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Asymptotic Behaviors of Nearest Neighbor Kernel Density Estimator in Left-truncated Data

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Abstract

Kernel density estimators are the basic tools for density estimation in non-parametric statistics. The k-nearest neighbor kernel estimators represent a special form of kernel density estimators, in which the bandwidth is varied depending on the location of the sample points. In this paper, we initially introduce the k-nearest neighbor kernel density estimator in the random left-truncation model, and then prove some of its asymptotic behaviors, such as strong uniform consistency and asymptotic normality. In particular, we show that the proposed estimator has truncation-free variance. Simulations are presented to illustrate the results and show how the estimator behaves for finite samples. Moreover, the proposed estimator is used to estimate the density function of a real data set.

Keywords: Asymptotic normality; Left-truncation; Nearest neighbor; Strong consistency.

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Introduction

Suppose Y and T are two continuous independent random variables with unknown cumulative distribution functions (d.f.) F and G respectively, and let $(Y_1,T_1),\ldots,(Y_N,T_N)$ be N independent and identically distributed (i.i.d.) copies of (Y,T) , where the sample size N is fixed, but unknown. In the random left-truncation (RLT) model, the random variable (r.v.) of interest Y is interfered by the truncation r.v. T , when both quantities Y and T are observable only if $Y \geq T$, whereas nothing is observed if $Y < T$. Without possible confusion, we still denote $(Y_i,T_i), i=1,\ldots,n\,(n \leq N)$, the observed i.i.d. pairs from the original N -sample. As a consequence of truncation, the size of the actual observed sample, n is a $Bin(N,\beta)$ random variable with $\beta := \mathrm{P}(Y \geq T)$. Obviously, if $\beta = 0$, no data is observed and therefore, we suppose $\beta > 0$ throughout this paper.

Truncation plays an important role in a variety of statistical applications including medicine, actuary, astronomy, demography, epidemiology, reliability testing and other studies. More examples and references dealing with truncated data can be found in Woodroofe [25], Wang et al. [24], Tsai et al. [2۲], Andersen et al. [1], He and Yang [7] and Chen et al. [3]. For instance, in medical studies, when one wants to study the length of survival after the start of the disease, if Y denotes the elapsed time between the onset of the disease and death, and if the follow-up period starts T units of time after the onset of the disease then, clearly, Y is left truncated by T. Denote by $f(\cdot)$ the probability density function of Y with respect to Lebesgue measure.

At first, some results from the literature for the univariate RLT model are presented, which will be used to define our nonparametric kernel density estimator with the nearest neighbor bandwidth. Since N is unknown and n is known (although random), our results will not be stated with respect to the probability measure P (related to the N -sample) but will involve the probability P (related to the n -sample).

Under the RLT sampling scheme, the conditional joint distribution of an observed (Y,T) (Stute, [21]), is given by

$$H^*(y,t) = \mathrm{P}\{Y \leq y, T \leq t \mid Y \geq T\}$$
$$= \beta^{-1}\int_{-\infty}^{y} G(t \wedge u)\, dF(u), \qquad (1)$$

where $t \wedge u = \min(t,u)$. The marginal distributions are defined by

$$F^*(y) := H^*(y,\infty) = \beta^{-1}\int_{-\infty}^{y} G(u)dF(u), \qquad (2)$$
$$G^*(t) := H^*(\infty,t) = \beta^{-1}\int_{-\infty}^{+\infty} G(t \wedge u)dF(u),$$

and their empirical estimators are given by

$$F_n^*(y) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \leq y) \quad and \quad G_n^*(t) = \frac{1}{n}\sum_{i=1}^{n} I(T_i \leq t),$$

respectively, where $I(\cdot)$ denotes the indicator function. Thus, F_n^* and G_n^* estimate the marginal functions F^* and G^*.

