2}(p_1\cdot N\cdot\varepsilon_3^S)_i\mathcal{Q} - \phi^i p_3^{a_2}(p_1\cdot N\cdot\varepsilon_3^A)^{a_3}\mathcal{Q} + \varepsilon_2{}^{a_3} p_3^{a_2}(p_2\cdot V\cdot\varepsilon_3^S)_i\mathcal{Q}_2 + \varepsilon_2{}^{a_3} p_3^{a_2}(p_3\cdot V\cdot\varepsilon_3^S)_i\mathcal{Q}_1 - \phi^i p_3^{a_2}(p_2\cdot V\cdot\varepsilon_3^A)^{a_3}\mathcal{Q}_2 - \phi^i p_3^{a_2}(p_3\cdot V\cdot\varepsilon_3^A)^{a_3}\mathcal{Q}_1\bigg] - (f_1)^{a_0}{}_{ij} p_3^{a_1} p_3{}^i[\phi^j(\varepsilon_3^A)^{a_2 a_3} - 2\varepsilon_2{}^{a_3}(\varepsilon_3^S)^{a_2 j}]\mathcal{Q}, \tag{17}$$

where the RR factors $(f_1)^{a_0 a_1 i} = -2p_1^{a_1}\varepsilon_1{}^{a_0 i}$ and $(f_1)^{a_0 ij} = p_1^j\varepsilon_1{}^{a_0 i} - p_1^i\varepsilon_1{}^{a_0 j}$. For simplicity, we have written the amplitude for $p=3$. In the above amplitude, $\phi$ is the polarization of the transverse scalar fields, and $\varepsilon^S$ is the polarization of the graviton. Note that each term has either one transverse polarization that is the T duality of the gauge field or one symmetric NSNS polarization that is the T dual of the antisymmetric NSNS polarization in (14). The above amplitude satisfies the Ward identity corresponding to the antisymmetric NSNS and the NS gauge symmetries. They are inherited from the amplitude (14). The graviton term in the last line also satisfies the Ward identity corresponding to the symmetric NSNS gauge transformation. However, the other graviton terms do not satisfy this Ward identity.

This indicates that the T-dual Ward identity could not capture all terms of the scattering amplitude of the RR $(p-1)$. In fact under the T duality, the RR potential $C^{(n)}$, which has no $y$ index, transforms to $(C^{(n+1)})^y$, which has one $y$ index. On the other hand, if this $y$ index is contracted with a polarization of the NSNS or the NS state in (14), the T duality then can capture it. However, if the $y$ index is contracted with the momentum of the NSNS polarization tensor in (14), then the T duality can not capture it because in the T-duality transformation it is implicitly assumed that fields are independent of the $y$ coordinate. Therefore, the T-dual Ward identity cannot capture the terms that have the RR factor $(f_1^{(n+1)})^i p_{3i}$. The terms in the second bracket in (17) have already one $p_{3i}$ that contracted with the RR factor. So it is impossible to have another $p_{3j}$ to contract the RR factor. However, the terms in the first bracket have no $p_{3i}$, so it is possible to include terms that have $(f_1)^{a_0 a_1 i} p_{3i}$. These terms could not be captured by the T-dual Ward identity.

To find such terms, we can consider all independent terms made of one momentum, $\varepsilon_2$, $\varepsilon_3^S$, or $\phi_2$, $\varepsilon_3^A$ which carry the indices $(\cdots)^{a_2 a_3}$. Each term should be invariant under the linear T duality when the world volume indices $a_2$ and $a_3$ are not the $y$ index; e.g., $(\varepsilon_2\cdot V\cdot\varepsilon_3^S)^{a_3} - (\phi_2\cdot N\cdot\varepsilon_3^A)^{a_3}$ is invariant under the linear T duality when $a_3\neq y$. Choosing all such terms that are seven terms, with unknown coefficients and imposing the condition that they should satisfy the Ward identity corresponding to the NSNS and the NS gauge transformations, one finds the following result:

$$A'_1 \sim (f_1)^{a_0 a_1 i} p_{3i}\bigg[-\frac{1}{2}p_3^{a_2}\varepsilon_2{}^{a_3}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{Q}_1 + p_3^{a_2}\mathcal{Q}_2((\varepsilon_2\cdot V\cdot\varepsilon_3^S)^{a_3} - (\phi\cdot N\cdot\varepsilon_3^A)^{a_3}) + \frac{1}{2}p_3\cdot N\cdot\phi(\varepsilon_3^A)^{a_2 a_3}\mathcal{Q}_2 - \varepsilon_2{}^{a_3}(p_1\cdot N\cdot\varepsilon_3^S)^{a_2}\mathcal{Q} - \varepsilon_2{}^{a_3}(p_2\cdot V\cdot\varepsilon_3^S)^{a_2}\mathcal{Q}_2\bigg]. \tag{18}$$

The combination of the above amplitude and amplitude (17) satisfies the Ward identity corresponding to the NSNS and the NS gauge transformation. They satisfy the T-dual Ward identity when the $y$ index is carried by RR potential. Otherwise, they are not invariant under the linear T duality. In the next subsection, we find the amplitudes that are needed to make the amplitudes (17) and (18) invariant under the T duality.

### 2. RR $(p+1)$ form with two transverse indices

The amplitude (17) makes the amplitude (14) to be invariant under the linear T duality when the $y$ index in the amplitude (14) is carried by the NSNS and the NS polarization tensors. However, the amplitude (17) is invariant under the T duality only when the $y$ index is carried by the RR potential; otherwise, it is not invariant. To fix this problem, we have to add the following amplitude to it:

$$A_2 \sim (f_1)^{a_0 a_1}{}_{ij}\bigg[\frac{1}{4}p_3\cdot V\cdot p_3(\varepsilon_2^{a_2}(\varepsilon_3^A)^{ij} + 2\phi^i(\varepsilon_3^S)^{a_2 j})\mathcal{Q}_1 - \frac{1}{2}p_3^{a_2}p_3\cdot V\cdot\varepsilon_2(\varepsilon_3^A)^{ij}\mathcal{Q}_2 + \phi^j p_3^{a_2}(p_1\cdot N\cdot\varepsilon_3^S)^i\mathcal{Q} + \phi^j p_3^{a_2}(p_2\cdot V\cdot\varepsilon_3^S)^i\mathcal{Q}_2 + \phi^j p_3^{a_2}(p_3\cdot V\cdot\varepsilon_3^S)^i\mathcal{Q}_1\bigg] - (f_1)^{a_0}{}_{ijk}p_3^{a_1}p_3{}^i[\varepsilon_2{}^{a_2}(\varepsilon_3^A)^{jk} + 2\phi^j(\varepsilon_3^S)^{a_2 k}]\mathcal{Q}, \tag{19}$$

where $(f_1)^{a_0 a_1 ij} = p_1^{a_0}\varepsilon_1{}^{a_1 ij} - p_1^{a_1}\varepsilon_1{}^{a_0 ij}$ and $(f_1)^{a_0 ijk} = -p_1^k\varepsilon_1{}^{a_0 ij} - p_1^j\varepsilon_1{}^{a_0 ki} - p_1^i\varepsilon_1{}^{a_0 jk}$. For simplicity, we have written the amplitude for $p=2$. Note that the RR factors are not the RR field strengths. The above amplitude, which has been found by imposing the T-dual Ward identity on the amplitude (17), is not the full amplitude for the RR $(p+1)$ form with two transverse indices because it is not invariant under the NSNS and the NS gauge transformations. However, the terms in the last line satisfy these Ward identities so that the T-dual Ward identity could capture all terms that have the RR factor $(f_1)^{a_0ijk}$. As in the previous section, there should be some terms in the first bracket that are proportional to $(f_1)^{a_0a_1ij}p_{3j}$. These term are not captured by the T duality.

One may either impose the Ward identity corresponding to the NSNS and the NS gauge transformations to find the gauge completion of the amplitude in the first bracket in (19), as we have done in the previous section, or impose the T-dual Ward identity to find the T-dual completion of the amplitude (18) when the $y$ index is carried by the NSNS and the NS polarization tensors. In both cases, one finds the following result:

$$\begin{aligned} A'_2 \sim (f_1)^{a_0a_1ij}p_{3i}\bigg[&-\frac{1}{2}(p_3)^{a_2}\phi_j\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{Q}_1 \\ &+ p_3\cdot N\cdot\phi(\varepsilon_3^S)^{a_2}{}_j\mathcal{Q}_2 + p_3^{a_2}((\varepsilon_2\cdot V\cdot\varepsilon_3^A)_j \\ &-(\phi\cdot N\cdot\varepsilon_3^S)_j)\mathcal{Q}_2 - (\phi_j(p_1\cdot N\cdot\varepsilon_3^S)^{a_2} \\ &-\varepsilon_2{}^{a_2}(p_1\cdot N\cdot\varepsilon_3^A)_j)\mathcal{Q} - (\phi_j(p_2\cdot V\cdot\varepsilon_3^S)^{a_2} \\ &-\varepsilon_2{}^{a_2}(p_2\cdot V\cdot\varepsilon_3^A)_j)\mathcal{Q}_2\bigg]. \end{aligned} \tag{20}$$

The combination of the above amplitude and the amplitude (19) satisfies the Ward identities corresponding to the NSNS and the NS gauge transformations. Neither the above amplitude nor the amplitude (19) are invariant under linear T duality when the $y$ index is carried by the NSNS and the NS polarization tensors in these amplitudes. In the next subsection, we find the T-dual completion of these amplitudes.

### *3. RR $(p+3)$ form with three transverse indices*

The symmetric NSNS and the gauge field polarization tensors in the first and last line of the amplitude (19) carry the world volume index $a_2$. So when the world volume index $y$ is carried by these tensors, the amplitude does not satisfy the T-dual Ward identity. So we must add the following amplitude to make it consistent with T-dual Ward identity:

$$\begin{aligned} A_3 \sim{}& (f_1)^{a_0a_1ijk}\left[\frac{1}{4}\phi_k p_3\cdot V\cdot p_3(\varepsilon_3^A)_{ij}\mathcal{Q}_1\right] \\ &-(f_1)^{a_0ijkl}[\phi_l p_3^{a_1}p_{3i}(\varepsilon_3^A)_{jk}\mathcal{Q}], \end{aligned} \tag{21}$$

where $(f_1)^{a_0a_1ijk} = -2p_1^{a_1}\varepsilon_1{}^{a_0ijk}$ and $(f_1)^{a_0ijkl} = p_1^l\varepsilon_1{}^{a_0ijk} - p_1^k\varepsilon_1{}^{a_0ikl} + p_1^j\varepsilon_1{}^{a_0ikl} - p_1^i\varepsilon_1{}^{a_0jkl}$. For simplicity, we have written the amplitude for $p=1$. Similarly, we have to add the following amplitude to (20) to make it invariant under the linear T duality:

$$\begin{aligned} A'_3 \sim{}& \frac{1}{2}(f_1)^{a_0a_1ijk}p_{3i}[p_3\cdot N\cdot\phi(\varepsilon_3^A)_{jk}\mathcal{Q}_2 \\ &-2\phi_j(p_1\cdot N\cdot\varepsilon_3^A)_k\mathcal{Q} - 2\phi_j(p_2\cdot V\cdot\varepsilon_3^A)_k\mathcal{Q}_2]. \end{aligned} \tag{22}$$

The combination of the above two amplitudes satisfies the Ward identity corresponding to the antisymmetric NSNS gauge transformation. The antisymmetric NSNS polarization tensor in these amplitudes does not carry any world volume index. So the above amplitudes are invariant under linear T duality. So there is no need for the amplitude $A_4(C_{ijkl}^{(p+5)})$ to be added. In fact, noting that the open-string momentum must be along the world volume directions, one can verify that it is impossible to have contraction between $\epsilon_{a_0\cdots a_p}$, one $C_{ijkl}^{(p+5)}$, three momenta, one NSNS, and one NS polarization tensors.

Therefore, the amplitudes that we have found so far, i.e.,

$$\begin{array}{ccccccc} A_0(C^{(p-3)}) & \rightarrow & A_1(C_i^{(p-1)}) & \rightarrow & A_2(C_{ij}^{(p+1)}) & \rightarrow & A_3(C_{ijk}^{(p+3)}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A'_1(C_i^{(p-1)}) & \rightarrow & A'_2(C_{ij}^{(p+1)}) & \rightarrow & A'_3(C_{ijk}^{(p+3)}) \end{array} \tag{23}$$

satisfy the Ward identity corresponding to the T duality, the NSNS, and the NS gauge transformations. However, they do not satisfy the Ward identity corresponding to the RR gauge transformation. In the next section, we find some other amplitudes by imposing the constraint that the amplitudes must satisfy the RR Ward identity as well.

### B. RR Ward identity

In this section, we are going to add the appropriate amplitudes to the amplitudes that have been found in the previous section to make them satisfy the Ward identity corresponding to the RR gauge transformations as well as the T duality and the NSNS and the NS gauge transformations. The RR Ward identity allows us to write the amplitudes in terms of the RR field strengths. The combination of the amplitudes $A_1(C_i^{(p-3)})$ in (14) and $A_0(C^{(p-3)})$ in (9) satisfies the RR Ward identity because they are the amplitudes that have been calculated explicitly in string theory [20,39]. The terms in the first three lines of (14) are in terms of the RR field strength $F_1^{a_0a_1}$. The combination of the terms in the last line of (14) and (9) can also be written in terms of the RR field strength $F_1^{a_0i} = p_1^{a_0}\varepsilon_1^i - p_1^i\varepsilon_1^{a_0}$.

Using the T-dual Ward identity, we have found the amplitudes for the RR potential $(p-1)$ form, which has two and one transverse indices. One can verify that it is impossible to have the amplitude for the RR $(p-1)$ form, which carries more that two transverse indices. However, there are possibilities for having amplitude for the RR $(p-1)$ form, which carries zero transverse index. This amplitude can be found by imposing the RR Ward identity on the amplitudes that we have found in the previous section. The combination of the terms in the last line of (17) and the terms in the first line of (10) satisfies the RR Ward identity. They can be written as in the last line of (17) but with the RR field strength $(F_1)^{a_0ij} = p_1^{a_0}\varepsilon_1^{ij} + p_1^j\varepsilon_1^{a_0i} + p_1^i\varepsilon^{ja_0}$ instead of $(f_1)^{a_0ij}$.

