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Using Neural Network Approach for Bifurcation Analysis of a Reaction-diffusion System

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Abstract. In this paper, we consider a reaction-diffusion system which is a cellular nonlinear model. Our aim is to study the system near the origin as the parameters are varying. For this purpose, we use neural network method for transforming the reaction-diffusion PDE to an ODE.

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1. Introduction

Reaction-diffusion systems are mathematical models which explain how the concentration of one or more substances distributed in space changes under the influence of two processes. These systems are naturally applied in chemistry. However, the system can also describe dynamical processes in biology, geology, physics and ecology.

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In [1], the edge of chaos phenomena for the following partial differential equation (PDE), generally referred in the literature as a reaction-diffusion equation, was studied:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u,v) \\ \frac{\partial v}{\partial t} = g(u,v) \end{cases} \tag{1.1}$$

where

$$\begin{cases} f(u,v) = -c_1\frac{u}{1+u^2} + b_1\frac{u}{(1+u^2-u)(1+v)} \\ g(u,v) = -c_2\frac{v}{1+v^2} + b_2\frac{v}{(1+v^2-v)(1+u)}. \end{cases}, \tag{1.2}$$

where c_1 and c_2 are parameters, while $b_i > 0$ for $i = 1, 2$, are constants. However, in this paper, we devote our attention to bifurcation analysis of system (1.1). In fact, we discuss the behavior of system (1.1) near their equilibria by using the bifurcation theory.

Since existence and behavior of the solutions of (1.1) are very difficult to establish, many aspects of qualitative behavior have to be investigated numerically. For this purpose, we use the numerical method proposed in [2] to result an ordinary differential equation (ODE) system that its dynamics are equivalent to the dynamics of (1.1). After that, the bifurcation analysis will be presented.

2. Main results

The reaction-diffusion system (1.1) has some difficulties in considering the equilibrium points, because of the zeros in the denominators. So to simplify (1.1), first, we introduce the variables $\tau = \int_0^t \frac{1}{1+v}dt$ and $\theta = \int_0^\tau \frac{1}{1+u}d\tau$, applying them to (1.1) and replacing θ by t, we have

$$\begin{cases} \frac{\partial u}{\partial t} = (1+u)(1+v)\frac{\partial^2 u}{\partial x^2} - c_1\frac{u(1+u)(1+v)}{1+u^2} + b_1\frac{u(1+u)}{1+u^2-u} \\ \frac{\partial v}{\partial t} = -c_2\frac{v(1+v)(1+u)}{1+v^2} + b_2\frac{v(1+v)}{1+v^2-v}. \end{cases} \tag{2.1}$$

In order to transform the system (2.1) to an ODE system, we apply a neural network (NN) approach. Let

$$u(x,t) = w_1(t)\sigma_1(x), \quad v(x,t) = w_2(t)\sigma_2(x) \tag{2.2}$$

where $w_j(t)$ $(j = 1, 2)$ are NN weights and $\sigma_j(x)$ $(j = 1, 2)$ are NN activation functions. The $\sigma_j(x)$ $(j = 1, 2)$ can be assumed to be orthogonal basis functions. In this paper, we choose chebyshev polynomials as activation functions. Applying (2.2) and their derivatives to (2.1), we can state the following lemma:

Lemma 2.1. *Suppose $\sigma_1(x) = \sigma_2(x) = 2x^2 - 1$, then the dynamics of (2.1) are equivalent to the dynamics of the following system:*

$$\begin{cases} \dot{w}_1(t) = P(w_1(t), w_2(t)) - c_1R(w_1(t), w_2(t)) + b_1Q(w_1(t)) \\ \dot{w}_2(t) = -c_2S(w_1(t), w_2(t)) + b_2G(w_2(t)) \end{cases} \tag{2.3}$$

where

$$P(w_1(t), w_2(t)) = 4w_1^2(t) + 4w_1(t)w_2(t) - 2w_1^2(t)w_2(t),$$

$$R(w_1(t), w_2(t)) = \frac{2}{3}\Big(\frac{2w_1(t) + w_1^2(t) + w_1(t)w_2(t) + \frac{w_1^2(t)w_2(t)}{2}}{4 + w_1^2(t)} + \frac{w_1(t) - w_1^2(t) - w_1(t)w_2(t) + w_1^2(t)w_2(t)}{1 + w_1^2(t)}\Big),$$

$$Q(w_1(t)) = \frac{2}{3}\Big(\frac{2w_1(t) + w_1^2(t)}{4 + w_1^2(t) - 2w_1(t)} + \frac{w_1(t) - w_1^2(t)}{1 + w_1^2(t) + w_1(t)}\Big),$$

$$S(w_1(t), w_2(t)) = \frac{2}{3}\Big(\frac{2w_2(t) + w_2^2(t) + w_1(t)w_2(t) + \frac{w_1(t)w_2^2(t)}{2}}{4 + w_2^2(t)} + \frac{w_2(t) - w_2^2(t) - w_1(t)w_2(t) + w_1(t)w_2^2(t)}{1 + w_2^2(t)}\Big),$$

$$G(w_2(t)) = \frac{2}{3}\Big(\frac{2w_2(t) + w_2^2(t)}{4 + w_2^2(t) - 2w_2(t)} + \frac{w_2(t) - w_2^2(t)}{1 + w_2^2(t) + w_2(t)}\Big).$$

Proof. For the proof, it should be noted that the neural network weights $w_j(t)\,(j = 1, 2)$ will be estimated in a way to minimize the residual error in a least-squares sense over a set of points within the region Ω. By applying the inner product $\langle ., \sigma_j(x)\rangle\,(j = 1, 2)$ and approximating them in such a way stated in Theorem 2.4 in [4], we can get system (2.3) which is an ODE. □

Now, we discuss the dynamics of the ODE system (2.3) by using the bifurcation theory [3]. The origin $(0, 0)$ is a trivial equilibrium for (2.3).

Theorem 2.2. *$c_1 = b_1$ and $c_2 = b_2$ are the bifurcation lines of* (2.1)*, and we have the following results:*
(i) If $b_1 < c_1$ and $b_2 < c_2$, then $(0, 0)$ is a sink.
(ii) If $b_1 > c_1$ and $b_2 < c_2$, or if $b_1 < c_1$ and $b_2 > c_2$ then $(0, 0)$ is a saddle point.
(iii) If $b_1 > c_1$ and $b_2 > c_2$ then $(0, 0)$ is a source for system (2.1)*.*

Proof. Using Lemma 2.1, we can see that system (2.1) is equivalent to system (2.3), and so we prove the results for system (2.3). Considering the right hand function as $f(w_1, w_2)$, it is easy to see that $\lambda_1 = -c_1 + b_1$ and $\lambda_2 = -c_2 + b_2$ are the eigenvalues of $Df(0, 0)$. Now, for the case (i), we have $\lambda_1, \lambda_2 < 0$, while in the case (ii) $\lambda_1 > 0$, $\lambda_2 < 0$ or $\lambda_1 < 0$, $\lambda_2 > 0$. However, for the case (iii), we have $\lambda_1, \lambda_2 > 0$. Then, using the bifurcation theory establishes the claim. □

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