For any d.f. W, let $a_W = \inf\{x : W(x) > 0\}$ and $b_W = \sup\{x : W(x) < 1\}$ be respectively left and right endpoints of its support. Woodroofe [25] pointed out that F and G can be estimated completely only if

$$a_G \leq a_F, \quad b_G \leq b_F \quad and \quad \int_{a_F}^{\infty} \frac{dF}{G} < \infty. \qquad (3)$$

Then, under (3),

$$\beta = \mathrm{P}(Y \geq T) = \int G(u)dF(u) > 0,$$

is the truncation probability. Define

$$C(y) := G^*(y) - F^*(y) = \beta^{-1}G(y)\bar{F}(y), \quad y \in [a_F, \infty), \qquad (4)$$

where $\bar{F} = 1 - F$, and consider its empirical estimate

$$C_n(y) := G_n^*(y) - F_n^*(y^-) = \frac{1}{n}\sum_{i=1}^{n} I(T_i \leq y \leq Y_i), \quad y \in [a_F, \infty). \qquad (5)$$

Assuming no ties in the data, the nonparametric maximum likelihood estimate (NPMLE) of F is given by

$$F_n(y) = 1 - \prod_{i:Y_i \leq y}\left[1 - \frac{1}{nC_n(Y_i)}\right], \qquad (6)$$

The estimator of F_n , was derived by Lynden-Bell [11]. Asymptotic properties of (6) have also been studied by Woodroofe [25] who established the uniform consistency result

$$\sup_{y \geq a_F} |F_n(y) - F(y)| \xrightarrow{P-a.s.} 0.$$

Additional results were obtained by Keiding and Gill [8].

Loftsgaarden and Quesenberry [10] defined a very simple and useful nonparametric estimation of a density $f(x)$ based on a random sample $X_1, \ldots, X_n$. If $k(n)$ is an integer, the nonparametric estimate $f_n(x)$ of $f(x)$ is defined by

$$f_n(x) = \frac{k(n)}{2nR_n(x)},$$

where

$R_n(x)$

$= \min\{a$: there exist at least $k(n)$ of $X_1, \ldots, X_n$ in $[x-a, x+a]\}$

They showed that $f_n(x)$ converges to $f(x)$ in probability, for each x at which f is continuous and positive, if

(7)

(a) $k(n) \to \infty,$

(b) $k(n)/n \to 0, \quad as\ n \to \infty.$

Moore and Henrichon [14] showed that

$$\sup_x |f_n(x) - f(x)| \to 0,$$

in probability, if f is uniformly continuous and positive on $\Re$ and if,

$$\frac{k(n)}{\log n} \to \infty, \quad as\ n \to \infty,$$

(8)

additionally. Wagner [23] showed that $\hat{f}_n(x)$ is a strongly consistent estimate of $f(x)$ at each continuity point of f if, in addition to $(7b)$,

$$\sum_{n=1}^{\infty} \exp\{-\alpha k(n)\} < \infty \qquad for\ all\ \alpha > 0. \tag{9}$$

Notice that (10) is always implied by (8) but $(7a)$ and (9) are needed to imply (8).

Moore and Yackel [16] considered a more general class of estimators defined by

$$f_n^*(x) = \frac{1}{nR_n(x)} \sum_{j=1}^{n} K\left(\frac{x - X_j}{R_n(x)}\right),$$

where $k(\cdot)$ is a bounded kernel function on $\Re$ and $R_n(x)$ is the Euclidean distance between x and the $k(n)$th nearest neighbor of x among the X_j's. Moore and Yackel [16] showed that, in general, any consistency result for the kernel estimator with bandwidth h_n remains correct for the nearest neighbor estimator with the same kernel and $k(n) = \alpha n h_n$ for any $\alpha > 0$. Mack and Rosenblatt [12] treated the problem of the optimum choice of $k(n)$ for different criteria. Based on the paper [12], Orava [18] has derived a practical method that could be used for choosing $k(n)$. Biau et al. [2] have introduced a weighted version of the *k*-nearest neighbor density estimate. They also establish some limit theorems of this estimator, such as pointwise consistency and the central limit theorem. In addition, they obtain strong approximation for their estimator. Furthermore, Ouadah [19] has proved a uniform-in-bandwidth limit law for the nearest-neighbor density estimator.

Under the right censorship model, Mielniczuk [13] based on Kaplan-Meier estimator, introduced the $k(n)$th nearest uncensored neighbor estimator. He also established strong uniform consistency (under Assumption (8)) and asymptotic normality of his proposed estimator. Furthermore, as indicated in Mielniczuk [13], the asymptotic variance of the $k(n)$th nearest uncensored neighbor estimator is "censor-free". It should be mentioned that the strong uniform consistency of $f(x)$ can be established under the weaker condition than (8) i.e.,

$$\frac{k(n)}{\log\log n} \to \infty,$$

as $n \to \infty$. (see, Moore and Yackel, [16]).