However, the other terms in (17) and the terms in (18) do not not satisfy the RR Ward identity because the RR factor $(f_1)^{a_0a_1i}$ is not the full RR field strength. So the obvious extension of these amplitudes to the RR invariant amplitudes is to extend this factor to the RR field strength $(F_1)^{a_0a_1i} = p_1^{a_0}\varepsilon_1^{a_1i} - p_1^{a_1}\varepsilon_1^{a_0i} + p_1^i\varepsilon_1^{a_0a_1}$. However, the new amplitude resulting from the last term in $(F_1)^{a_0a_1i}$ would not be invariant under the NSNS and the NS gauge transformations. To remedy this failure, one has to still add another amplitude that should be proportional to the RR field strength $(F_1)^{a_0a_1a_2}$ and should make the above terms invariant under the NSNS and the NS gauge transformations. We consider all independent terms $(\cdots)^{a_3}$ containing two momenta and the NSNS and the NS polarization tensors which are invariant under the linear T duality when the world volume index $a_3$ is not the $y$ index. Fixing the coefficients of these terms by combining them with the above nongauge-invariant terms and requiring that they should satisfy the Ward identity corresponding to the NSNS and the NS gauge transformations, one finds the following result:

$$\begin{aligned}
A_0 \sim (F_1)^{a_0a_1a_2}\bigg[&\frac{1}{3}\varepsilon_2^{a_3}\bigg(3p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{Q}_3 + p_1\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{Q} + p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot p_3\mathcal{Q}_1 + 3p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_3\mathcal{Q}_4\\
&+ 2p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{Q}_2 - \frac{1}{2}(p_1\cdot N\cdot p_3\mathcal{Q}_1 + 3p_2\cdot p_3\mathcal{Q}_4)\mathrm{Tr}[\varepsilon_3^S\cdot V]\bigg)\\
&-\frac{1}{3}p_3\cdot V\cdot\varepsilon_2\bigg((p_1\cdot N\cdot\varepsilon_3^S)^{a_3}\mathcal{Q}_2 + 3(p_2\cdot V\cdot\varepsilon_3^S)^{a_3}\mathcal{Q}_3 + \frac{3}{2}p_3^{a_3}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{Q}_4\bigg) - \frac{1}{3}p_3\cdot N\cdot\phi((p_1\cdot N\cdot\varepsilon_3^A)^{a_3}\mathcal{Q}_2\\
&+ 3(p_2\cdot V\cdot\varepsilon_3^A)^{a_3}\mathcal{Q}_3 + 3(p_3\cdot V\cdot\varepsilon_3^A)^{a_3}\mathcal{Q}_4) - \frac{1}{2}p_3\cdot V\cdot p_3((\varepsilon_2\cdot V\cdot\varepsilon_3^S)^{a_3} - (\phi\cdot N\cdot\varepsilon_3^A)^{a_3})\mathcal{Q}_4\\
&+ p_3^{a_3}\bigg((p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2 + p_2\cdot V\cdot\varepsilon_3^A\cdot N\cdot\phi)\mathcal{Q}_3 + (p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2 + p_3\cdot V\cdot\varepsilon_3^A\cdot N\cdot\phi)\mathcal{Q}_4\\
&+ \frac{1}{3}(p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2 + p_1\cdot N\cdot\varepsilon_3^A\cdot N\cdot\phi)\mathcal{Q}_2\bigg)\bigg],
\end{aligned}\tag{24}$$

where the new integrals $\mathcal{Q}_{,3}$, $\mathcal{Q}_{,4}$ satisfy the following relation:

$$3p_3\cdot V\cdot p_3\mathcal{Q}_4 + 6p_2\cdot p_3\mathcal{Q}_3 + 2p_1\cdot N\cdot p_3\mathcal{Q}_2 = 0. \tag{25}$$

The above constraint cannot be used to find the integrals $\mathcal{Q}_3$, $\mathcal{Q}_4$. Unlike the Ward identities corresponding to the T duality and the NSNS/NS gauge transformations, which do not produce new integrals, the Ward identities corresponding to the RR and the NSNS/NS gauge transformations produce new integrals $\mathcal{Q}_3$, $\mathcal{Q}_4$. In order to study the above amplitude at low energy, one has to perform the explicit string-theory calculation to find these integrals. In this paper, we are not interested in the explicit form of these integrals.

The amplitude (24) satisfies the T-dual Ward identity when the $y$ index is carried by the RR field strength; however, when $a_3$ is the $y$ index, it is not invariant under the linear T duality. To make (24) invariant under the linear T duality, one has to include the following amplitude:

$$\begin{aligned}
A_1 \sim (f_1)^{a_0a_1a_2}{}_i\bigg[&\frac{1}{3}\phi^i\bigg(3p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{Q}_3 + p_1\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{Q} + p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot p_3\mathcal{Q}_1\\
&+ 2p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{Q}_2 + 3p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_3\mathcal{Q}_4 - \frac{1}{2}(p_1\cdot N\cdot p_3\mathcal{Q}_1 + 3p_2\cdot p_3\mathcal{Q}_4)\mathrm{Tr}[\varepsilon_3^S\cdot V]\bigg)\\
&-\frac{1}{3}p_3\cdot V\cdot\varepsilon_2((p_1\cdot N\cdot\varepsilon_3^A)^i\mathcal{Q}_2 + 3(p_2\cdot V\cdot\varepsilon_3^A)^i\mathcal{Q}_3) - \frac{1}{2}p_3\cdot V\cdot p_3((\varepsilon_2\cdot V\cdot\varepsilon_3^A)^i - (\phi\cdot N\cdot\varepsilon_3^S)^i)\mathcal{Q}_4\\
&-\frac{1}{3}p_3\cdot N\cdot\phi((p_1\cdot N\cdot\varepsilon_3^S)^i\mathcal{Q}_2 + 3(p_3\cdot V\cdot\varepsilon_3^S)^i\mathcal{Q}_4 + 3(p_2\cdot V\cdot\varepsilon_3^S)^i)\mathcal{Q}_3\bigg],
\end{aligned}\tag{26}$$

where $(f_1)^{a_0a_1a_2i} = p_1^{a_0}\varepsilon_1^{a_1a_2i} + p_1^{a_2}\varepsilon_1^{a_0a_1i} + p_1^{a_1}\varepsilon_1^{a_2a_0i}$. The world volume form does not contract with the NSNS or the NS polarization tensors, so the above amplitude satisfies the T-dual Ward identity. However, it does not satisfy the Ward identity corresponding to the NSNS or the NS gauge transformations. So there are some missing terms that are not captured by the T-dual Ward identity. The missing terms are the following:

$$A'_1 \sim (f_1)^{a_0a_1a_2i} p_{3i}\bigg[\frac{1}{3}(p_1\cdot N\cdot \varepsilon_3^A\cdot V\cdot \varepsilon_2 + p_1\cdot N\cdot\varepsilon_3^S\cdot N\cdot\phi)\mathcal{Q}_2 + \frac{1}{2}p_3\cdot N\cdot\phi\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{Q}_4 + (p_2\cdot V\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2 + p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot\phi)\mathcal{Q}_3\bigg]. \tag{27}$$

The combination of the above amplitude and (26) is invariant under the NSNS and the NS gauge transformations.

Since the RR factor $(f_1)^{a_0a_1a_2i}$ is not the RR field strength, the amplitude $A_1 + A'_1$ does not satisfy the RR Ward identity. It can be easily extended to satisfy this Ward identity by extending the RR factor to the RR field strength $(F_1)^{a_0a_1a_2i}$. The new terms in this amplitude, the terms proportional to $(F_1)^{a_0a_1a_2i} - (f_1)^{a_0a_1a_2i}$, should satisfy the Ward identity corresponding to the NSNS and the NS gauge transformations.

We now study the RR Ward identity of the amplitudes in Sec. III. A. 2. The combination of the terms in the last line of (19) and in the second line of (10) satisfies the RR Ward identity. They can be written as the last line in (19) in which the RR factor is replaced by the RR field strength $(F_1)^{a_0ijk}$. The terms in the first three lines of (19) and the terms in (20), however, do not satisfy the RR Ward identity because the RR factor in these amplitudes, $(f_1)^{a_0a_1ij}$, is not the RR field strength. They can easily be extended to the RR invariant amplitudes by extending the RR factor to the RR field strength $(F_1)^{a_0a_1ij}$. The new terms in this amplitude, the terms proportional to $(F_1)^{a_0a_1ij} - (f_1)^{a_0a_1ij}$, should satisfy the Ward identity corresponding to the NSNS and the NS gauge transformations. In the next section, we show that when the NSNS state is antisymmetric, the amplitude at order $\alpha'^2$, which has only contact terms, can be written in terms of field strengths.

The RR Ward identity of the amplitudes in Sec. III. A. 3 is as follows: The combination of terms in the last line of (21) and in the third line of (10) satisfies the RR Ward identity. They can be written as the last line in (21) in which the RR factor is replaced by the RR field strength $(F_1)^{a_0ijkl}$. The terms in the first line of (21) and the terms in (22), however, do not satisfy the RR Ward identity because the RR factor in these amplitudes, $(f_1)^{a_0a_1ijk}$, is not the RR field strength. They can easily be extended to the RR invariant amplitudes by extending the RR factor to the RR field strength $(F_1)^{a_0a_1ijk}$. The new terms, the terms proportional to $(F_1)^{a_0a_1ijk} - (f_1)^{a_0a_1ijk}$, should satisfy the Ward identity corresponding to the NSNS and the NS gauge transformations.

Therefore, the amplitudes that are invariant under the linear T duality and all of the gauge transformations can be written as three multiplets in terms of the RR field strength. The first multiplet is the following:

$$A_1(F_i^{(p-2)}) \to A_2(F_{ij}^{(p)}) \to A_3(F_{ijk}^{(p+2)}) \to A_4(F_{ijkl}^{(p+4)}), \tag{28}$$

where $A_1$, $A_2$, $A_3$, and $A_4$ are the terms in the last lines of (14), (17), (19), and (21), respectively, in which the RR factor $f_1$ is replaced by the RR field strength $F_1$. The other multiplet is

$$\begin{array}{ccccccc} A_0(F^{(p-2)}) & \to & A_1(F_i^{(p)}) & \to & A_2(F_{ij}^{(p+2)}) & \to & A_3(F_{ijk}^{(p+4)}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A'_1(F_i^{(p)}) & \to & A'_2(F_{ij}^{(p+2)}) & \to & A'_3(F_{ijk}^{(p+4)}) \end{array} \tag{29}$$

where $A_0$ appears in the first bracket in (14). The amplitudes $A_1$, $A_2$, and $A_3$ appear in the first brackets in (17), (19), and (21), respectively, in which the RR factor $f_1$ is replaced by the RR field strength $F_1$. The amplitudes $A'_1$, $A'_2$, and $A'_3$ are the amplitudes in (18), (20), and (22), respectively, in which the RR factor $f_1$ is replaced by the RR field strength $F_1$. The last multiplet is

$$\begin{array}{ccc} A_0(F^{(p)}) & \to & A_1(F_i^{(p+2)}) \\ & & \downarrow \\ & & A'_1(F_i^{(p+2)}) \end{array} \tag{30}$$

where $A_0$ appears in (24). The amplitudes $A_1$ and $A'_1$ are the same as the amplitudes (26) and (27), respectively, in which the RR factor $f_1$ is replaced by the RR field strength $F_1$.

### C. Low-energy couplings

The S-matrix elements that we have found in the previous sections can be analyzed at low energy to extract the appropriate couplings in field theory at order $\alpha'^2$. To this end, we need the $\alpha'$ expansion of the integrals that appear in the amplitudes. The $\alpha'$ expansion of the integrals $J_1$, $J_2$, $J_3$ has been found in [20]. Using the relation $p_1\cdot D\cdot p_1 + 4p_1\cdot p_2 = p_3\cdot D\cdot p_3$, one finds

$$\begin{aligned} J_1 &= -\frac{1}{p_1.p_3} - \frac{4}{p_3.D.p_3} + \frac{2}{3}\pi^2 p_1.p_3 + \frac{1}{6}\pi^2 p_3.D.p_3 - \frac{8\pi^2(p_2.p_3)^2}{3p_3.D.p_3} + \cdots \\ J_2 &= -\frac{1}{p_1.p_3} - \frac{4}{p_3.D.p_3} + \frac{2}{3}\pi^2 p_1.p_3 - \frac{1}{6}\pi^2 p_1.D.p_1 + \frac{1}{3}\pi^2 p_3.D.p_3 - \frac{8\pi^2(p_2.p_3)^2}{3p_3.D.p_3} + \cdots \\ J_3 &= -\frac{3}{p_1.p_3} - \frac{4}{p_3.D.p_3} + \frac{2}{3}\pi^2 p_1.p_3 + \frac{1}{2}\pi^2 p_3.D.p_3 - \frac{8\pi^2(p_2.p_3)^2}{3p_3.D.p_3} + \cdots, \end{aligned} \tag{31}$$

where the dots refer to the terms with more than two momenta. They are related to the couplings at order $O(\alpha'^3)$ in which we are not interested.

It is interesting to note that the massless closed-string pole $1/p_1 \cdot p_3$ appears only at the leading order, which resultediii from the fact that there is neither the higher derivative couplings between three closed strings in the bulk nor the higher derivative couplings between one closed and one open string on the D-brane world volume. The above expansions can be used to find the low energy expansion of the integrals $\mathcal{Q}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, which appear in the S-matrix elements in multiplets (28) and (29). The massless poles at the leading order should be reproduced by the supergravity couplings in the bulk and the DBI and CS couplings on the D-brane. The next to leading order terms have massless open-string pole and contact terms. The contact terms do not produce, in general, the couplings of field theory at order $\alpha'^2$. One has to first calculate the massless pole in field theory, which is produced by the couplings of one closed- and two open-string states at order $\alpha'^2$ [43,44] and by the couplings of one closed- and one open-string states, which are given by the DBI and CS actions. Then one should subtract it from the above massless pole. This subtraction may cancel some of the contact terms as well. The leftover contact terms then produce new couplings at order $\alpha'^2$ between one RR, one NSNS, and one NS state in the field theory.

We are interested in this paper in finding the couplings of one $F^{(p+2)}$ with two transverse indices, one B field, and one gauge boson. There is no coupling between one RR $(p+1)$ form and two gauge bosons [44], so we expect that the amplitude in the string theory side has no massless open string pole. The string theory amplitude is given by the sum of (19) and (20) in which the RR factors are replaced by the RR field strength and the NSNS polarization is antisymmetric. Using the expansion (31), one can easily verify that it has no massless open string pole, as expected. The amplitude at order $\alpha'^2$ then has only contact terms. These contact terms are the following:

$$\begin{aligned}
A_c(\alpha'^2) \sim \frac{\pi^2}{3}(F_1)^{a_0a_1}{}_{ij}[&4p_2.p_3p_3^ip_3^{a_2}(\varepsilon_2.V.\varepsilon_3^A)^j + 2p_3.V.p_3p_3^ip_3^{a_2}(\varepsilon_2.V.\varepsilon_3^A)^j - 2p_3.V.p_3\varepsilon_2{}^{a_2}p_3^i(p_1.N.\varepsilon_3^A)^j \\
&+ 4p_2.p_3\varepsilon_2{}^{a_2}p_3^i(p_2.V.\varepsilon_3^A)^j + 2p_3.V.p_3\varepsilon_2{}^{a_2}p_3^i(p_2.V.\varepsilon_3^A)^j - p_2.p_3p_3.V.p_3\varepsilon_2{}^{a_2}(\varepsilon_3^A)^{ij} \\
&+ p_1.N.p_3p_3.V.p_3\varepsilon_2{}^{a_2}(\varepsilon_3^A)^{ij} - 2p_2.p_3p_3.V.\varepsilon_2p_3^{a_2}(\varepsilon_3^A)^{ij} - p_3.V.\varepsilon_2p_3.V.p_3p_3^{a_2}(\varepsilon_3^A)^{ij} - 2(p_2.p_3)^2\varepsilon_2{}^{a_2}(\varepsilon_3^A)^{ij}] \\
&+ \frac{2\pi^2}{3}(F_1)^{a_0}{}_{ijk}p_3.V.p_3\varepsilon_2{}^{a_2}p_3^ip_3^{a_1}(\varepsilon_3^A)^{jk}. \qquad (32)
\end{aligned}$$

They satisfy the Ward identity corresponding to the gauge transformations.

