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# Exponentiality Test Based on the Progressive Type II Censoring via Cumulative Entropy

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*In this article, we use cumulative residual Kullback-Leibler information (CRKL) and cumulative Kullback-Leibler information (CKL) to construct two goodness-of-fit test statistics for testing exponentiality with progressively Type-II censored data. The power of the proposed tests are compared with the power of goodness-of-fit test for exponentiality introduced by Balakrishnan et al. (2007). We show that when the hazard function of the alternative is monotone decreasing, the test based on CRKL has higher power and when the hazard function of the alternative is non-monotone, the test based on CKL has higher power. But, when it is monotone increasing the power difference between test based on CKL and their proposed test is not so remarkable. The use of the proposed tests is shown in an illustrative example.*

**Keywords** Cumulative residual entropy; Exponential distribution; Kullback-Leibler divergence; Maximum entropy; Power study.

**Mathematics Subject Classification** 62G10; 62E10; 94A17; 65C05.

## 1. Introduction

In the context of probability theory, entropy describes the amount of uncertainty associated with a random variable. Entropy as a baseline concept in the field of information theory, was introduced by Shannon (1948). For a non-negative absolutely continuous random variable $X$, Shannon entropy called differential entropy, is defined as

$$H(X) = -\int_0^\infty f(x) \ln f(x) \mathrm{d}x,$$

where "ln″ means natural logarithm and $f(x)$ is the probability density function (pdf) of $X$. Recent years have witnessed a growing interest in utilizing information-theoretic measures for distributional disparities as a tool for statistical inference in a variety of fields. For testing problems, the earliest work dates back to Vasicek (1976) which used Shannon maximum entropy to construct a goodness-of-fit test for normality. Vasicek approach has much affected the development of entropy-based tests of fit for several parametric models; for example, see Grzegorzewski and Wieczorkowski (1999), Taufer (2002), and Alizadeh Noughabi and Arghami (2011). In probability theory and information theory, the Kullback-Leibler (KL) divergence [Kullback and Leibler (1951) and Kullback (1959, 1987)] is a

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non-symmetric measure of the difference between two distributions $F(x)$ and $G(x)$ which is defined as follows:

$$\mathrm{KL}(F:G)=\int_{-\infty}^{\infty} f(x)\ln\frac{f(x)}{g(x)}\mathrm{d}x,$$

where $f(x)$ and $g(x)$ are pdfs of distributions $F(x)$ and $G(x)$, respectively. Tests of fit based on KL information have been developed; see Ebrahimi et al. (1992), Choi et al. (2004), and Gurevich and Davidson (2008).

Another type of entropy, the cumulative residual entropy (CRE) was introduced by Rao et al. (2004) in order to provide a way to accommodate random variables that do not have a defined density function. This measure dose not have limitations of the use of Shannon entropy in measuring the randomness of certain systems (see also Rao, 2005) and is based on the complementary cumulative distribution function (ccdf), $\bar{F}(x)=1-F(x)$, which in the reliability is called survival function. This measure is defined for the non-negative random variables as follows:

$$CRE(X)=-\int_{0}^{\infty}\bar{F}(x)\ln\bar{F}(x)\mathrm{d}x.$$

Asadi and Zohrevand (2007) proposed a dynamic form of CRE and obtain some of its properties (also see Navarro et al., 2010). Sunoj and Linu (2012) introduced a generalized measure of dynamic form of CRE, namely cumulative residual Renyis entropy, and studied its properties. Baratpour (2010) characterized the first order statistics based on the CRE.

Baratpour and Habibi Rad (2011) defined a new measure of distance between two non-negative and continuous distributions based on CRE, called cumulative residual Kullback-Leibler (CRKL) divergence and construct a goodness-of-fit test for exponentiality. They proved that CRKL is non-negative and equality holds if and only if $F(x)=G(x)$, *a.e.* This measure is defined as follows:

$$CRKL(F:G)=\int_{0}^{\infty}\bar{F}(x)\ln\frac{\bar{F}(x)}{\bar{G}(x)}\mathrm{d}x-[E(X)-E(Y)],$$

where $\bar{F}(x)$ and $\bar{G}(x)$ are ccdf of $X$ and $Y$, respectively.

Di Crescenzo and Longobardi (2009) introduced and studied the cumulative entropy (CE) which is suitable to measure information when uncertainty is related to the past, a dual concept of CRE which relates to uncertainty on the future lifetime of a system. For the non-negative and continuous random variables $X$ with distribution function $F(x)$, CE is defined as

$$CE(X)=-\int_{0}^{\infty}F(x)\ln F(x)\mathrm{d}x.$$

Park et al. (2012) considered another extension to the cumulative distribution, called cumulative KL information (CKL) which is defined as

$$CKL(F:G)=\int_{0}^{\infty}F(x)\ln\frac{F(x)}{G(x)}\mathrm{d}x-[E(Y)-E(X)].$$

By noting that $\ln x\le x-1, x>0$ and equality holds if and only if $x=1$, we conclude that $CKL(F:G)\ge 0$ and equality holds if and only if $F=G$, *a.e.*

In many life-testing and reliability studies, the experimenter may be unable to obtain complete information on failure times for all experimental units. There are also situations wherein the removal of units prior to failure is preplanned in order to reduce the cost and time associated with testing. For these and other reasons, progressive censoring has been discussed by Nelson (1982).