Many authors have investigated the asymptotic properties of nearest neighbor estimators under dependent samples. Yang [26] proved the consistency of f_n for samples based on negatively associated r.v., also Csörgö and Szyszkowicz [4] established an invariance principle of f_n* for long-rang dependent samples. Consistency and asymptotic normality of f_n based on strong mixing assumption were investigated by Yanyan and Yanli [27].

In this article, we introduce a $k(n)$th nearest truncated neighbor estimator of f that is based on the Lynden-Bell estimator:

$$f_n(y) = \frac{1}{R_n(y)} \int_{\Re} K\left(\frac{y-u}{R_n(y)}\right) dF_n(u), \qquad (10)$$

where K is a kernel function, $R_n(y)$ is the distance from y to its $k(n)$th nearest truncated neighbor and $k(n)$ is a given sequence of integers such that $k(n) \to \infty$ and $k(n)/n \to 0$, as $n \to \infty$.

We show some properties of the estimator (10) which might be deduced from the properties of classic kernel estimators when the observations are not truncated. We will make use of the assumptions gathered together hereafter for easy reference. In what follows, we suppose that $a_G < a_F, b_G \le b_F$.

Assumptions

A1: $\{Y_j; j \ge 1\}$ is a sequence of i.i.d. interesting variables with continuous d.f. F and density function f.

A2: $\{T_j; j \ge 1\}$ is a sequence of i.i.d. truncating variables with common continuous d.f. G, density function g and are independent from $\{Y_j; j \ge 1\}$.

A3: (i) K is a bounded kernel function with support in $[-1,1]$.

(ii) $K(cu) \ge K(u)$ for any $0 \le c \le 1$.

A4: Let fG be continuous and positive on $[a-\varepsilon, b+\varepsilon]$ for some $\varepsilon > 0$, where $a_F < a-\varepsilon < b+\varepsilon < b_F$ and g is continuous on $[a-\varepsilon, b+\varepsilon]$.

A5: The sequence $k(n)$ satisfies

(i) $k(n) \uparrow \infty$, and $k(n)/n \to 0$ as $n \to \infty$,

(ii) $k(n)/\log n \to \infty$ as $n \to \infty$,

(iii) $k(n)/\log\log n \to \infty$ as $n \to \infty$,

(iv) $\sum_{n=1}^{\infty} \exp(-ck(n)) < \infty$ for any $c > 0$.

The rest of the paper is as follows. In the next section, we give the main results. Some simulations are drawn to grant further support of our theoretical results regarding the consistency as well as the asymptotic normality. Proofs of the main results are deferred to the Appendix.

Results

1.1. Strong consistency

Following Moore and Yackel [16], for any fixed

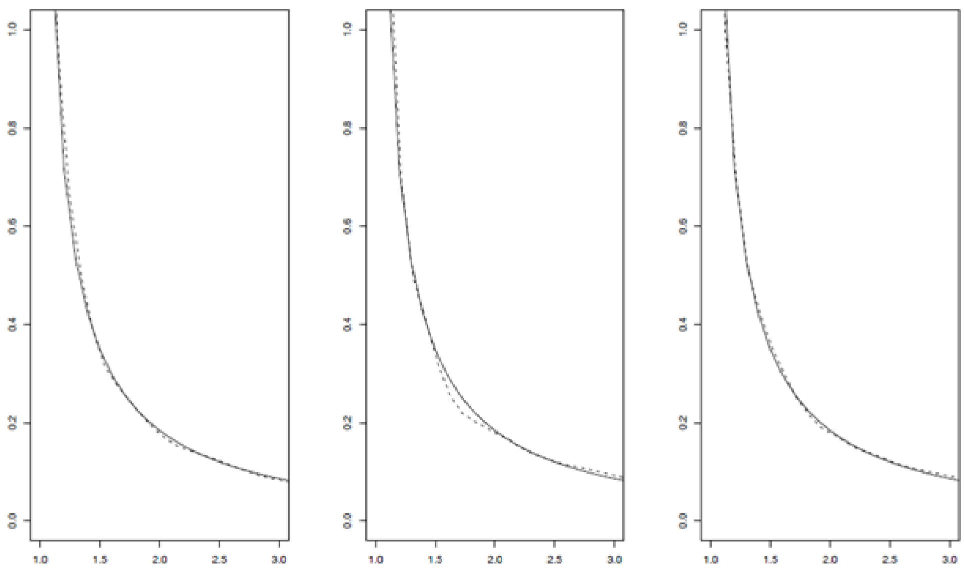

Figure. 1. True density, black line and its estimates f_n, dashed line, for Weibull distribution with $k = 0.5$, $\beta = 0.6, 0.75, 0.9$, respectively.