The above amplitude is in terms of the RR field strength. Since they are contact terms, one should be able to rewrite them in terms of the field strengths $H = dB$ and $\mathcal{F} = dA$ as well. To this end, we first write $p_3^{a_1}$ in the last line in terms of $-p_2^{a_1} - p_1^{a_1}$. The term in the last line corresponding to $-p_2^{a_1}$ can easily be written in terms of the field strengths. The term in the last line corresponding to $-p_1^{a_1}$ can be combined with the terms in the first bracket to write them in terms of the field strengths. After some algebra, one can write the above amplitude as

$$\begin{aligned}
A_c(\alpha'^2) \sim &\frac{\pi^2}{3}(F_1)^{a_0a_1}{}_{ij}p_1.V.p_1(\mathcal{F}_2.V.H_3)^{a_2ij} \\
&- \frac{\pi^2}{9}(F_1)^{a_0}{}_{ijk}p_3.V.p_3\mathcal{F}_2^{a_1a_2}H_3^{ijk}. \qquad (33)
\end{aligned}$$

It is interesting that the complicated amplitude (32) in terms of the polarization tensors has such a simple form in terms of the corresponding field strengths.

By transforming the above contact terms in the momentum space to the coordinate space, one finds the following $\alpha'^2$ couplings on the world volume of the $D_p$-brane:

$$\begin{aligned}
S \supset \pi^2\alpha'^2T_p \int d^{p+1}x\epsilon^{a_0\cdots a_p} &\\
\times \bigg(&\frac{1}{2!3!(p-1)!}F^{(p+2)}_{a_2\cdots a_pikj}H^{ijk,a}{}_a(2\pi\alpha'\mathcal{F}_{a_0a_1}) \\
&+ \frac{1}{2!p!}F^{(p+2)}_{a_1\cdots a_pij,a}{}^aH^{bij}(2\pi\alpha'\mathcal{F}_{a_0b})\bigg), \qquad (34)
\end{aligned}$$

where the RR field strength is $F^{(n)} = dC^{(n-1)}$. The first term has been found in [21] by analyzing the amplitude for the RR $(p+1)$ form with three transverse indices. The other coupling is a new coupling that should be added to the $D_p$-brane action at order $\alpha'^2$.

Extending the above couplings to be covariant under the coordinate transformations and invariant under the RR and NSNS gauge transformations, one finds the following nonlinear couplings at order $\alpha'^2$:

$$\begin{aligned}
S \supset \pi^2\alpha'^2T_p \int d^{p+1}x\epsilon^{a_0\cdots a_p} &\\
\times \bigg(&\frac{1}{2!3!(p-1)!}\tilde{F}^{(p+2)}_{a_2\cdots a_pikj}H^{ijk;a}{}_a\tilde{B}_{a_0a_1} \\
&+ \frac{1}{4}\frac{1}{2!p!}\tilde{F}^{(p+2)}_{a_1\cdots a_pij;a}{}^aH^{bij}\tilde{B}_{a_0b}\bigg), \qquad (35)
\end{aligned}$$

where the nonlinear RR field strength and the NSNS gauge invariant $\tilde{B}$ are

$$\tilde{F}^{(n)} = dC^{(n-1)} + H \wedge C^{(n-3)}; \qquad \tilde{B} = B + 2\pi\alpha' \mathcal{F}. \quad (36)$$

The action (35) predicts, among other things, the couplings with structure $C^{(p+1)}HB$. These couplings can be confirmed by the S-matrix elements of one RR and two NSNS states, which we find them in the next section.

There should be another term in the action (35) in which the RR field strength is $\tilde{F}^{(p+2)}_{a_0\cdots a_p i}$. This term is resulted from the low energy expansion of amplitudes (26) and (27). The explicit form of the integrals $\mathcal{Q}_3$, $\mathcal{Q}_4$, which appear in these amplitude, can be found by the string-theory calculation of the amplitudes (26) and (27) in which we are not interested in this paper.

## IV. THREE CLOSED-STRING AMPLITUDES

The disk-level S-matrix element of one RR $(p-3)$ form and two NSNS states has been calculated in [34,38–41]. The amplitude is nonzero when the RR polarization has two, one, and zero transverse indices. The T-dual multiplets corresponding to the first two cases have been found in [21]. They satisfy the Ward identity corresponding to the T duality and the NSNS gauge transformations. However, since the RR $(p-3)$ form with no transverse index was not considered in this study, the multiplets found in [21] are not invariant under the RR gauge transformations. In this section, we are going to consider the RR $(p-3)$ form with no transverse index. The requirement that this amplitude should satisfy the Ward identity corresponding to the T duality and the NSNS gauge transformation allows us to find various S-matrix elements of one RR and two NSNS states.

### A. RR $(p-3)$ form with no transverse index

When the RR $(p-3)$ form has no transverse index, the amplitude is nonzero for the case that the NSNS polarizations are both antisymmetric or symmetric [34,38,39]. The amplitude can be written as two parts. One part is in terms of the RR field strength $(F_1)^{a_0 a_1} = p_1^{a_0}\varepsilon_1^{a_1} - p_1^{a_1}\varepsilon_1^{a_0}$, and the other part is in terms of the RR factor $(f_1)^{i a_0} = p_1^i \varepsilon_1^{a_0}$,

$$\mathbf{A}_0 = A_0(F_1) + \mathcal{A}_0(f_1). \quad (37)$$

The subscript 0 refers to the number of transverse indices of the RR potential. For simplicity, we consider the case that $p = 4$.

The second part in (37) which is nonzero when both NSNS states are antisymmetric, includes the following terms [38] (see Eq. (34) in [38][2]):

$$\begin{aligned}\mathcal{A}_0(f_1) \sim \frac{1}{2}(f_1)_i^{a_4}\bigg[&\frac{1}{2}p_3\cdot V\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1 a_3}(p_2^i p_2^{a_0}\mathcal{I}_3 + p_3^i p_3^{a_0}\mathcal{I}_2) + \frac{1}{2}p_3\cdot N\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1 a_3}(p_3^i p_2^{a_0}\mathcal{I}_3 + p_2^i p_3^{a_0}\mathcal{I}_2)\\ &- 2p_3^i p_3^{a_0} p_2\cdot V\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1 a_3}\mathcal{I}_7 + p_3^i p_3^{a_0} p_1\cdot N\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1 a_3}\mathcal{I}_1 + (2\leftrightarrow 3)\\ &+\frac{1}{4}\varepsilon_2^{a_0 a_1}\varepsilon_3^{a_2 a_3}\left(p_2^i p_3\cdot V\cdot p_3\mathcal{I}_4 + \frac{1}{2}p_3^i p_2\cdot V\cdot p_3\mathcal{I}_2 - \frac{1}{2}p_3^i p_2\cdot N\cdot p_3\mathcal{I}_3\right)\bigg]. \end{aligned} \quad (38)$$

Our notation for the first $(2\leftrightarrow 3)$ in the above amplitude and in all other amplitudes in this paper is that the expressions from the beginning up to that point, including the overall factor, should be interchanged under $2\leftrightarrow 3$. In above equation, $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$, and $\mathcal{I}_7$ are the integrals that represent the open- and closed-string channels. The explicit form of these integrals is given in [38,40]. They satisfy the relation [40]

$$\begin{aligned}&-2p_1\cdot N\cdot p_2\mathcal{I}_1 + 2p_2\cdot V\cdot p_2\mathcal{I}_7 + p_2\cdot N\cdot p_3\mathcal{I}_3\\ &- p_2\cdot V\cdot p_3\mathcal{I}_2 = 0,\end{aligned} \quad (39)$$

and similar relation under the interchange of $(2\leftrightarrow 3)$. The symmetries of the integrals under the interchange of $(2\leftrightarrow 3)$ are such that $\mathcal{I}$ is invariant, $\mathcal{I}_2\leftrightarrow\mathcal{I}_3$, and $\mathcal{I}_4\leftrightarrow\mathcal{I}_7$. Using the above relations, one finds that the amplitude (38) satisfies the Ward identity corresponding to the antisymmetric NSNS gauge transformation [40]. If one adds to (38) the amplitude of the RR $(p-3)$ form with one transverse index, then the RR factor in the amplitude (38) is extended to the RR field strength $(F_1)_i^{a_4}$; the amplitude $\mathcal{A}_0(C)$ is extended to $\mathcal{A}_1(F_i)$ where the subscript 1 in the latter amplitude refers to the number of the transverse index of the RR field strength. As a result, it would satisfy the RR Ward identity. The amplitude of the RR $(p-3)$ form with one transverse index has also some terms that are combined with the amplitude of the RR $(p-3)$ form with two transverse indices to become RR invariant [21]. The amplitudes in terms of the RR field strength, however, do not satisfy the NSNS Ward identity unless one considers the first part in (37).

The first part of (37) which is nonzero when the NSNS states are both symmetric or antisymmetric, is given as [38]

[2]Note that there is a typo in the last line of Eq. (34): the overall factor 2 should be 1/4.

$$\begin{aligned}
A_0 \sim -\frac{1}{4}(F_1)^{a_0a_4}\bigg[&2p_3^{a_1}(-p_2\cdot N\cdot p_3(\varepsilon_2^S\cdot N\cdot\varepsilon_3^S)^{a_2a_3}+p_2\cdot V\cdot p_3(\varepsilon_2^S\cdot V\cdot\varepsilon_3^S)^{a_2a_3}+(p_3\cdot N\cdot\varepsilon_2^S)^{a_2}(p_2\cdot N\cdot\varepsilon_3^S)^{a_3}\\
&-(p_3\cdot V\cdot\varepsilon_2^S)^{a_2}(p_2\cdot V\cdot\varepsilon_3^S)^{a_3})\mathcal{J}-2p_2\cdot V\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}p_3\cdot V\cdot p_3\mathcal{J}_3+p_3^{a_1}(p_2\cdot N\cdot p_3(\varepsilon_2\cdot V\cdot\varepsilon_3)^{a_2a_3}\\
&-p_2\cdot V\cdot p_3(\varepsilon_2\cdot N\cdot\varepsilon_3)^{a_2a_3}-\varepsilon_3^{a_2a_3}p_3\cdot V\cdot\varepsilon_2\cdot N\cdot p_3)\mathcal{J}\\
&+\bigg(p_3\cdot V\cdot\varepsilon_2\cdot V\cdot p_2\mathcal{J}_1-\frac{1}{2}p_3\cdot V\cdot\varepsilon_2\cdot N\cdot p_1\mathcal{I}_3+p_2\cdot V\cdot\varepsilon_2\cdot N\cdot p_3\mathcal{J}_2-p_3\cdot V\cdot\varepsilon_2\cdot N\cdot p_3\mathcal{J}_5\\
&+\frac{1}{2}p_3\cdot N\cdot\varepsilon_2\cdot N\cdot p_1\mathcal{I}_2\bigg)p_3^{a_1}\varepsilon_3^{a_2a_3}-p_3^{a_1}p_1\cdot N\cdot\varepsilon_2^{a_2}p_2\cdot V\cdot\varepsilon_3^{a_3}\mathcal{I}_3-p_3^{a_1}p_1\cdot N\cdot\varepsilon_2^{a_2}p_1\cdot N\cdot\varepsilon_3^{a_3}\mathcal{I}_1\\
&+p_3^{a_1}p_1\cdot N\cdot\varepsilon_2^{a_2}p_2\cdot N\cdot\varepsilon_3^{a_3}\mathcal{I}_2+2p_3^{a_1}p_3\cdot V\cdot\varepsilon_2^{a_2}p_2\cdot N\cdot\varepsilon_3^{a_3}\mathcal{J}_5+4p_3^{a_1}p_1\cdot N\cdot\varepsilon_2^{a_2}p_3\cdot V\cdot\varepsilon_3^{a_3}\mathcal{I}_4\\
&-2p_3^{a_1}p_2\cdot V\cdot\varepsilon_2^{a_2}p_2\cdot N\cdot\varepsilon_3^{a_3}\mathcal{J}_2+2p_3^{a_1}p_2\cdot V\cdot\varepsilon_2^{a_2}p_2\cdot V\cdot\varepsilon_3^{a_3}\mathcal{J}_1-4p_3^{a_1}p_2\cdot V\cdot\varepsilon_2^{a_2}p_3\cdot V\cdot\varepsilon_3^{a_3}\mathcal{J}_3\\
&+\frac{1}{2}p_3\cdot N\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}(\mathcal{J}_{15}p_2\cdot N\cdot p_3-(2\mathcal{J}-\mathcal{J}_{16})p_2\cdot V\cdot p_3)+p_1\cdot N\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}p_3\cdot V\cdot p_3\mathcal{I}_4\\
&+\frac{1}{4}p_3\cdot V\cdot\varepsilon_2^{a_2}\varepsilon_3^{a_1a_3}(2p_2\cdot V\cdot p_2\mathcal{J}_1+2p_3\cdot V\cdot p_3\mathcal{J}_4-4p_2\cdot N\cdot p_3\mathcal{J})\bigg]+(2\leftrightarrow 3),
\end{aligned}\tag{40}$$

where $\mathcal{J}_{15}=\mathcal{J}_{13}-\mathcal{J}_{14}$ and $\mathcal{J}_{16}=\mathcal{J}_{13}+\mathcal{J}_{14}$. In above amplitude, $\mathcal{J}$, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$, $\mathcal{J}_5$, $\mathcal{J}_{12}$, $\mathcal{J}_{13}$, and $\mathcal{J}_{14}$ are some integrals in which the explicit forms are given in [38]. The symmetries of the integrals under the interchange of $(2\leftrightarrow 3)$ are such that $\mathcal{J}$, $\mathcal{J}_3$, $\mathcal{J}_{13}$, $\mathcal{J}_{14}$ are invariant, $\mathcal{J}_1\leftrightarrow\mathcal{J}_4$, $\mathcal{J}_2\leftrightarrow\mathcal{J}_{12}$, and $\mathcal{J}_5\leftrightarrow-\mathcal{J}_5$. The terms in the first two lines, which are proportional to the integral $\mathcal{J}$, are the amplitude for the symmetric NSNS polarization tensors. This part satisfies the Ward identity corresponding to the NSNS gauge transformation. In fact, the low energy limit of this part at order $\alpha'^2$ produces only contact terms that are the couplings $C^{(p-3)}\wedge R\wedge R$ in the momentum space [34,38]. The other terms are the amplitude for the antisymmetric NSNS polarization tensors. The combination of these terms and the terms considered in the previous paragraph satisfies the Ward identity corresponding to the NSNS gauge transformation provided that the integrals satisfy the relations

$$\begin{aligned}
-2\mathcal{I}_2p_1\cdot N\cdot p_2+\mathcal{J}_{15}p_2\cdot N\cdot p_3+2\mathcal{J}_2p_2\cdot V\cdot p_2+(-4\mathcal{J}+\mathcal{J}_{16}-2\mathcal{J}_5)p_2\cdot V\cdot p_3&=0\\
2\mathcal{I}_3p_1\cdot N\cdot p_2-2\mathcal{J}_1p_2\cdot V\cdot p_2+\mathcal{J}_{15}p_2\cdot V\cdot p_3+(\mathcal{J}_{16}+2\mathcal{J}_5)p_2\cdot N\cdot p_3&=0\\
-2\mathcal{I}_4p_1\cdot N\cdot p_2+\mathcal{J}_{12}p_2\cdot N\cdot p_3+2\mathcal{J}_3p_2\cdot V\cdot p_2-\mathcal{J}_4p_2\cdot V\cdot p_3&=0
\end{aligned}\tag{41}$$

and similar relations under the interchange of $(2\leftrightarrow 3)$. One may use these relations and the relation in (39) to eliminate eight integrals and then one left with six independent integrals. We prefer to work with the 14 integrals and the eight constraints.