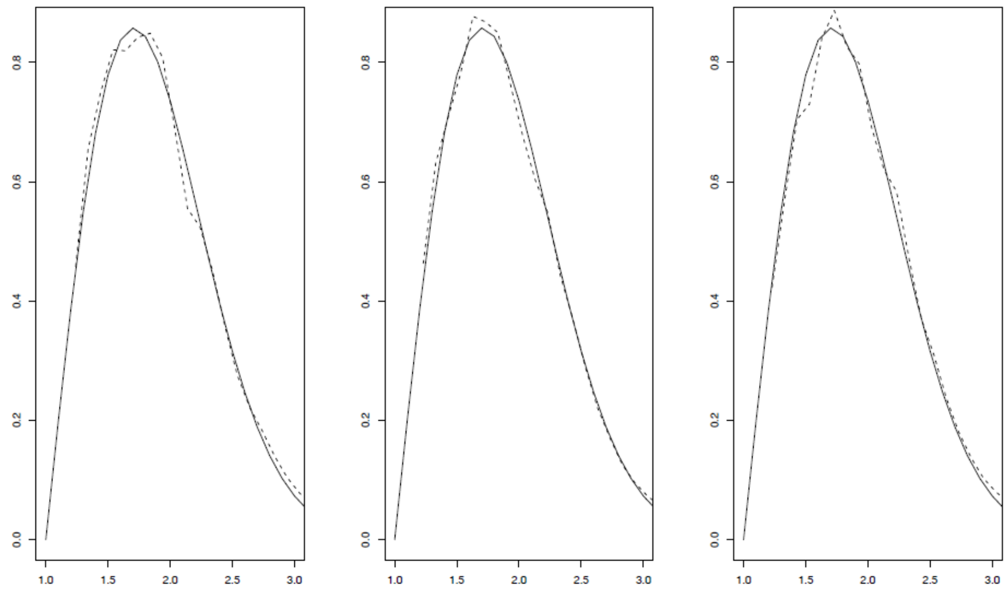

Figure. 2. True density, black line and its estimates f_n, dashed line, for Weibull distribution with $k = 2$, $\beta = 0.6, 0.75, 0.9,$ respectively.

sequence $k(n)$, we consider an arbitrary consistency result holding for the estimator with kernel K (that satisfies Assumption A3) and bandwidth $h_n = k(n)/n.$ Then this result holds for the nearest neighbor estimator with kernel K and the bandwidth based on $k(n)$. The only prerequisite for this argument is that the conditions on h_n must also be satisfied by αh_n for any $\alpha > 0$.

Theorem 1. *Under Assumptions* ***A1-A4*** *and* ***A5(i), (iii) , (iv),*** *for* $a \le y \le b$, *we have*

$$\lim_{n\to\infty}[f_n(y) - f(y)] = 0 \quad a.s. \tag{11}$$

Proof. **See Appendix.**

In what follows, we prove strong uniform consistency of the nearest neighbor density estimator in the RLT model.

Theorem 2. *Under Assumptions* ***A1-A4*** *and* ***A5 (i),(ii),***

$$\lim_{n\to\infty}\sup_{a\le y\le b}|f_n(y) - f(y)| = 0 \quad a.s. \tag{12}$$

Proof. **See Appendix.**

1.2. Asymptotic Normality

Let f^* be a density function of F^* and

$$\tilde{f}_n(y) = \frac{1}{R_n(y)}\int_{\Re} K\left(\frac{y-u}{R_n(y)}\right)dF_n^*(u).$$

To state and prove the asymptotic normality, we need the following assumption on density function of observed data.

A6: $(k(n))^{1/2}\left(f^*(x_n) - f^*(x)\right) \to 0,$ in probability, when $|x_n - x| = O(k(n)/n).$

Theorem 3. *Under Assumptions* ***A1-A4, A5 (i)-(ii)****, and* ***A6,*** *for* $a \le y \le b$, *we have*

$$(k(n))^{1/2}\left(f_n(y) - f(y)\right) \xrightarrow{D} N(0, \sigma^2(y)), \tag{13}$$

where

$$\sigma^2(y) = 2f^2(y)\int_{\Re} K^2(y)dy.$$

Proof. See the Appendix.