The amplitude (37) does not satisfy the T-dual Ward identity. When the $y$ index is carried by the RR polarization tensor, the amplitude is invariant under the linear T duality. However, when the $y$ index is one of the indices of the NSNS polarization tensors, one finds that the amplitude (37) is not invariant under the T duality. The invariance requires the amplitude for the RR $(p-1)$ form with one or two transverse indices. In the next subsection, we are going to find such amplitudes by constraining the amplitude to satisfy the T-dual Ward identity, as we have done for the case of two closed- and one open-string amplitude in the previous section.

### B. RR $(p-1)$ form

The amplitude for the RR $(p-1)$ form is nonzero when the RR potential has three, two, one, and zero transverse indices. When the RR potential has one transverse index, the amplitude can be found by applying the T-dual Ward identity on the amplitude (37) in which the RR potential has zero transverse index. The T-dual completion of the amplitude (37) can be written as

$$\mathbf{A}_1=A_1(f_1)+\mathcal{A}_1(f_1),\tag{42}$$

where $\mathcal{A}_1(f_1)$ is the T-dual completion of the amplitude $\mathcal{A}_0(f_1)$ in (38) and $A_1(f_1)$ is the T-dual completion of the amplitude $A_0(F_1)$ in (40). The subscript 1 refers to the number of the transverse index of the RR potential. The T-dual Ward identity on the amplitude (38) requires the following amplitude for $\mathcal{A}_1(f_1)$:

$$\mathcal{A}_1 \sim \frac{1}{4}(f_1)_{ij}{}^{a_3}[((p_2\cdot V\cdot \varepsilon_3^S)^i(\varepsilon_2^A)^{a_1a_2} - 2(p_3\cdot V\cdot \varepsilon_2^A)^{a_2}(\varepsilon_3^S)^{a_1i})(p_2^jp_2^{a_0}\mathcal{I}_3 + p_3^jp_3^{a_0}\mathcal{I}_2)$$
$$+ ((p_2\cdot N\cdot \varepsilon_3^S)^i(\varepsilon_2^A)^{a_1a_2} - 2(p_3\cdot N\cdot \varepsilon_2^A)^{a_2}(\varepsilon_3^S)^{a_1i})(p_3^jp_2^{a_0}\mathcal{I}_3 + p_2^jp_3^{a_0}\mathcal{I}_2)$$
$$- 4(p_2^jp_2^{a_0}(p_3\cdot V\cdot \varepsilon_3^S)^i(\varepsilon_2^A)^{a_1a_2}\mathcal{I}_4 - 2p_3^jp_3^{a_0}(p_2\cdot V\cdot \varepsilon_2^A)^{a_2}(\varepsilon_3^S)^{a_1i}\mathcal{I}_7)$$
$$+ 2(p_2^jp_2^{a_0}(p_1\cdot N\cdot \varepsilon_3^S)^i(\varepsilon_2^A)^{a_1a_2} - 2p_3^jp_3^{a_0}(p_1\cdot N\cdot \varepsilon_2^A)^{a_2}(\varepsilon_3^S)^{a_1i})\mathcal{I}_1 + (2\leftrightarrow 3)$$
$$- ((\varepsilon_3^S)^{a_0i}(\varepsilon_2^A)^{a_1a_2} + (2\leftrightarrow 3))(2p_2^jp_3\cdot V\cdot p_3\mathcal{I}_4 + p_3^j(p_2\cdot V\cdot p_3\mathcal{I}_2 - p_2\cdot N\cdot p_3\mathcal{I}_3))], \tag{43}$$

where the second $(2\leftrightarrow 3)$ is only for the term in the parentheses. The RR factor $(f_1)_{ij}{}^{a_3} = p_{1i}\varepsilon_{1j}{}^{a_3} - p_{1j}\varepsilon_{1i}{}^{a_3}$. The above amplitude does not satisfy the Ward identity corresponding to the NSNS and the RR gauge transformations.

The asymmetry under the NSNS transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p-1)$ form, which have $(f_1)^{ija_3}p_{2i}$ or $(f_1)^{ija_3}p_{3i}$. Since all terms in (38) have either $p_{2i}$ or $p_{3i}$, one finds that the missing terms should have the factor $(f_1)^{ija_3}p_{2i}p_{3j}$. Considering all such terms that have one momentum and the two NSNS polarization tensors, with unknown coefficients and imposing the condition that when they combine with the amplitude (43) they should satisfy the NSNS Ward identity, one finds the following result:

$$\mathcal{A}'_1 \sim \frac{1}{4}p_2^ip_3^j(f_1)_{ij}{}^{a_3}[2p_3^{a_0}\mathcal{I}_4(\varepsilon_2^A)^{a_1a_2}\,\mathrm{Tr}[\varepsilon_3^S\cdot V] + \mathcal{I}_2(p_2\cdot N\cdot \varepsilon_3^S)^{a_2}(\varepsilon_2^A)^{a_0a_1} - \mathcal{I}_3(p_2\cdot V\cdot \varepsilon_3^S)^{a_2}(\varepsilon_2^A)^{a_0a_1} - (2\leftrightarrow 3)$$
$$- 2\mathcal{I}_1(p_1\cdot N\cdot \varepsilon_3^S)^{a_2}(\varepsilon_2^A)^{a_0a_1} + 2p_3^{a_0}\mathcal{G}(\varepsilon_2^A\cdot V\cdot \varepsilon_3^S)^{a_1a_2}\mathcal{I}_3 + 2p_3^{a_0}\mathcal{G}(\varepsilon_3^A\cdot V\cdot \varepsilon_2^S)^{a_1a_2}\mathcal{I}_2], \tag{44}$$

where the operator $\mathcal{G}$, which appears in the above amplitude and in the subsequent amplitudes, is defined as

$$\mathcal{G}(\varepsilon_n^A\cdot V\cdot \varepsilon_m^S)^{\mu\nu} \to (\varepsilon_n^A\cdot V\cdot \varepsilon_m^S)^{\mu\nu} - (\varepsilon_n^S\cdot N\cdot \varepsilon_m^A)^{\mu\nu}$$
$$\mathcal{G}(\varepsilon_n^S\cdot V\cdot \varepsilon_m^A)^{\mu\nu} \to (\varepsilon_n^S\cdot V\cdot \varepsilon_m^A)^{\mu\nu} - (\varepsilon_n^A\cdot N\cdot \varepsilon_m^S)^{\mu\nu}$$
$$\mathcal{G}(\varepsilon_n^A\cdot N\cdot \varepsilon_m^S)^{\mu\nu} \to (\varepsilon_n^A\cdot N\cdot \varepsilon_m^S)^{\mu\nu} - (\varepsilon_n^S\cdot V\cdot \varepsilon_m^A)^{\mu\nu}$$
$$\mathcal{G}(\varepsilon_n^S\cdot N\cdot \varepsilon_m^A)^{\mu\nu} \to (\varepsilon_n^S\cdot N\cdot \varepsilon_m^A)^{\mu\nu} - (\varepsilon_n^A\cdot V\cdot \varepsilon_m^S)^{\mu\nu}, \tag{45}$$

where $n$, $m$ are the particle labels of the polarization tensors. The right-hand side expressions are invariant under the linear T duality when $\mu, \nu \neq y$. One may multiply each term with a momentum, e.g.,

$$\mathcal{G}(p_3\cdot \varepsilon_2^A\cdot V\cdot \varepsilon_3^S)^{\mu} = (p_3\cdot \varepsilon_2^A\cdot V\cdot \varepsilon_3^S)^{\mu} - (p_3\cdot \varepsilon_2^S\cdot N\cdot \varepsilon_3^A)^{\mu}. \tag{46}$$

The result is still invariant under the linear T duality.

The combination of the amplitudes (43) and (44) satisfies the NSNS Ward identity; however, they do not satisfy the RR Ward identity. If one includes the amplitude of the RR $(p-1)$ form with two transverse indices, which has been found in [21], then the RR factor in the above amplitude is extended to the RR field strength $(F_1)_{ij}{}^{a_3} = p_{1i}\varepsilon_{1j}{}^{a_3} - p_{1j}\varepsilon_{1i}{}^{a_3} + p_1^{a_3}\varepsilon_{1ij}$, the amplitudes $\mathcal{A}_1(C_i) + \mathcal{A}'_1(C_i)$ is extended to $\mathcal{A}_2(F_{ij}) + \mathcal{A}'_2(F_{ij})$. The amplitude of the RR $(p-1)$ form with two transverse indices has also some terms that become RR gauge invariant after including the amplitude of the RR $(p-1)$ form with three transverse indices [21].

The amplitude $A_1(f_1)$ in (42) can be found from imposing the invariance of the amplitude (40) under the linear T duality when the Killing index $y$ is carried by the NSNS polarization tensors in (40). The result is the following:

$$A_1 \sim -\frac{1}{2}(f_1)_i{}^{a_0a_3}\Bigg[p_3^{a_1}(-p_2\cdot N\cdot p_3\mathcal{G}(\varepsilon_2^S\cdot V\cdot \varepsilon_3^A)^{ia_2} - p_2\cdot V\cdot p_3\mathcal{G}(\varepsilon_2^A\cdot V\cdot \varepsilon_3^S)^{ia_2}$$
$$+ (p_3\cdot N\cdot \varepsilon_2^A)^i(p_2\cdot N\cdot \varepsilon_3^S)^{a_2} - (p_3\cdot V\cdot \varepsilon_2^A)^i(p_2\cdot V\cdot \varepsilon_3^S)^{a_2})\mathcal{J}$$
$$- (\varepsilon_3^S)^{a_2i}\Bigg(p_3^{a_1}\Bigg(p_3\cdot V\cdot \varepsilon_2^A\cdot V\cdot p_2\mathcal{J}_1 - \frac{1}{2}p_3\cdot V\cdot \varepsilon_2^A\cdot N\cdot p_1\mathcal{I}_3$$
$$+ p_2\cdot V\cdot \varepsilon_2^A\cdot N\cdot p_3\mathcal{J}_2 - p_3\cdot V\cdot \varepsilon_2^A\cdot N\cdot p_3(\mathcal{J}+\mathcal{J}_5) + \frac{1}{2}p_3\cdot N\cdot \varepsilon_2^A\cdot N\cdot p_1\mathcal{I}_2\Bigg)$$

$$
\begin{aligned}
&+\frac{1}{2}(p_3\cdot V\cdot\varepsilon_2^A)^{a_1}(p_2\cdot V\cdot p_2\mathcal{J}_1+p_3\cdot V\cdot p_3\mathcal{J}_4-2p_2\cdot N\cdot p_3\mathcal{J})-2(p_2\cdot V\cdot\varepsilon_2^A)^{a_1}p_3\cdot V\cdot p_3\mathcal{J}_3\\
&+\frac{1}{2}(p_3\cdot N\cdot\varepsilon_2^A)^{a_1}(p_2\cdot N\cdot p_3\mathcal{J}_{15}+p_2\cdot V\cdot p_3(\mathcal{J}_{16}-2\mathcal{J}))+(p_1\cdot N\cdot\varepsilon_2^A)^{a_1}p_3\cdot V\cdot p_3\mathcal{I}_4\Big)\\
&+(p_1\cdot N\cdot\varepsilon_3^S)^i\bigg(p_3^{a_1}\bigg((p_1\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_1+\frac{1}{2}(p_3\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_2-\frac{1}{2}(p_3\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_3-2(p_2\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_7\bigg)\\
&-\frac{1}{2}(\varepsilon_2^A)^{a_1a_2}p_2\cdot V\cdot p_2\mathcal{I}_7\bigg)+(p_2\cdot N\cdot\varepsilon_3^S)^i\bigg(p_3^{a_1}\bigg(-\frac{1}{2}(p_1\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_2-(p_3\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_5+(p_2\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_2\bigg)\\
&-\frac{1}{4}(\varepsilon_2^A)^{a_1a_2}(p_2\cdot N\cdot p_3\mathcal{J}_{15}+p_2\cdot V\cdot p_3(\mathcal{J}_{16}-2\mathcal{J}))\bigg)\\
&+(p_2\cdot V\cdot\varepsilon_3^S)^i\bigg(p_3^{a_1}\bigg(\frac{1}{2}(p_1\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_3-(p_2\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_1+(p_3\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_5\bigg)\\
&-\frac{1}{4}(\varepsilon_2^A)^{a_1a_2}(p_2\cdot V\cdot p_2\mathcal{J}_1+p_3\cdot V\cdot p_3\mathcal{J}_4-2p_2\cdot N\cdot p_3\mathcal{J})\bigg)+(p_3\cdot V\cdot\varepsilon_3^S)^i(p_3^{a_1}(-2(p_1\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_4\\
&+4(p_2\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_3+(p_3\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_{12}-(p_3\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_4)+(\varepsilon_2^A)^{a_1a_2}p_2\cdot V\cdot p_2\mathcal{J}_3)\Big]+(2\leftrightarrow 3),
\end{aligned}
\tag{47}
$$

where the RR factor is $(f_1)_i{}^{a_0a_3}=p_1{}^{a_3}\varepsilon_{1i}{}^{a_0}-p_1{}^{a_0}\varepsilon_{1i}{}^{a_3}$. The above amplitude is invariant under the linear T duality when the world volume $y$ index is carried by the RR potential. Note that the two NSNS polarization tensors in the first line contract with each other in such a way that they are invariant under the T duality when $a_2\neq y$. The above amplitude does not satisfy the Ward identity corresponding to the NSNS and the RR gauge transformations.