Remark 1. *We observe that asymptotic variance of* f_n *does not depend on truncation distribution.*

Remark 2. *A sufficient condition for Assumption* ***A6*** *to be true is that* f^* *has an abounded derivative in a neighborhood of* $a \le y \le b,$ *in turn is satisfied when* $k(n) = o(n^{2/3}).$ *This suboptimal choice for* $k(n)$ *was used by Mielniczuk [1 r] in the random censorship model.*

Remark 3. *In complete data, the best rate for* $k(n)$ *is obtained about* $n^{4/5}$ *(in MSE sense, see e.g., Mack and Rosenblatt, [12]).*

Corollary 1. *It is possible to construct confidence interval for* f *using Theorem 3. For that purpose, a plug-in estimate*

$$\sigma_n^2(y) := 2 f_n^2(y) \int_{\Re} K^2(y)dy,$$

for the asymptotic variance $\sigma^2(y)$ can be easily obtained using (10). This is a consistent estimator and yields a confidence interval of asymptotic level $1-\alpha,$ namely,

$$\left[f_n(y) - \left(\sigma_n^2(y)/k(n)\right)^{1/2} \times z_{1-\alpha/2},\ f_n(y) + \left(\sigma_n^2(y)/k(n)\right)^{1/2} \times z_{1-\alpha/2} \right] \quad (14)$$

where $z_{1-\alpha/2}$ denotes the $(1-\alpha/2)-$quantile of the standard normal distribution.

Application

This section has two parts. The first part shows the behavior of the proposed estimator for finite samples and the second one deals with the density estimator and confidence bound for real data.

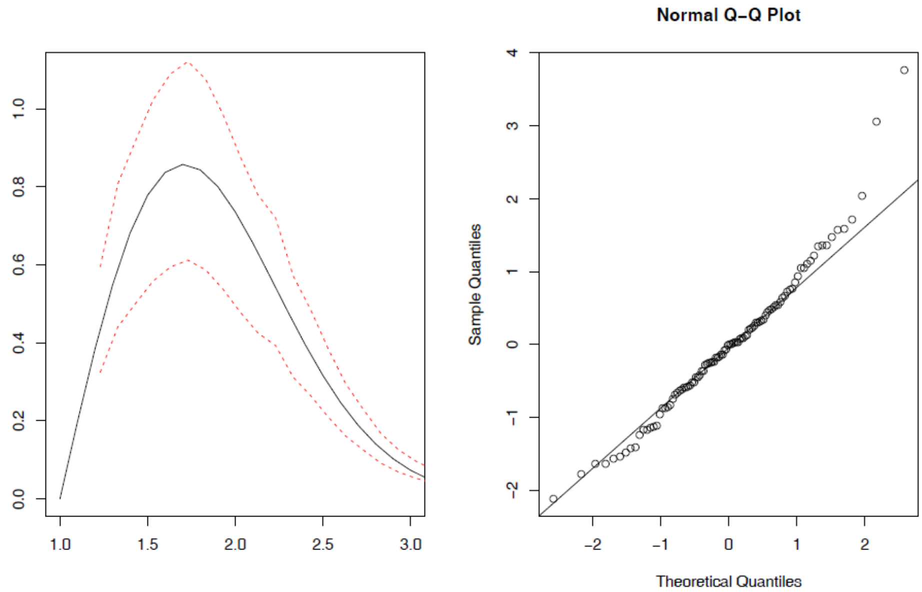

Figure. 5. Confidence bound and Q-Q plot, for Weibull distribution with $k = 2$, $\beta = 0.75$.