The asymmetry under the NSNS gauge transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p-1)$ form, which have $(f_1)^{ia_0a_3}p_{2i}$ or $(f_1)^{ia_0a_3}p_{3i}$. Since no term in (40) has $p_{2i}$ or $p_{3i}$, one finds that there are two types of missing terms. One type is the terms that have $(f_1)^{ia_0a_3}p_{2i}$, and the other type is the terms that have $(f_1)^{ia_0a_3}p_{3i}$. Therefore, the missing terms can be separated as $A'_1=A'_{12}+A'_{13}$. Considering all such terms that have two momentum and the two NSNS polarization tensors, with unknown coefficients and imposing the condition that when they combine with the amplitude (47) they should satisfy the NSNS Ward identity, one finds the following result for the terms of the first type:

$$
\begin{aligned}
A'_{12}\sim&-\frac{1}{4}(f_1)^{ia_0a_3}p_{2i}\Big[p_3^{a_1}(\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^A)^{a_2}(\mathcal{J}_{16}-2\mathcal{J}_5)+2\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_2\\
&+\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{a_2}(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5)+4\mathcal{G}(p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_{12}\\
&+(\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^A)^{a_2}+\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{a_2})\mathcal{J}_{15}-2\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{a_2}\mathcal{I}_3\\
&-4\mathcal{G}(p_3\cdot V\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^A)^{a_2}\mathcal{J}_4-2p_3\cdot V\cdot p_3\mathcal{G}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^S)^{a_1a_2}\mathcal{J}_{12})\\
&+\frac{1}{2}(\varepsilon_2^A)^{a_1a_2}(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16}-2\mathcal{J}_5)-4\mathcal{J}_4p_3\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2+4\mathcal{I}_2p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_1\\
&+p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5)+4p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{J}_{12}-2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_3\\
&+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2\mathcal{J}_{15}+2(-2\mathcal{I}_4p_1\cdot N\cdot p_3+\mathcal{J}_4p_2\cdot N\cdot p_3-\mathcal{J}_{12}p_2\cdot V\cdot p_3)\mathrm{Tr}[\varepsilon_3^S\cdot V])\\
&+(p_3\cdot N\cdot\varepsilon_2^A)^{a_2}(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_4-2\mathcal{I}_2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}-(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}-2\mathcal{J}_5)\\
&-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15})-(p_3\cdot V\cdot\varepsilon_2^A)^{a_2}(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_{12}+(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15}\\
&+(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5))\Big]+(2\leftrightarrow 3).
\end{aligned}
\tag{48}
$$

The terms of the second type, which have the RR factor $(f_1)^{ia_0a_3}p_{3i}$, are the following:

$$\begin{aligned}A'_{13} \sim &-\frac{1}{4}(f_1)^{ia_0a_3}p_{3i}\Big[p_3^{a_1}(\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^A)^{a_2}(\mathcal{J}_{16}-2\mathcal{J}_5)+\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{a_2}\\&\times(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5)+(\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{a_2}+\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^A)^{a_2})\mathcal{J}_{15})\\&+\frac{1}{2}(\varepsilon_2^A)^{a_1a_2}(4(p_1\cdot N\cdot p_2\mathcal{I}_4-p_2\cdot V\cdot p_2\mathcal{J}_3)\mathrm{Tr}[\varepsilon_3^S\cdot V]+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16}-2\mathcal{J})\\&+(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2+p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2)\mathcal{J}_{15}+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_2)\\&+(p_3\cdot V\cdot\varepsilon_2^A)^{a_2}(2p_3^{a_1}\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_4-2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_2-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15}\\&-(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}-4\mathcal{J}-2\mathcal{J}_5))+2(p_1\cdot N\cdot\varepsilon_2^A)^{a_2}(2p_3^{a_1}\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{I}_4\\&-2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_1+(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_2-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_3)-4(p_2\cdot V\cdot\varepsilon_2^A)^{a_2}\\&\times((p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_2-2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_7-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_1+2p_3^{a_1}\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_3)\\&-(p_3\cdot N\cdot\varepsilon_2^A)^{a_2}((p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15}+(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}+2\mathcal{J}_5)+2p_3^{a_1}\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_{12})\\&-2p_2\cdot V\cdot p_2\mathcal{G}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^S)^{a_1a_2}\mathcal{J}_1+(2\leftrightarrow3)+2p_1\cdot N\cdot p_2(\mathcal{G}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^S)^{a_1a_2}\mathcal{I}_3+(2\leftrightarrow3))\Big].\end{aligned}\tag{49}$$

The coefficient of the first term in the second line of (48) and the coefficient of the last term in the first line of (49) are not fixed by the condition that the sum of the above amplitudes and the amplitude (47) to be invariant under the NSNS gauge transformation. That coefficients are fixed in the next section by using the appropriate Ward identity. In above equations, we have written the final result.

The sum of the amplitudes (47), (48), and (49) satisfies the NSNS Ward identities. However, it does not satisfy the RR Ward identity because the RR factor $(f_1)^{ia_0a_3}$ is not the RR field strength. It can easily be extended to the RR invariant amplitude by extending the RR factor to the RR field strength $(F_1)^{ia_0a_3}=p_1^i\varepsilon_1^{a_0a_3}+p_1^{a_3}\varepsilon_1^{ia_0}-p_1^{a_0}\varepsilon_1^{ia_3}$. The amplitude corresponding to the first term does not satisfy the NSNS Ward identity. So one has to add the amplitude of the RR $(p-1)$ form with no transverse index. The RR invariance requires this amplitude to be in terms of the RR field strength $(F_1)^{a_0a_1a_3}$. One may consider all independent terms containing $(F_1)^{a_0a_1a_3}$, which are 196 terms, and then impose the condition that when they combine with the above nongauge-invariant amplitude, the combination satisfies the NSNS Ward identity. We have done this calculation and have found that the NSNS Ward identity fixes 171 unknown integrals, and the 25 remaining integrals appear in some constraint equations, like the constraint in (25). However, to analyze the amplitude at low energy, one needs the explicit form of the integrals that may be found by performing the string-theory calculations to find the amplitude of the RR $(p-1)$ form, which have the structure $(F_1)^{a_0a_1a_3}(\cdots)^{a_2}$. We leave the details of these calculations for the future work.

### C. RR $(p+1)$ form

The amplitude for the RR $(p+1)$ form is nonzero when the RR potential has four, three, two, and one transverse indices. When the RR potential has two transverse indices, the amplitude can be found by applying the T-dual Ward identity on the amplitude (42),

$$\mathbf{A}_2 = A_2(f_1)+\mathcal{A}_2(f_1).\tag{50}$$

The amplitude $\mathcal{A}_2(f_1)$ is the T-dual completion of the amplitude $\mathcal{A}_1(f_1)$ in (43), and $A_2(f_1)$ is the T-dual completion of the amplitude $A_1(f_1)$ in (47). The amplitude $\mathcal{A}_2(f_1)$ is

$$\begin{aligned}\mathcal{A}_2 \sim &-\frac{1}{4}(f_1)_{ijk}{}^{a_2}\Big[(2(p_3\cdot V\cdot\varepsilon_2^S)^i(\varepsilon_3^S)^{a_1j}+(p_3\cdot V\cdot\varepsilon_2^A)^{a_1}(\varepsilon_3^A)^{ij})(p_2^kp_2^{a_0}\mathcal{I}_3+p_3^kp_3^{a_0}\mathcal{I}_2)+(2(p_3\cdot N\cdot\varepsilon_2^S)^i(\varepsilon_3^S)^{a_1j}\\&+(p_3\cdot N\cdot\varepsilon_2^A)^{a_1}(\varepsilon_3^A)^{ij})(p_3^kp_2^{a_0}\mathcal{I}_3+p_2^kp_3^{a_0}\mathcal{I}_2)-4p_3^kp_3^{a_0}(2(p_2\cdot V\cdot\varepsilon_2^S)^i(\varepsilon_3^S)^{a_1j}+(p_2\cdot V\cdot\varepsilon_2^A)^{a_1}(\varepsilon_3^A)^{ij})\mathcal{I}_7\\&+2p_3^kp_3^{a_0}(2(p_1\cdot N\cdot\varepsilon_2^S)^i(\varepsilon_3^S)^{a_1j}+(p_1\cdot N\cdot\varepsilon_2^A)^{a_1}(\varepsilon_3^A)^{ij})\mathcal{I}_1+(2\leftrightarrow3)\\&+\frac{1}{2}((\varepsilon_2^A)^{ij}(\varepsilon_3^A)^{a_0a_1}+2(\varepsilon_2^S)^{a_0i}(\varepsilon_3^S)^{a_1j}+(2\leftrightarrow3))\\&\times(2p_2^kp_3\cdot V\cdot p_3\mathcal{I}_4+p_3^kp_2\cdot V\cdot p_3\mathcal{I}_2-p_3^kp_2\cdot N\cdot p_3\mathcal{I}_3)\Big],\end{aligned}\tag{51}$$

where the RR factor is $(f_1)^{ijka_2} = p_1^i \varepsilon_1^{jka_2} + p_1^k \varepsilon_1^{ija_2} + p_1^j \varepsilon_1^{kia_2}$. For simplicity, we have considered the amplitude for $p=2$. The amplitudes in the previous section are nonzero when one of the NSNS polarization tensors is symmetric and the other one is antisymmetric. The amplitudes in this section, which are the T-dual completion of the amplitude in the previous section, are then nonzero when both tensors are symmetric or both are antisymmetric. The above amplitude does not satisfy the Ward identity corresponding to the NSNS and the RR gauge transformations.

The asymmetry under the NSNS transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p+1)$, which have $(f_1)^{ijka_2} p_{2i}$ or $(f_1)^{ijka_2} p_{3i}$. Since all terms in (43) have either $p_{2i}$ or $p_{3i}$, one finds that the missing terms corresponding to the above amplitude should have the factor $(f_1)^{ijka_2} p_{2i} p_{3j}$. Considering all such terms that have one momentum and the two NSNS polarization tensors, with unknown coefficients and imposing the condition that when they combine with the amplitude (51) they should satisfy the NSNS Ward identity, one finds the following result:

$$\begin{aligned}\mathcal{A}'_2 \sim{}& \frac{1}{4} p_2^i p_3^j (f_1)_{ijk}{}^{a_2} [4 p_3^{a_0} \mathcal{I}_4 (\varepsilon_2^S)^{a_1 k} \,\mathrm{Tr}[\varepsilon_3^S \cdot V] + \mathcal{I}_2 ((p_2 \cdot N \cdot \varepsilon_3^A)^k (\varepsilon_2^A)^{a_0 a_1} - 2 (p_2 \cdot N \cdot \varepsilon_3^S)^{a_1} (\varepsilon_2^S)^{a_0 k})\\ &+ \mathcal{I}_3 (2 (p_2 \cdot V \cdot \varepsilon_3^S)^{a_1} (\varepsilon_2^S)^{a_0 k} - (p_2 \cdot V \cdot \varepsilon_3^A)^k (\varepsilon_2^A)^{a_0 a_1}) - (2 \leftrightarrow 3) + 2 p_3^{a_0} \mathcal{G} ((\varepsilon_2^S \cdot V \cdot \varepsilon_3^S)^{k a_1} \mathcal{I}_3\\ &+ (\varepsilon_2^A \cdot V \cdot \varepsilon_3^A)^{a_1 k} \mathcal{I}_3 + (2 \leftrightarrow 3)) - 2 \mathcal{I}_1 ((p_1 \cdot N \cdot \varepsilon_3^A)^k (\varepsilon_2^A)^{a_0 a_1} - 2 (p_1 \cdot N \cdot \varepsilon_3^S)^{a_1} (\varepsilon_2^S)^{a_0 k})].\end{aligned} \tag{52}$$

The above amplitude is also the T-dual completion of the amplitude (44). In fact, the amplitude (44) is invariant/covariant under the linear T duality when the world volume Killing $y$ index is carried by the RR potential. However, when the world volume $y$ index is carried by the NSNS polarization tensors in (44), the amplitude does not transform to itself under the linear T duality. It produces the above amplitude for $k=y$ under T duality. Completing the transverse $y$ index, one finds the above amplitude.

The combination of the amplitudes (51) and (52) satisfies the NSNS Ward identity; however, it does not satisfy the RR Ward identity. If one includes the amplitude of the RR $(p+1)$ form with three transverse indices, which has been found in [21], then the RR factor in the above amplitude is extended to the RR field strength $(F_1)^{ijka_2} = p_1^i \varepsilon_1^{jka_2} + p_1^k \varepsilon_1^{ija_2} + p_1^j \varepsilon_1^{kia_2} - p_1^{a_2} \varepsilon_1^{ijk}$; the amplitudes $\mathcal{A}_2(C_{ij}) + \mathcal{A}'_2(C_{ij})$ are extended to $\mathcal{A}_3(F_{ijk}) + \mathcal{A}'_3(F_{ijk})$. The amplitude of the RR $(p+1)$ form with three transverse indices has also some terms that become RR gauge invariant after combining them with the amplitude of the RR $(p+1)$ form with four transverse indices [21]. However, the RR invariant amplitude does not satisfy the NSNS Ward identity anymore. Here also, unlike the amplitudes $\mathcal{A}_2(C_{ij}) + \mathcal{A}'_2(C_{ij})$, which satisfy the NSNS Ward identity but do not satisfy the RR Ward identity, the amplitudes $\mathcal{A}_3(F_{ijk}) + \mathcal{A}'_3(F_{ijk})$ do not satisfy the NSNS Ward identity but satisfy the RR Ward identity. To make the amplitudes $\mathcal{A}_3(F_{ijk}) + \mathcal{A}'_3(F_{ijk})$ to satisfy the NSNS Ward identity, one requires taking into account the other amplitudes for the RR $(p+1)$ form, which we are going to consider now.

The amplitude $A_2(f_1)$ in (50) can be found by imposing the invariance of the amplitude (47) under the linear T duality. The amplitude (47) is invariant under the linear T duality when the world volume index $y$ is contracted with the RR potential; however, when the Killing index $y$ is contracted with the NSNS polarization tensors in (47), $a_1 = y$ or $a_2 = y$, then the amplitude produces new terms under the linear T duality. Completing the transverse $y$ index in the new terms, one finds the following result:

$$\begin{aligned}A_2 \sim{}& \frac{1}{2} (f_1)_{ij}{}^{a_0 a_2} \Big[ p_3^{a_1} (-p_2 \cdot N \cdot p_3 \mathcal{G} (\varepsilon_2^S \cdot V \cdot \varepsilon_3^S)^{ij} - p_2 \cdot V \cdot p_3 \mathcal{G} (\varepsilon_2^A \cdot V \cdot \varepsilon_3^A)^{ij} + (p_3 \cdot N \cdot \varepsilon_2^A)^i (p_2 \cdot N \cdot \varepsilon_3^A)^j\\ &- (p_3 \cdot V \cdot \varepsilon_2^A)^i (p_2 \cdot V \cdot \varepsilon_3^A)^j) \mathcal{J} - (\varepsilon_3^A)^{ij} \Big( p_3^{a_1} \Big( p_3 \cdot V \cdot \varepsilon_2^A \cdot V \cdot p_2 \mathcal{J}_1 - \frac{1}{2} p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot p_1 \mathcal{I}_3 + p_2 \cdot V \cdot \varepsilon_2^A \cdot N \cdot p_3 \mathcal{J}_2\\ &- p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot p_3 (\mathcal{J} + \mathcal{J}_5) + \frac{1}{2} p_3 \cdot N \cdot \varepsilon_2^A \cdot N \cdot p_1 \mathcal{I}_2 \Big) + \frac{1}{2} (p_3 \cdot V \cdot \varepsilon_2^A)^{a_1} (p_2 \cdot V \cdot p_2 \mathcal{J}_1 + p_3 \cdot V \cdot p_3 \mathcal{J}_4\\ &- 2 p_2 \cdot N \cdot p_3 \mathcal{J}) + (p_1 \cdot N \cdot \varepsilon_2^A)^{a_1} p_3 \cdot V \cdot p_3 \mathcal{I}_4 - (p_2 \cdot V \cdot \varepsilon_2^A)^{a_1} p_3 \cdot V \cdot p_3 \mathcal{J}_3 + \frac{1}{2} (p_3 \cdot N \cdot \varepsilon_2^A)^{a_1} (\mathcal{J}_{15} p_2 \cdot N \cdot p_3\\ &+ (\mathcal{J}_{16} - 2\mathcal{J}) p_2 \cdot V \cdot p_3) \Big) - (p_1 \cdot N \cdot \varepsilon_2^S)^i (p_3^{a_1} ((p_2 \cdot V \cdot \varepsilon_3^S)^j \mathcal{I}_3 + (p_1 \cdot N \cdot \varepsilon_3^S)^j \mathcal{I}_1 - (p_2 \cdot N \cdot \varepsilon_3^S)^j \mathcal{I}_2\\ &- 4 (p_3 \cdot V \cdot \varepsilon_3^S)^j \mathcal{I}_4) + 2 (\varepsilon_3^S)^{a_1 j} p_3 \cdot V \cdot p_3 \mathcal{I}_4) - 2 (p_2 \cdot V \cdot \varepsilon_2^S)^i (p_3^{a_1} ((p_2 \cdot N \cdot \varepsilon_3^S)^j \mathcal{J}_2 - (p_2 \cdot V \cdot \varepsilon_3^S)^j \mathcal{J}_1\\ &+ 2 (p_3 \cdot V \cdot \varepsilon_3^S)^j \mathcal{J}_3) - 2 (\varepsilon_3^S)^{a_1 j} p_3 \cdot V \cdot p_3 \mathcal{J}_3) + (p_3 \cdot V \cdot \varepsilon_2^S)^i (2 p_3^{a_1} (p_2 \cdot N \cdot \varepsilon_3^S)^j \mathcal{J}_5 - (\varepsilon_3^S)^{a_1 j} (p_2 \cdot V \cdot p_2 \mathcal{J}_1\\ &+ p_3 \cdot V \cdot p_3 \mathcal{J}_4 - 2 p_2 \cdot N \cdot p_3 \mathcal{J})) - (p_3 \cdot N \cdot \varepsilon_2^S)^i (\varepsilon_3^S)^{a_1 j} (\mathcal{J}_{15} p_2 \cdot N \cdot p_3 + (\mathcal{J}_{16} - 2\mathcal{J}) p_2 \cdot V \cdot p_3) \Big] + (2 \leftrightarrow 3),\end{aligned} \tag{53}$$

where the RR factor is $(f_1)^{ija_0a_2} = p_1^{a_0}\varepsilon_1^{ija_2} - p_1^{a_2}\varepsilon_1^{ija_0}$. The above amplitude does not satisfy the Ward identity corresponding to the NSNS and the RR gauge transformations.

The asymmetry under the NSNS gauge transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p+1)$, which have $(f_1)^{ija_0a_2}p_{2i}$ or $(f_1)^{ija_0a_2}p_{3i}$. Since the RR factor carries two transverse indices, one finds that there are three types of missing terms in this case. The first type is the terms that have $(f_1)^{ija_0a_2}p_{2i}$. The second type is the terms that have $(f_1)^{ija_0a_2}p_{3i}$, and the third type is the terms that have $(f_1)^{ija_0a_2}p_{2i}p_{3j}$. None of these terms can be captured by the T-dual Ward identity because the momentum along the $y$ direction is zero in the T-duality transformation. In fact, under the T-duality rules, one has $(f_1)^{iya_0a_2}p_{2i}p_{3y} = 0$ or $(f_1)^{yja_0a_2}p_{2y}p_{3j} = 0$. Therefore, the missing terms can be separated as $A'_2 = A'_{22} + A'_{23} + A''_2$. One may consider all such terms with unknown coefficients and impose the NSNS Ward identity to find the coefficients. Alternatively, one may find the amplitudes $A'_{22} + A'_{23}$ by imposing the T-dual Ward identity on the amplitudes (48) and (49) and then finding the amplitude $A''_2$ from the NSNS Ward identity. Note that the RR factor of $A''_2$, which is $(f_1)^{ija_0a_2}p_{2i}p_{3j}$, does not allow the NSNS polarization tensors to carry the transverse index of the RR factor. As a result, these terms cannot be the T-dual completion of the amplitudes in the previous section.

The amplitude (48) has the overall RR factor $(f_1)^{ia_0a_3}p_{2i}$. Therefore, the T-dual completion of this amplitude produces new terms in the first type. In fact, the amplitude (48) is invariant under the linear T duality when the world volume index $y$ is contracted with the RR potential; however, when the Killing index $y$ is contracted with the NSNS polarization tensors in (48), $a_1 = y$ or $a_2 = y$, then the amplitude produces new terms under the linear T duality. Completing the transverse $y$ index in the new terms, one finds the following result:

$$\begin{aligned}
A'_{22} \sim &-\frac{1}{4}(f_1)_{ij}{}^{a_0a_2}p_2^i[-2p_3\cdot V\cdot p_3(\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^S)^{ja_1} + \mathcal{G}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^A)^{a_1j})\mathcal{J}_{12}\\
&+ p_3^{a_1}(\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^S)^j(\mathcal{J}_{16}-2\mathcal{J}_5) + \mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^S)^j(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5)\\
&+ 4\mathcal{G}(p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^S)^j\mathcal{J}_{12} + (\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^S)^j + \mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^S)^j)\mathcal{J}_{15}\\
&- 2\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^S)^j\mathcal{I}_3 + 2\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^S)^j\mathcal{I}_2 - 4\mathcal{G}(p_3\cdot V\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^S)^j\mathcal{J}_4)\\
&+ (p_3\cdot N\cdot\varepsilon_2^A)^{a_1}(2\mathcal{I}_2(p_1\cdot N\cdot\varepsilon_3^A)^j + (p_2\cdot N\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}-2\mathcal{J}_5) + (p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{J}_{15})\\
&+ (p_3\cdot N\cdot\varepsilon_2^S)^j(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_4 - 2\mathcal{I}_2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1} - (p_2\cdot N\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}-2\mathcal{J}_5)\\
&- (p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15}) + (p_3\cdot V\cdot\varepsilon_2^A)^{a_1}((p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{J}_{15} + (p_2\cdot V\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5))\\
&- (p_3\cdot V\cdot\varepsilon_2^S)^j(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_{12} + (p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15} + (p_2\cdot V\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5))\\
&+ (\varepsilon_2^S)^{a_1j}(2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2\mathcal{J}_{15} - 2(2\mathcal{I}_4p_1\cdot N\cdot p_3 - \mathcal{J}_4p_2\cdot N\cdot p_3 + \mathcal{J}_{12}p_2\cdot V\cdot p_3)\mathrm{Tr}[\varepsilon_3^S\cdot V]\\
&+ p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5) + 4p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{J}_{12} - 2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_3\\
&+ p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16}-2\mathcal{J}_5) - 4\mathcal{J}_4p_3\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2 + 4\mathcal{I}_2p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_1)] + (2\leftrightarrow 3).
\end{aligned}\tag{54}$$

This amplitude is invariant under the linear T duality when the $y$ index is carried by the RR potential; otherwise, it is not invariant. We consider the T-dual completion of this amplitude in the next section.

The terms of the second type, which have the RR factor $(f_1)^{ija_0a_2}p_{3i}$, can be found from the T-dual completion of the amplitude (49) because this amplitude has the overall RR factor $(f_1)^{ia_0a_3}p_{3i}$. Here also one realizes that the amplitude (49) is invariant under the linear T duality only when the world volume index $y$ is contracted with the RR potential. When it is contracted with the NSNS polarization tensors, the amplitude produces new terms, which have the transverse $y$ index, under the linear T duality. Completing this index, one finds the following result:

$$\begin{aligned}
A'_{23} \sim &-\frac{1}{4}(f_1)_{ij}{}^{a_0a_2}p_3^i[-2p_2\cdot V\cdot p_2(\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^S)^{ja_1} + \mathcal{G}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^A)^{a_1j})\mathcal{J}_1\\
&+ p_3^{a_1}(\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^S)^j(\mathcal{J}_{16}-2\mathcal{J}_5) + \mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^S)^j(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5)\\
&+ (\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot\varepsilon_2^S)^j + \mathcal{G}(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot\varepsilon_2^S)^j)\mathcal{J}_{15}) + (p_3\cdot V\cdot\varepsilon_2^A)^{a_1}(2(p_1\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_2\\
&+ (p_2\cdot N\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}-4\mathcal{J}-2\mathcal{J}_5) + (p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{J}_{15}) + (p_3\cdot V\cdot\varepsilon_2^S)^j(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_4
\end{aligned}$$

$$
\begin{aligned}
&-2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_2-(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}-4\mathcal{J}-2\mathcal{J}_5)-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15})\\
&+2(p_1\cdot N\cdot\varepsilon_2^A)^{a_1}(2(p_1\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_1-(p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_2+(p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{I}_3)\\
&+2(p_1\cdot N\cdot\varepsilon_2^S)^j(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{I}_4-2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_1+(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_2-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_3)\\
&-4(p_2\cdot V\cdot\varepsilon_2^A)^{a_1}(2(p_1\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_7-(p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{J}_2+(p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{J}_1)\\
&-4(p_2\cdot V\cdot\varepsilon_2^S)^j(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_3-2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_7+(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_2-(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_1)\\
&+(p_3\cdot N\cdot\varepsilon_2^A)^{a_1}((p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{J}_{15}+(p_2\cdot V\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}+2\mathcal{J}_5))\\
&-(p_3\cdot N\cdot\varepsilon_2^S)^j(2p_3^{a_1}\,\mathrm{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_{12}+(p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15}+(p_2\cdot V\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16}+2\mathcal{J}_5))\\
&+(\varepsilon_2^S)^{a_1j}(4(p_1\cdot N\cdot p_2\mathcal{I}_4-p_2\cdot V\cdot p_2\mathcal{J}_3)\mathrm{Tr}[\varepsilon_3^S\cdot V]+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16}-2\mathcal{J})\\
&+(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2+p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2)\mathcal{J}_{15}+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_2)+(2\leftrightarrow 3)\\
&+2p_1\cdot N\cdot p_2((\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^S)^{ja_1}+\mathcal{G}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^A)^{a_1j})\mathcal{I}_3+(2\leftrightarrow 3))].\quad(55)
\end{aligned}
$$

The above amplitude is invariant under the linear T duality when the $y$ index is carried by the RR potential; otherwise, it is not invariant. We consider the T-dual completion of this amplitude in the next section. We have checked that the sum of the amplitudes (53), (54), and (55) does not satisfy the NSNS Ward identity. So the amplitude $A_2''$ is required to make the amplitudes invariant under the NSNS gauge transformations.

Since the amplitude $A_2''$ has the RR factor $(f_1)^{ija_0a_2}p_{2i}p_{3j}$, one has to consider all independent terms with one world volume index $(\cdots)^{a_1}$, which contain one momentum and the two NSNS polarization tensors. In this case, there are terms in which the two tensors contract with each other. All possible such terms are

$$
\begin{aligned}
&\mathrm{Tr}[\varepsilon^S\cdot V\cdot\varepsilon^S\cdot V],\quad \mathrm{Tr}[\varepsilon^S\cdot N\cdot\varepsilon^S\cdot N],\quad \mathrm{Tr}[\varepsilon^S\cdot V\cdot\varepsilon^S\cdot N]\\
&\mathrm{Tr}[\varepsilon^A\cdot V\cdot\varepsilon^A\cdot V],\quad \mathrm{Tr}[\varepsilon^A\cdot N\cdot\varepsilon^A\cdot N],\quad \mathrm{Tr}[\varepsilon^A\cdot V\cdot\varepsilon^A\cdot N].
\end{aligned}\tag{56}
$$

Since the independent terms must be invariant under the linear T duality when $a_1\neq y$, we have to consider the combination of the above terms that are invariant under the T duality. The only possibility is the following combination:

$$
\mathrm{Tr}[\varepsilon_2^A\cdot V\cdot\varepsilon_3^A\cdot V]+\mathrm{Tr}[\varepsilon_2^A\cdot N\cdot\varepsilon_3^A\cdot N]-2\,\mathrm{Tr}[\varepsilon_2^S\cdot V\cdot\varepsilon_3^S\cdot N].\tag{57}
$$

However, the NSNS Ward identity requires other traces as well. The only way that we can make the T-duality invariant combination is to consider dilaton terms as well as the gravitons. Using the T-duality transformation of the dilaton in the string frame, one finds that the following combination is invariant under the linear T duality:

$$
\begin{aligned}
&\mathrm{Tr}[\varepsilon_2^S\cdot V\cdot\varepsilon_3^S\cdot V]+\mathrm{Tr}[\varepsilon_2^S\cdot N\cdot\varepsilon_3^S\cdot N]\\
&-2\,\mathrm{Tr}[\varepsilon_2^A\cdot V\cdot\varepsilon_3^A\cdot N]+4\Phi_2\Phi_3,
\end{aligned}\tag{58}
$$

where $\Phi$ is the polarization of the dilaton, which is one; however, we keep it for clarity.

Using the above two T-duality invariant combinations, as well as the structures in which the polarization tensors contract with the momentum, one finds that the NSNS Ward identity is satisfied provided that the amplitude $A_2''$ has the following terms:

$$
\begin{aligned}
A''_2\sim\frac{1}{4}(f_1)_{ij}{}^{a_0a_2}p_2^ip_3^j\Big[&-\mathcal{J}_{15}\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^A)^{a_1}-\mathcal{J}_{15}\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^A)^{a_1}\\
&+(4\mathcal{J}-\mathcal{J}_{16}-2\mathcal{J}_5)\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^A)^{a_1}-(\mathcal{J}_{16}-2\mathcal{J}_5)\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^A)^{a_1}\\
&-2(\mathcal{J}_4(p_3\cdot V\cdot\varepsilon_2^S)^{a_1}-\mathcal{J}_{12}(p_3\cdot N\cdot\varepsilon_2^S)^{a_1}+\mathcal{J}_3p_3^{a_1}\,\mathrm{Tr}[\varepsilon_2^S\cdot V])\mathrm{Tr}[\varepsilon_3^S\cdot V]\\
&+\frac{1}{2}p_3^{a_1}((2\mathcal{J}-\mathcal{J}_{16})(\mathrm{Tr}[\varepsilon_2^A\cdot N\cdot\varepsilon_3^A\cdot N]+\mathrm{Tr}[\varepsilon_2^A\cdot V\cdot\varepsilon_3^A\cdot V]-2\,\mathrm{Tr}[\varepsilon_2^S\cdot V\cdot\varepsilon_3^S\cdot N])\\
&+\mathcal{J}_{15}(\mathrm{Tr}[\varepsilon_2^S\cdot N\cdot\varepsilon_3^S\cdot N]+\mathrm{Tr}[\varepsilon_2^S\cdot V\cdot\varepsilon_3^S\cdot V]-2\,\mathrm{Tr}[\varepsilon_2^A\cdot V\cdot\varepsilon_3^A\cdot N]+4\Phi_2\Phi_3)-(2\leftrightarrow 3)\\
&-2\mathcal{I}_2\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^A)^{a_1}+2\mathcal{I}_3\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^A)^{a_1}\Big].
\end{aligned}\tag{59}
$$

This amplitude is also invariant under the linear T duality when the $y$ index is carried by the RR potential; otherwise, it is not invariant. We consider the T-dual completion of this amplitude in the next section.

Note that we have included the dilaton term in above amplitude based on the fact that the amplitude should be consistent with the T-dual Ward identity. As a result, the above amplitude is correct in the string frame. However, the direct string-theory calculation produces amplitudes in the Einstein frame. Therefore, if one is interested in verifying the dilaton amplitude by the direct string-theory S-matrix element of one RR and two dilaton vertex operators, one has to transform the S-matrix element to the string frame and then compare it with the above result.

The combination of the amplitudes (53), (54), (55), and (59) satisfies the Ward identity corresponding to the NSNS gauge transformations. However, it does not satisfy the RR Ward identity because the RR factor $(f_1)^{ija_0a_2}$ in them is not the RR field strength. It can easily be extended to the RR invariant amplitudes by extending the RR factor to the RR field strength $(F_1)^{ija_0a_2} = p_1^{a_0}\varepsilon_1^{ija_2} - p_1^{a_2}\varepsilon_1^{ija_0} + p_1^i\varepsilon_1^{a_2ja_0} - p_1^j\varepsilon_1^{a_2ia_0}$. The amplitude corresponding to the last two terms does not satisfy the NSNS Ward identity. So one has to add the amplitude of the RR $(p+1)$ form with one transverse index. The RR invariance requires the amplitude to be in terms of RR field strength, $(F_1)^{ia_0a_1a_2}(\cdots)_i$. These couplings may be found by imposing the T-dual Ward identity on the RR $(p-1)$ form amplitude with structure $(F_1)^{a_0a_1a_3}(\cdots)^{a_2}$ when $a_2 = y$. As we have discussed in the previous section, one needs explicit calculation to find the RR $(p-1)$-form amplitude with structure $(F_1)^{a_0a_1a_3}(\cdots)^{a_2}$. We leave the details of theses calculations for the future work.