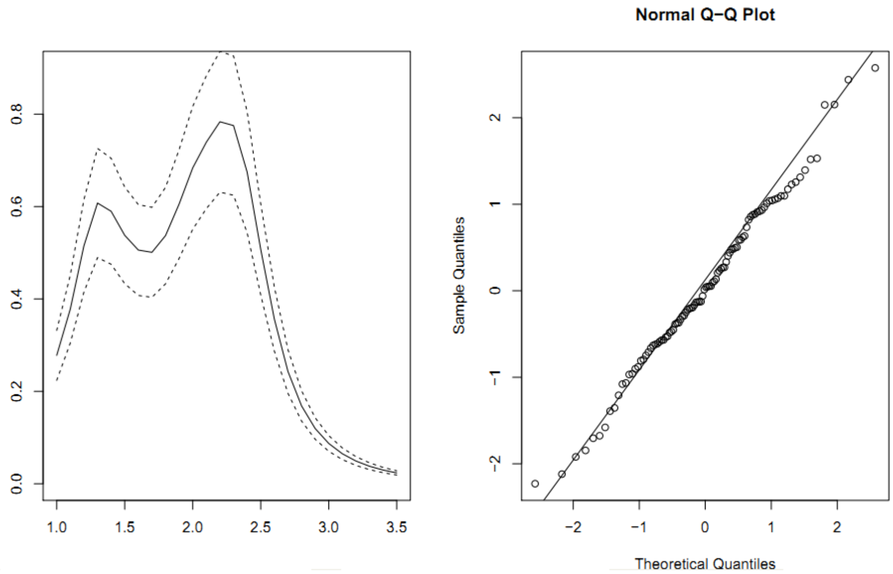

Figure. 6. Confidence bound and Q-Q plot, for mixture distribution, $\beta = 0.75$.

1. Simulation

To show the performance of the proposed estimator, we present simulated models to compute the estimator $f_n(x)$ that is presented in (10). We simulated N $i.i.d.$ random variables (Y_i, T_i). Here it is assumed that truncating variable T is distributed as an exponential random variable with parameter λ. The exponential parameter is needed to obtain different values of the theoretical proportion. The variable Y is distributed as a Weibull distribution with density

$$f(y;k,\delta) = k\delta(y-1)^{k-1}\exp(-\delta(y-1)^{k-1})\, y > 1,$$

for $k = 0.5, 2$ and $\delta = 1$,

and, a mixture distribution with the following density function

$f(y) = \frac{1}{2} f(y; 1.4,2) + \frac{1}{2} f(y; 5,0.25)$, y>1.

We then kept the data (Y_i, T_i) such that $Y_i \geq T_i$. Using this scheme, $m = 10$ independent samples of size n were generated. For each sample, plug-in estimates σ_n and $f_n(\cdot)$ for $\sigma(\cdot)$ and $f(\cdot)$, were used respectively. The following figures represent the average of m density estimations and their the confidence bounds. The kernel function

$$K(x) = \frac{70}{81}\left(1 - |x|^3\right)^3 I_{(|x| \leq 1)}(x), \qquad (15)$$

where $I_A(x)$ is indicator function for set A, is used to construct a density estimator. It should be noted that the applied kernel (15) satisfies the Assumption **A3.** In the case where $n = 500$, we give the confidence bound and Q-Q plot.

As one can see from Figures 1, 2 and 3, the quality of the estimator does not seem to be affected by β. By employing (13), 95% asymptotic confidence band for the true Weibull distributions and mixture model are constructed and plotted in Figures 4, 5 and 6.

According to the Q-Q- normal plots, we trivially notice that

$$\frac{(k(n)^{1/2}(f_n(2) - f(2))}{\sigma_n(2)}$$

has asymptotically standard Normal distribution. Furthermore Kolmogorov-Smirnov test gives the p-values 0.59, 0.7892, 0.9306 respectively, for Weibull distributions with k=0. 5, 2 and mixture distribution, which suggests not to reject the Normality distribution.

2. Real Data

In this subsection, a real data set of length-biased lifetimes with a size of 98 is used in order to estimate the density function $f(\cdot)$. These are real data that are mentioned here from [9] and are recruited from brake pads in 1000 kilometer units. It should be mentioned that the length-biased data are special cases of left-truncation data when T has a uniform distribution. We regenerated these data 10 times and the average of 10 density estimators is obtained. The estimator is graphed in Figure 7 in addition to the 95% confidence band for the true density. This confidence band is formulated in (14). It should be mentioned that these data appear to be distributed as Gamma distribution.