### D. RR $(p+3)$ form

The amplitude for the RR $(p+3)$ form is nonzero when the RR potential has five, four, and three transverse indices. When the RR potential has three transverse indices, the amplitude can be found by applying the T-dual Ward identity on the amplitude (50),

$$\mathbf{A}_3 = A_3(f_1) + \mathcal{A}_3(f_1), \tag{60}$$

where the subscript 3 refers to the number of the transverse indices of the RR potential. The amplitude $\mathcal{A}_3(f_1)$ is the T-dual completion of the amplitude $\mathcal{A}_2(f_1)$ in (51), and $A_3(f_1)$ is the T-dual completion of the amplitude $A_2(f_1)$ in (53). The amplitude (51) does not satisfy the T-dual Ward identity when the $y$ index is carried by the NSNS polarization tensors. The consistency with the T-dual Ward identity requires the following amplitude:

$$\begin{aligned}\mathcal{A}_3 \sim -\frac{1}{4}(f_1)_{ijkl}{}^{a_1}[&(\varepsilon_2^A)^{ij}(2p_2^kp_2^{a_0}(p_1\cdot N\cdot\varepsilon_3^S)^l\mathcal{I}_1 + (p_3^kp_2^{a_0}\mathcal{I}_3 + p_2^kp_3^{a_0}\mathcal{I}_2)(p_2\cdot N\cdot\varepsilon_3^S)^l\\ &+ (p_2^kp_2^{a_0}\mathcal{I}_3 + p_3^kp_3^{a_0}\mathcal{I}_2)(p_2\cdot V\cdot\varepsilon_3^S)^l - 4p_2^kp_2^{a_0}(p_3\cdot V\cdot\varepsilon_3^S)^l\mathcal{I}_4) + (2\leftrightarrow 3)\\ &- (2p_2^kp_3\cdot V\cdot p_3\mathcal{I}_4 + p_3^k(-p_2\cdot N\cdot p_3\mathcal{I}_3 + p_2\cdot V\cdot p_3\mathcal{I}_2))((\varepsilon_2^A)^{ij}(\varepsilon_3^S)^{a_0l} + (2\leftrightarrow 3))],\end{aligned} \tag{61}$$

where the RR factor is $(f_1)^{ijkla_1} = p_1^i\varepsilon_1^{jkla_1} - p_1^j\varepsilon_1^{ikla_1} + p_1^k\varepsilon_1^{ijla_1} - p_1^l\varepsilon_1^{ijka_1}$. For simplicity, we have considered the amplitude for $p=1$. The above amplitude is invariant under the T-dual Ward identity when the $y$ index is carried by the RR potential. However, it does not satisfy this identity when the Killing index is carried by the NSNS polarization tensors, when $a_0 = y$. We find the T-dual completion of this amplitude in the next section.

The above amplitude does not satisfy the Ward identity corresponding to the NSNS and the RR gauge transformations. The asymmetry under the NSNS transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p+3)$, which have $(f_1)^{ijkla_1}p_{2i}$ or $(f_1)^{ijkla_1}p_{3i}$. Since all terms in (51) have either $p_{2i}$ or $p_{3i}$, one finds that the missing terms should have the factor $(f_1)^{ijkla_1}p_{2i}p_{3j}$. Considering all such terms that have one momentum and the two NSNS polarization tensors, with unknown coefficients and imposing the condition that when they combine with the amplitude (61) they should satisfy the NSNS Ward identity, one finds the following result:

$$\begin{aligned}\mathcal{A}'_3 \sim \frac{1}{4}(f_1)^{ijkla_1}p_{2i}p_{3j}[&2p_3^{a_0}\mathcal{I}_4(\varepsilon_2^A)^{kl}\,\mathrm{Tr}[\varepsilon_3^S\cdot V] + \mathcal{I}_2(2(p_2\cdot N\cdot\varepsilon_3^A)^k(\varepsilon_2^S)^{a_0l} + (p_2\cdot N\cdot\varepsilon_3^S)^{a_0}(\varepsilon_2^A)^{kl})\\ &- \mathcal{I}_3(2(p_2\cdot V\cdot\varepsilon_3^A)^k(\varepsilon_2^S)^{a_0l} + (p_2\cdot V\cdot\varepsilon_3^S)^{a_0}(\varepsilon_2^A)^{kl}) - (2\leftrightarrow 3) + 2p_3^{a_0}\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^A)^{kl}\mathcal{I}_3\\ &+ 2p_3^{a_0}\mathcal{G}(\varepsilon_3^S\cdot V\cdot\varepsilon_2^A)^{kl}\mathcal{I}_2 - 2\mathcal{I}_1(2(p_1\cdot N\cdot\varepsilon_3^A)^k(\varepsilon_2^S)^{a_0l} + (p_1\cdot N\cdot\varepsilon_3^S)^{a_0}(\varepsilon_2^A)^{kl})].\end{aligned} \tag{62}$$

One can verify that the above amplitude is the T-dual completion of the amplitude (52), as expected. The combination of the amplitudes (61) and (62) satisfies the NSNS Ward identity; however, they do not satisfy the RR Ward identity. If one includes the amplitude of the RR $(p+3)$ form with four transverse indices, which has been found in [21], then the RR factor in the above amplitude is extended to the RR field strength $(F_1)^{ijkla_1} = (f_1)^{ijkla_1} + p_1^{a_1}\varepsilon_1^{ijkl}$, the amplitudes $\mathcal{A}_3(C_{ijk}) + \mathcal{A}'_3(C_{ijk})$ are extended to $\mathcal{A}_4(F_{ijkl}) + \mathcal{A}'_4(F_{ijkl})$. The amplitude of the RR $(p+3)$

form with four transverse indices has also some terms that become RR gauge invariant after including the amplitude of the RR $(p+3)$ form with five transverse indices [21]. The RR gauge invariant amplitudes $\mathcal{A}_4(F_{ijkl})+\mathcal{A}'_4(F_{ijkl})$ do not satisfy the NSNS Ward identity, which indicates the presence of other amplitude in (60).

The amplitude $A_3(f_1)$ in (60) can be found from imposing the invariance of the amplitude (53) under the linear T duality when the Killing index $y$ is carried by the NSNS polarization tensors in (53). The result is the following:

$$
\begin{aligned}
A_3 \sim \frac{1}{4}(f_1)_{ijk}{}^{a_0a_1}(\varepsilon_2^A)^{ij}[&2p_2\cdot V\cdot p_2(p_1\cdot N\cdot\varepsilon_3^S)^k\mathcal{I}_7\\
&-4p_2\cdot V\cdot p_2(p_3\cdot V\cdot\varepsilon_3^S)^k\mathcal{J}_3+(p_2\cdot N\cdot p_3\mathcal{J}_{15}\\
&+p_2\cdot V\cdot p_3(\mathcal{J}_{16}-2\mathcal{J}))(p_2\cdot N\cdot\varepsilon_3^S)^k\\
&-(2p_2\cdot N\cdot p_3\mathcal{J}-p_2\cdot V\cdot p_2\mathcal{J}_1\\
&-p_3\cdot V\cdot p_3\mathcal{J}_4)(p_2\cdot V\cdot\varepsilon_3^S)^k]+(2\leftrightarrow 3),
\end{aligned}
\tag{63}
$$

where the RR factor is $(f_1)^{ijka_0a_1}=p_1^{a_0}\varepsilon_1^{ijka_1}-p_1^{a_1}\varepsilon_1^{ijka_0}$. Since the NSNS polarization tensors do not carry the world volume index, the above amplitude is invariant under the linear T duality. However, it does not satisfy the Ward identity corresponding to the NSNS and RR gauge transformations.

The asymmetry under the NSNS gauge transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p+3)$ which have $(f_1)^{ijka_0a_1}p_{2i}$ or $(f_1)^{ijka_0a_1}p_{3i}$. Since the RR factor is $(f_1)^{ijka_0a_1}$, one finds that there are three types of missing terms. The first type is the terms that have $(f_1)^{ijka_0a_1}p_{2i}$. The second type is the terms that have $(f_1)^{ijka_0a_1}p_{3i}$, and the third type is the terms that have $(f_1)^{ijka_0a_1}p_{2i}p_{3j}$. Therefore, the missing terms can be separated as $A'_3=A'_{32}+A'_{33}+A''_3$. One may consider all such terms with unknown coefficients and impose the NSNS Ward identity to find the coefficients. Alternatively, one may find these amplitudes by imposing the T-dual Ward identity on the amplitudes (54), (55), and (59). We are going to preform the latter calculations. The T-dual completion of the amplitude (54) which has the terms of the first type, is the following:

$$
\begin{aligned}
A'_{32} \sim \frac{1}{4}(f_1)_{ijk}{}^{a_0a_1}p_2^i\Big[&(p_3\cdot V\cdot\varepsilon_2^S)^k((p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{J}_{15}+(p_2\cdot V\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5))\\
&+(p_3\cdot N\cdot\varepsilon_2^S)^k(2\mathcal{I}_2(p_1\cdot N\cdot\varepsilon_3^A)^j+(p_2\cdot N\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}-2\mathcal{J}_5)+(p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{J}_{15})\\
&+\frac{1}{2}(\varepsilon_2^A)^{jk}(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16}-2\mathcal{J}_5)-4\mathcal{J}_4p_3\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2+4\mathcal{I}_2p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_1\\
&+p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2(\mathcal{J}_{16}-4\mathcal{J}+2\mathcal{J}_5)+4p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{J}_{12}-2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_3\\
&+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2\mathcal{J}_{15}+(-2\mathcal{I}_4p_1\cdot N\cdot p_3+\mathcal{J}_4p_2\cdot N\cdot p_3-\mathcal{J}_{12}p_2\cdot V\cdot p_3)\mathrm{Tr}[\varepsilon_3^S\cdot V])\\
&-2p_3\cdot V\cdot p_3\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^A)^{jk}\mathcal{J}_{12}\Big]+(2\leftrightarrow 3).
\end{aligned}
\tag{64}
$$

Since the NSNS polarization tensors do not carry the world volume index, the above amplitude is invariant under the linear T duality.

The T-dual completion of the amplitude (55) which has the terms of the second type, is the following:

$$
\begin{aligned}
A'_{33} \sim \frac{1}{4}(f_1)_{ijk}{}^{a_0a_1}p_3^i\Big[&-2p_2\cdot V\cdot p_2\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^A)^{jk}\mathcal{J}_1+(p_3\cdot V\cdot\varepsilon_2^S)^k(2(p_1\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_2\\
&+(p_2\cdot N\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}-4\mathcal{J}-2\mathcal{J}_5)+(p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{J}_{15})\\
&+2(p_1\cdot N\cdot\varepsilon_2^S)^k(2(p_1\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_1-(p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_2+(p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{I}_3)\\
&-4(p_2\cdot V\cdot\varepsilon_2^S)^k(2(p_1\cdot N\cdot\varepsilon_3^A)^j\mathcal{I}_7-(p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{J}_2+(p_2\cdot V\cdot\varepsilon_3^A)^j\mathcal{J}_1)\\
&+(p_3\cdot N\cdot\varepsilon_2^S)^k((p_2\cdot N\cdot\varepsilon_3^A)^j\mathcal{J}_{15}+(p_2\cdot V\cdot\varepsilon_3^A)^j(\mathcal{J}_{16}+2\mathcal{J}_5))\\
&+\frac{1}{2}(\varepsilon_2^A)^{jk}(4(p_1\cdot N\cdot p_2\mathcal{I}_4-p_2\cdot V\cdot p_2\mathcal{J}_3)\mathrm{Tr}[\varepsilon_3^S\cdot V]+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16}-2\mathcal{J})\\
&+(p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2+p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2)\mathcal{J}_{15}+2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_2)+(2\leftrightarrow 3)\\
&+2p_1\cdot N\cdot p_2p_3^i(\mathcal{G}(\varepsilon_2^S\cdot V\cdot\varepsilon_3^A)^{jk}\mathcal{I}_3+(2\leftrightarrow 3))\Big].
\end{aligned}
\tag{65}
$$

The NSNS polarization tensors do not carry the world volume index, so the above amplitude is also invariant under the linear T duality. While the first $(2 \leftrightarrow 3)$ means the interchange of the labels 2, 3 for all expressions from the beginning up to that point, including the overall factor, the second $(2 \leftrightarrow 3)$ means the interchange of the labels 2, 3 only for the term in the parenthesis in the last line.

The T-dual completion of the amplitude (59) which has the terms of the third type, is the following:

$$\begin{aligned} A''_3 \sim \frac{1}{4}(f_1)^{ijka_0a_1} p_{2i}p_{3j}[&-2(\mathcal{J}_4(p_3\cdot V\cdot \varepsilon_2^A)_k - \mathcal{J}_{12}(p_3\cdot N\cdot\varepsilon_2^A)_k)\mathrm{Tr}[\varepsilon_3^S\cdot V] \\ &- \mathcal{J}_{15}\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^S)_k + (4\mathcal{J}-\mathcal{J}_{16}-2\mathcal{J}_5)\mathcal{G}(p_2\cdot V\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^S)_k \\ &- (\mathcal{J}_{16}-2\mathcal{J}_5)\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^S)_k - \mathcal{J}_{15}\mathcal{G}(p_2\cdot N\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^S)_k - (2\leftrightarrow 3) \\ &- 2\mathcal{I}_2\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^S)_k + 2\mathcal{I}_3\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^S)_k]. \end{aligned} \tag{66}$$

This amplitude is also invariant under the linear T duality. The sum of the amplitudes (63), (64), (65), and (66) satisfies the Ward identity corresponding to the NSNS gauge transformations. However, it does not satisfy the RR Ward identity because the RR factor $(f_1)^{ijka_0a_1}$ is not the RR field strength. It can easily be extended to the RR invariant amplitude by extending the RR factor to the RR field strength $(F_1)^{ijka_0a_1} = (f_1)^{ijka_0a_1} + p_1^i\varepsilon_1^{a_1jka_0} - p_1^j\varepsilon_1^{a_1ika_0} + p_1^k\varepsilon_1^{a_1ija_0}$. In this case, the amplitudes corresponding to the last three terms satisfy the NSNS Ward identity.

### E. RR $(p+5)$ form

The amplitude for the RR $(p+5)$ form is nonzero when the RR potential has six, five, and four transverse indices. When the RR potential has four transverse indices, the amplitude can be found by applying the T-dual Ward identity on the amplitude (60),

$$\mathbf{A}_4 = \mathcal{A}_4(f_1), \tag{67}$$

where the subscript 4 refers to the number of the transverse indices of the RR potential. The amplitude $\mathcal{A}_4(f_1)$, which is the T-dual completion of the amplitude $\mathcal{A}_3(f_1)$ in (61) is

$$\begin{aligned} \mathcal{A}_4 \sim \frac{1}{8}(f_1)_{ijklm}{}^{a_0}(&2p_2^k p_3\cdot V\cdot p_3\mathcal{I}_4 + p_3^k(\mathcal{I}_2 p_2\cdot V\cdot p_3 \\ &- \mathcal{I}_3 p_2\cdot N\cdot p_3))(\varepsilon_2^A)^{ij}(\varepsilon_3^A)^{lm}, \end{aligned} \tag{68}$$

where the RR factor is $(f_1)^{ijklma_0} = p_1^{[i}\varepsilon_1^{jklm]a_0}$. Since the NSNS polarization tensors do not carry the world volume index, the above amplitude is invariant under the linear T duality. However, it does not satisfy the Ward identity corresponding to the NSNS and RR gauge transformations.