Conclusion

In this paper, a nearest neighbor kernel density estimator is proposed for the density function in the Left-truncation model. Uniform strong consistency and asymptotic normality of the proposed estimator is established. The performance of the estimator is illustrated through simulation studies. All simulations are drawn for different cases to demonstrate both consistency, and asymptotic normality and the method

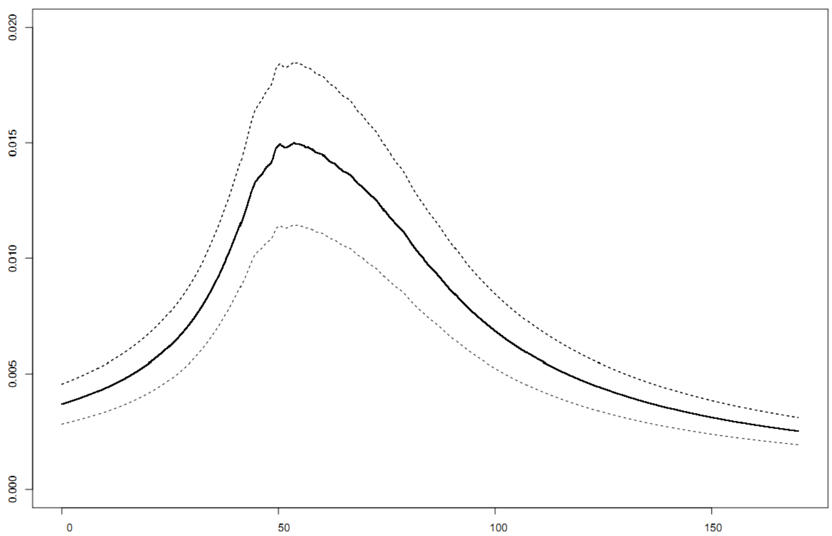

Figure. 7. Kernel density estimator and confidence bounds for the lifetime of automobile brake pads.

is illustrated by real automobile brake pad data.

Appendix

In order to make the proofs easier, we need some auxiliary results and notation.

Lemma 1. *Under Assumptions **A1, A2, A4** and **A5 (i) ,(iii),** for* $a \le y \le b$, *we have*

$$\frac{k(n)}{2nR_n(y)} \to \frac{f(y)G(y)}{\beta} \quad a.s. \tag{16}$$

Proof. Since distribution F^* has a density which is continuous at point y, by Theorem 1 of Moore and Yackel [15] we obtain the result.

Lemma 2. *Under Assumptions **A1-A2, A3(i), A4** and **A5(i)** , **(iii),** for* $a \le y \le b$, *we have*

$$f_n(y) - \frac{\beta}{G(y)}\tilde{f}_n(y) = O\left(\log n\sqrt{\frac{\text{loglog } n}{n}}\right) + O\left(\frac{k(n)}{n}\right) \quad a.s.$$

Proof. Let $S(y,r) = \{x : |x-y| \le r\}$, we have

$$|f_n(y) - \frac{\beta}{G(y)}\tilde{f}_n(y)| = \left|\frac{1}{R_n(y)}\sum_{i=1}^{n} K\left(\frac{y-Y_i}{R_n(y)}\right)[a_n(Y_i) - \frac{\beta}{G(y)n}]\right|$$

$$\le \left(\sup_y K(y)\right)\frac{k(n)}{nR_n(y)} \max_{Y_i \in S(y,R_n(y))} |na_n(Y_i) - \frac{\beta}{G(y)}|, \tag{17}$$

where $a_n(Y_i)$ is the value of the jump of the Lynden-Bell estimator in Y_i, that is

$$a_n(Y_i) = F_n(Y_i) - F_n(Y_i^-) = \frac{1}{nC_n(Y_i)}\bar{F}_n(Y_i^-), \tag{18}$$

where $\bar{F}_n = 1 - F_n$, and $F_n(y^-) = \lim_{t \to y^-} F_n(t)$ (cf. Woodroofe, [25]).

Now, using Lemma 1 it is enough to show that

$$\max_{Y_i \in S(y,R_n(y))} |na_n(Y_i) - \frac{\beta}{G(y)}| = O\left(\log n\sqrt{\frac{\text{loglog } n}{n}}\right) + O\left(\frac{k(n)}{n}\right) \quad a.s. \tag{19}$$

By using (18), we have

$$\max_{Y_i \in S(y,R_n(y))} |na_n(Y_i) - \frac{\beta}{G(y)}| \le \max_{Y_i \in S(y,R_n(y))} |\frac{1}{C_n(Y_i)}\bar{F}_n(Y_i^-) - \frac{1}{C(Y_i)}\bar{F}_n(Y_i^-)|$$

$$+ \sup_{\upsilon \in S(y,R_n(y))} |\frac{1}{C(\upsilon)}\bar{F}_n(\upsilon) - \frac{1}{C(\upsilon)}\bar{F}(\upsilon)|$$

$$+ \sup_{\upsilon \in S(y,R_n(y))} \beta |\frac{1}{G(y)} - \frac{1}{G(\upsilon)}| =: I_1 + I_2 + I_3. \tag{20}$$