The asymmetry under the NSNS transformation indicates that the T-dual Ward identity could not capture terms of the scattering amplitude of the RR $(p+5)$, which have $(f_1)^{ijklma_0}p_{2i}$ or $(f_1)^{ijklma_0}p_{3i}$. Since all terms in (61) have either $p_{2i}$ or $p_{3i}$, one finds that the missing terms corresponding to the above amplitude should have the factor $(f_1)^{ijklma_0}p_{2i}p_{3j}$. Considering all such terms that have one momentum and the two NSNS polarization tensors, with unknown coefficients and imposing the condition that when they combine with the amplitude (68) they should satisfy the NSNS Ward identity, one finds the following result:

$$\begin{aligned} \mathcal{A}'_4 \sim \frac{1}{4}(f_1)_{ijklm}{}^{a_0}p_2^ip_3^j(\varepsilon_2^A)^{lm}[&(p_2\cdot N\cdot\varepsilon_3^A)^k\mathcal{I}_2 \\ &- (p_2\cdot V\cdot\varepsilon_3^A)^k\mathcal{I}_3] + (2\leftrightarrow 3) \\ - \frac{1}{2}(f_1)_{ijklm}{}^{a_0}p_2^ip_3^j&(\varepsilon_2^A)^{lm}(p_1\cdot N\cdot\varepsilon_3^A)^k\mathcal{I}_1. \end{aligned} \tag{69}$$

The above amplitude is also the T-dual completion of the amplitude (62). There is no contraction between the NSNS polarization tensors and the world volume form, so this amplitude, like (68), is invariant under the linear T duality.

The combination of the above two amplitudes satisfies the NSNS Ward identity; however, they do not satisfy the RR Ward identity. To extend the amplitude $\mathcal{A}_4 + \mathcal{A}'_4$ to satisfy the RR Ward identity, one has to extend the RR factor to the RR field strength $(F_1)^{ijklma_0} = (f_1)^{ijklma_0} - p_1^{a_0}\varepsilon_1^{ijklm}$, the amplitudes $\mathcal{A}_4(C_{ijkl}) + \mathcal{A}'_4(C_{ijkl})$ are extended to $\mathcal{A}_5(F_{ijklm}) + \mathcal{A}'_5(F_{ijklm})$. This can be done by including the amplitude of the RR $(p+5)$ form with five transverse indices, which has been found in [21]. The amplitude of the RR $(p+5)$ form with five transverse indices has also some terms that become RR gauge invariant after including the amplitude of the RR $(p+5)$ form with six transverse indices [21]. These amplitudes and the amplitudes $\mathcal{A}_5(F_{ijklm}) + \mathcal{A}'_5(F_{ijklm})$ are exactly equal to the amplitudes that has been calculated explicitly in string theory for the case that the RR potential is $(p+5)$ form [38]. So they satisfies the NSNS Ward identity as well as the RR Ward identity.

Therefore, the S-matrix elements of one RR and two NSNS can be classified into the following multiplets in terms of the RR field strength. One T-dual multiplet, which has been found in [21] (see Eq. (15) in [21]), has the following structure:

$$A_2(F^{(p-2)}_{ij}) \to A_3(F^{(p)}_{ijk}) \to A_4(F^{(p+2)}_{ijkl}) \to A_5(F^{(p+4)}_{ijklm}) \to A_6(F^{(p+6)}_{ijklmn}). \tag{70}$$

The T-dual multiplet satisfies the RR Ward identity; however, it does not satisfy the NSNS Ward identity. Another multiplet has the following structure:

$$\begin{array}{ccccccccc} \mathcal{A}_1(F^{(p-2)}_i) & \to & \mathcal{A}_2(F^{(p)}_{ij}) & \to & \mathcal{A}_3(F^{(p+2)}_{ijk}) & \to & \mathcal{A}_4(F^{(p+4)}_{ijkl}) & \to & \mathcal{A}_5(F^{(p+6)}_{ijklm}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{A}'_2(F^{(p)}_{ij}) & \to & \mathcal{A}'_3(F^{(p+2)}_{ijk}) & \to & \mathcal{A}'_4(F^{(p+4)}_{ijkl}) & \to & \mathcal{A}'_5(F^{(p+6)}_{ijklm}) \end{array} \tag{71}$$

where $\mathcal{A}_1$ is the amplitude (38), the amplitudes $\mathcal{A}_2 + \mathcal{A}'_2$ are the amplitudes (43) and (44), the amplitudes $\mathcal{A}_3 + \mathcal{A}'_3$ are the amplitudes (51) and (52), the amplitudes $\mathcal{A}_4 + \mathcal{A}'_4$ are the amplitudes (61) and (62), and the amplitudes $\mathcal{A}_5 + \mathcal{A}'_5$ are the amplitudes (68) and (69) in which the RR factor $f_1$ is replaced by the RR field strength $F_1$. The third multiplet has the following structure:

$$\begin{array}{ccccccc} A_0(F^{(p-2)}) & \to & A_1(F^{(p)}_i) & \to & A_2(F^{(p+2)}_{ij}) & \to & A_3(F^{(p+4)}_{ijk}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A'_1(F^{(p)}_i) & \to & A'_2(F^{(p+2)}_{ij}) & \to & A'_3(F^{(p+4)}_{ijk}) \end{array} \tag{72}$$

where $A_0$ is the amplitude (40). The amplitudes $A_1 + A'_1$ are the amplitudes (47), (48), and (49), the amplitudes $A_2 + A'_2$ are the amplitudes (53), (54), (55), and (59), and the amplitudes $A_3 + A'_3$ are the amplitudes (63), (64), (65), and (66) in which the RR factor $f_1$ is replaced by the RR field strength $F_1$. The last multiplet would have the following structure:

$$\begin{array}{ccc} A_0(F^{(p)}) & \to & A_1(F^{(p+2)}_i) \\ & & \downarrow \\ & & A'_1(F^{(p+2)}_i) \end{array} \tag{73}$$

The first component of the above multiplet may be found from the explicit string-theory calculation in which we are not interested in this paper. Using the T-dual Ward identity on the first component, the second component then would be easily found, as we have done for many other cases in this paper.

## V. DISCUSSION

In this paper, we have used the constraints that the S-matrix elements should satisfy the Ward identity corresponding to the gauge symmetries and the T duality, to find the $D_p$-brane world volume amplitude of various RR $n$ forms from the known amplitudes of the RR $(p-3)$ form. Using this constraint, we have found various S-matrix elements of one RR, one NSNS, and one NS state and the S-matrix elements of one RR and two NSNS states.

We have found that the Ward identities corresponding to the combination of the T duality and the gauge transformations are powerful enough to find all of the S-matrix multiplets that are connected by these Ward identities. However, the Ward identities corresponding to the gauge transformations are not powerful enough to find all of the amplitudes that are connected by these Ward identities. For the case of two closed and one open strings, the T-dual multiplets are (28), (29), and (30) and for the case of three closed strings, the T-dual multiplets are (70), (71), (72), and (73).

In each multiplet, the different components are connected by the T-dual and the NSNS Ward identities. On the other hand, the components of all of the T-dual multiplets, which have a specific RR field strength, are connected by the NSNS/NS Ward identity, e.g., the $F^{(p)}$ component in the multiplets (28), (29), and (30), and the $F^{(p)}$ component in the multiplets (70), (71), (72), and (73) should satisfy the Ward identities corresponding to the gauge transformations. In the former case, the amplitude $A_0(F^{(p)})$ has been found by these Ward identities, (24); however, there are two integrals and one constraint. The explicit form of the integrals can be found only by direct calculation of the corresponding S-matrix element in the string theory. In the latter case, the Ward identities produce many new integrals and constraint equations. It would be interesting to find this amplitude by the explicit string-theory calculations and then find its corresponding multiplet (73). It would be also interesting to confirm the amplitudes that we have found in this paper by explicit string-theory calculations.

The S-matrix elements of three closed strings that have been found in this paper can be analyzed at low energy to extract the appropriate couplings of one RR and two NSNS states in the field theory at order $\alpha'^2$. In performing this calculation, one needs the $\alpha'$ expansion of the integrals that appear in the amplitudes. The $\alpha'$ expansion of the integrals $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_7$ and $\mathcal{J}$, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_5$, $\mathcal{J}_{13}$, $\mathcal{J}_{14}$ have been found in [38,40] for the special kinematic setup where $p_2 \cdot D \cdot p_3 = 0$ and $p_2 \cdot p_3 = 0$. The above integrals are similar to the integrals $I_0, \ldots, I_{10}$ that have been found in [39]. The relation between the two set of integrals is

$$\mathcal{I}_1 = I_{10}, \qquad \mathcal{I}_2 = I_5 - I_9, \qquad \mathcal{I}_7 = -\frac{1}{2} I_4, \qquad \mathcal{J} = 2I_0, \qquad \mathcal{J}_1 = -(I_6 + I_7),$$
$$\mathcal{J}_2 = -2I_0 - I_8 + I_{10}, \qquad \mathcal{J}_3 = -I_3, \qquad \mathcal{J}_5 = I_8, \mathcal{J}_{13} = 2I_0 + I_2, \qquad \mathcal{J}_{14} = 2I_0 + I_1.$$

The low energy expansion of the integrals $I_0, \ldots, I_{10}$, for the general setup, has been found in [39]. Using them, one can find the $\alpha'$ expansion of the amplitudes that contain various massless poles as well as contact terms. To find the couplings of one RR and two NSNS states at order $\alpha'^2$, one has to first calculate the massless poles in field theory and then subtract them from the massless poles of the string-theory amplitude. The massless poles at order $\alpha'^2$ are simple closed string poles, simple open-string poles, and double open-string poles. The closed-string poles should be reproduced by the supergravity and the brane couplings of two closed strings at order $\alpha'^2$ [25,29,37]. The simple open-string poles should be reproduced by the DBI or CS action and the brane couplings of two closed and one open string at order $\alpha'^2$, which can be found from the amplitudes in Sec. III. The double open-string poles should be reproduced by the DBI or CS action and the brane couplings of one closed and two open strings at order $\alpha'^2$ [43,44].

The subtraction of field-theory massless poles from the string-theory amplitude may add some extra contact terms to the contact terms of the string-theory amplitude. For the amplitudes that involve only the antisymmetric NSNS states, one may expect the extra contact terms to be avoided by writing both the string-theory amplitude and the field-theory massless poles, in terms of B field strength $H$. This can be done based on the fact that the S-matrix elements must satisfy the Ward identity corresponding to the B-field gauge transformation. In the field-theory side, the bulk couplings are in terms of $H$, and the brane couplings are either in terms of $H$ or in terms of $\tilde{B} = B + 2\pi\alpha'\mathcal{F}$. As a result, one can calculate the massless poles in the field theory in terms of $H$. In fact, the open-string poles of the scattering amplitude in which the gauge boson part of $\tilde{B}$ propagates can be combined with the contact terms resulting from the $B$-field part of $\tilde{B}$ to write the amplitude in terms of $H$ [38,40]. While the field-theory massless poles can be calculated uniquely in terms of $H$, there is no unique way, in general, to write the string-theory amplitude in terms of $H$.

For the case of the RR $(p-3)$ form that has been studied in [38], there is a unique way to write the string-theory amplitude in terms of $H$. Hence, in that case, one does not need to calculate the field-theory massless poles. The contact terms of the string theory in terms of $H$ gives the correct couplings in field theory. For the case of the RR $(p+1)$ form, we have checked that there is no unique way to write the amplitude in terms of $H$. Therefore, even in this, one has to calculate the massless poles and subtract them from the string-theory amplitude to find the contact terms. After finding all contact terms, one should be able to write them in terms of the field strengths of the external states. We leave the details of this calculation for the future works.

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[1] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[2] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[3] K. Kikkawa and M. Yamasaki, Phys. Lett. **149B**, 357 (1984).
[4] T. Buscher, Phys. Lett. B **194**, 59 (1987); **201**466 (1988).
[5] A. Giveon, M. Porrati, and E. Rabinovici, Phys. Rep. **244**, 77 (1994).
[6] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, Nucl. Phys. B, Proc. Suppl. **41**, 1 (1995).
[7] P. Meessen and T. Ortin, Nucl. Phys. **B541**, 195 (1999).
[8] E. Bergshoeff, C. M. Hull, and T. Ortin, Nucl. Phys. **B451**, 547 (1995).
[9] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos, and P. K. Townsend, Nucl. Phys. **B470**, 113 (1996).
[10] S. F. Hassan, Nucl. Phys. **B568**, 145 (2000).
[11] A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, Phys. Lett. B **249**, 35 (1990).
[12] S. J. Rey, Phys. Rev. D **43**, 526 (1991).
[13] A. Sen, Int. J. Mod. Phys. A **09**, 3707 (1994).
[14] A. Sen, Phys. Lett. B **329**, 217 (1994).
[15] J. H. Schwarz, arXiv:hep-th/9307121.
[16] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995).
[17] M. R. Garousi, J. High Energy Phys. 11 (2011) 016.
[18] M. R. Garousi, Phys. Rev. D **84**, 126019 (2011).
[19] M. R. Garousi, Nucl. Phys. **B862**, 107 (2012).
[20] M. R. Garousi, J. High Energy Phys. 04 (2012) 140.
[21] K. B. Velni and M. R. Garousi, Nucl. Phys. **B869**, 216 (2013).
[22] M. R. Garousi, J. High Energy Phys. 02 (2010) 002.
[23] K. Becker, G. Guo, and D. Robbins, J. High Energy Phys. 09 (2010) 029.
[24] M. R. Garousi, Nucl. Phys. **B852**, 320 (2011).
[25] M. R. Garousi, Phys. Lett. B **701**, 465 (2011).
[26] H. Godazgar and M. Godazgar, J. High Energy Phys. 09 (2013) 140.
[27] M. R. Garousi, A. Ghodsi, T. Houri, and G. Jafari, J. High Energy Phys. 10 (2013) 103.
[28] M. R. Garousi, arXiv:1310.7377.
[29] C. P. Bachas, P. Bain, and M. B. Green, J. High Energy Phys. 05 (1999) 011.

[30] M. R. Garousi and R. C. Myers, Nucl. Phys. **B475**, 193 (1996).
[31] M. B. Green, J. A. Harvey, and G. W. Moore, Classical Quantum Gravity **14**, 47 (1997).
[32] Y. K. Cheung and Z. Yin, Nucl. Phys. **B517**, 69 (1998).
[33] R. Minasian and G. W. Moore, J. High Energy Phys. 11 (1997) 002.
[34] B. Craps and F. Roose, Phys. Lett. B **445**, 150 (1998).
[35] J. F. Morales, C. A. Scrucca, and M. Serone, Nucl. Phys. **B552**, 291 (1999).
[36] B. J. Stefanski, Nucl. Phys. **B548**, 275 (1999).
[37] M. R. Garousi, J. High Energy Phys. 03 (2010) 126.
[38] M. R. Garousi and M. Mir, J. High Energy Phys. 05 (2011) 066.
[39] K. Becker, G. Guo, and D. Robbins, J. High Energy Phys. 12 (2011) 050.
[40] M. R. Garousi and M. Mir, J. High Energy Phys. 02 (2011) 008.
[41] K. Becker, G.-Y. Guo, and D. Robbins, J. High Energy Phys. 01 (2012) 127.
[42] W. Taylor and M. Van Raamsdonk, Nucl. Phys. **B573**, 703 (2000).
[43] A. Hashimoto and I. R. Klebanov, Phys. Lett. B **381**, 437 (1996).
[44] M. R. Garousi and R. C. Myers, Nucl. Phys. **B542**, 73 (1999).