It is easy to see that

$$I_1 \le \sup_{\upsilon} |C_n(\upsilon) - C(\upsilon)| \left(\inf_{a \le y \le b} C(y)\right)^{-2} \max_{Y_i \le y + R_n(y)} \frac{C(Y_i)}{C_n(Y_i)}.$$

Note that $\inf_{a \le y \le b} C(y) > 0$. By (4), (5) and the classical Law of the Iterated Logarithm for empirical processes, (see for example [5]) we have

$$\sup_{\upsilon} |C_n(\upsilon) - C(\upsilon)| = O\left(\sqrt{\frac{\text{loglog } n}{n}}\right), \quad a.s.$$

hence, by Corollary 1.3 of Stute [21]

$$I_1 = O\left(\log n\sqrt{\frac{\text{loglog } n}{n}}\right), \quad a.s. \tag{21}$$

To deal with I_2, by Corollary 2.2. of Zhou and Yip [28] and for large enough n, we have

$$I_2 \le \left(\inf_{a \le y \le b} C(y)\right)^{-1} \sup_{a_F < \upsilon < y + R_n(y)} |\bar{F}_n(\upsilon) - \bar{F}(\upsilon)|$$

$$= O\left(\sqrt{\frac{\text{loglog } n}{n}}\right), \quad a.s. \tag{22}$$

Since I_3 is majorized by

$$\frac{G(y + R_n(y)) - G(y - R_n(y))}{(G(y - R_n(y)))^2},$$

then it follows from

$$\frac{G(y + R_n(y)) - G(y - R_n(y))}{R_n(y)} \to 2g(y), \quad a.s.$$

and

$$\frac{nR_n(y)}{k(n)} = O(1), \quad a.s.,$$

that

$$I_3 = O\left(\frac{k(n)}{n}\right), \quad a.s. \tag{23}$$

Now using (20)-(23), we obtain (19), and we get the result.

The following lemma can be easily obtained by the main result of Devroye and Wagner [6].

Lemma 3. *Under Assumptions **A1, A2, A4** and **A5 (i), (ii)***

$$\sup_{a\le y\le b} \left| \frac{k(n)}{nR_n(y)} - \frac{2f(y)G(y)}{\beta} \right| \to 0 \quad a.s. \qquad (24)$$

Proof of Theorem 1. Using Lemma 2, and the result of Nadaraya [17] we obtain the result.

Proof of Theorem 2. First, we show that the strong convergence in Lemma 2 can be replaced by uniform strong convergence on $[a,b]$. To prove this, it is enough considering the last term of the majorant occurring in the proof of Lemma 2 and to show that

$$\sup_{a\le y\le b} G(y+R_n(y)) - G(y-R_n(y)) = O\left(\frac{k(n)}{n}\right) \quad a.s.$$

We have

$$\sup_{a\le y\le b} G(y+R_n(y)) - G(y-R_n(y))$$

$$= \sup_{a\le y\le b} \frac{k(n)}{n} \frac{nR_n(y)}{k(n)} \frac{G(y+R_n(y)) - G(y-R_n(y))}{R_n(y)}$$

Since by Lemma 3, $\sup_y R_n(y)$ on $[a,b]$ tends to 0 a.s. and g is uniformly continuous, we have

$$\sup_{a\le y\le b} \left| \frac{G(y+R_n(y)) - G(y-R_n(y))}{R_n(y)} - 2g(y) \right| \to 0 \quad a.s.$$

Thus the proof of Theorem 1 is completed in view of Lemma 3, Theorem A of Silverman [20] and the fact that $f.G$ is positive on $[a,b]$. ■

Proof of Theorem 3. Observe that for $\tilde{f}_n(y)$, we have

$$(k(n))^{1/2}\left(\tilde{f}_n(y) - \frac{f(y)G(y)}{\beta}\right) \xrightarrow{D} N\left(0, 2\left(\frac{f(y)G(y)}{\beta}\right)^2 \int_{\Re} K^2(y)dy\right) \qquad (25)$$

(Moore and Yackel [15]). (13) follows from (25) and Lemma 2.

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