A progressively Type-II censored data arises in the following manner. $n$ units are placed on a life-testing experiment and when the first failure occurs, $R_1$ of the $n-1$ surviving units are withdrawn from the experiment. When the next failure occurs, $R_2$ of the $n-2-R_1$ surviving units are withdrawn from the experiment, and so on. Finally, at the time of the $m$-th failure, all the remaining $R_m = n-m-R_1-\cdots-R_{m-1}$ surviving units are withdrawn from the experiment. It is clear that when we set $m = n$ and all $R_i = 0$, we obtain the complete sample situation; when we set $R_1 = \cdots = R_{m-1} = 0$ and $R_m = n-m$, we obtain the Type-II censored sample. Progressive censoring and its applications have been discussed by a number of authors including Viveros and Balakrishnan (1994), Balasooria and Balakrishnan (2000), Ng et al. (2002), and Balakrishnan (2007). A book-length account is available in Balakrishnan and Aggarwala (2000). The goodness-of-fit test based on progressively Type-II censored sample is widely used as a tool for testing distributional hypotheses. Some key references are Marohn (2002), Balakrishnan et al. (2002, 2004), and Wang (2008). Also, entropy/Kullback-Leibler information with progressively Type-II censored order statistics was investigated by Cramer and Bagh (2011).

The goal of this article is to propose CE tests of fit for the exponential distribution with progressively Type-II censored data. Two test statistics are derived from CRKL and CKL, and the powers of them are studied. The article is organized as follows: In Section 2, based on the CRKL and CKL and using progressively Type-II censored data, two test statistics are constructed. In Section 3, we obtain the power of the proposed tests by Mont Carlo simulation. We show that when the alternative has monotone decreasing hazard function, the test based on CRKL has good power and when the alternative has monotone increasing or non-monotone hazard function, the test based on CKL has high power. The use of the proposed tests is illustrated in Section 4.

## 2. The Test Procedure

In this section, we construct two test statistics for testing exponentiality versus some alternatives. These test statistics are based on CRKL and CKL which are EDF test statistics. The EDF tests are in term of some distance measures between the empirical distribution function $F_n(x)$ and $F_{\hat{\theta}}(x)$, where $\hat{\theta}$ denotes a consistent estimator of $\theta$ under $H_0$. Large values of this measure will lead to the rejection of $H_0$.

Let $x_{1;m:n} < x_{2;m:n} < \ldots < x_{m;m:n}$ be a progressively Type-II right censored sample with progressive censoring scheme $R = (R_1, R_2, \ldots, R_m)$ from a continuous distribution function $F(x)$ and $F_0(x) = 1 - \exp(-\frac{x}{\theta})$, $x > 0, \theta > 0$, denotes exponential distribution function, where $\theta$ is the unknown parameter. The aim of this article is testing the hypothesis

$$H_0 : F(x) = F_\theta(x), vs. H_a : F(x) \neq F_\theta(x),$$

based on the $x_{1;m:n} < x_{2;m:n} < \ldots < x_{m;m:n}$ and using CRKL and CKL information. We use the empirical distribution function $F_{m:n}(x)$ for the estimation of distribution function $F(x)$. The empirical distribution function $F_{m:n}(x)$ is given by

$$\begin{aligned} F_{m:n}(x) &= 0, \quad x < x_{1:m:n} \\ &= \alpha_{i:m:n}, \quad x_{i:m:n} \leq x < x_{i+1:m:n}, i = 1, \ldots, m-1 \\ &= \alpha_{m:m:n}, \quad x \geq x_{i:m:n}, \end{aligned}$$

**Table 1**
Power of the $T_1$ test for monotone decreasing hazard alternatives at 5% and 10% significance levels for several progressively censored schemes when the sample sizes are $n = 10, 20$, and 30

| | | schemes | 5% significance level | | | 10% significance level | | |
|---|---|---|---|---|---|---|---|---|
| n | m | $(R_1, \cdots, R_m)$ | G(0.5) | W(0.5) | LN(2) | G(0.5) | W(0.5) | LN(2) |
| 10 | 5 | 5,0,0,0,0 | .328 | .422 | .222 | .457 | .550 | .309 |
| | 5 | 0,5,0,0,0 | .348 | .449 | .231 | .493 | .589 | .324 |
| | 5 | 0,0,5,0,0 | .344 | .446 | .235 | .459 | .548 | .328 |
| | 5 | 0,0,0,5,0 | .246 | .327 | .206 | .337 | .410 | .288 |
| | 5 | 0,0,0,0,5 | .276 | .362 | .212 | .384 | .472 | .296 |
| | 5 | 1,1,1,1,1 | .293 | .382 | .221 | .396 | .480 | .308 |
| | 7 | 3,0,0,0,0,0,0 | .325 | .462 | .331 | .464 | .595 | .439 |
| | 7 | 0,3,0,0,0,0,0 | .345 | .480 | .339 | .478 | .615 | .446 |
| | 7 | 0,0,0,3,0,0,0 | .348 | .496 | .355 | .480 | .619 | .453 |
| | 7 | 0,0,0,0,0,3,0 | .241 | .352 | .289 | .335 | .447 | .378 |
| | 7 | 0,0,0,0,0,0,3 | .308 | .449 | .330 | .443 | .584 | .440 |
| | 7 | 1,0,0,1,0,0,1 | .331 | .479 | .337 | .468 | .606 | .444 |
| 20 | 10 | 10,0,0,...,0,0,0 | .422 | .567 | .334 | .572 | .699 | .440 |
| | 10 | 0,10,0,...,0,0,0 | .459 | .598 | .342 | .608 | .728 | .453 |
| | 10 | 0,...,0,10,0,...,0 | .472 | .641 | .367 | .637 | .760 | .470 |
| | 10 | 0,0,0,...,0,10,0 | .234 | .331 | .253 | .327 | .423 | .339 |
| | 10 | 0,0,0,...,0,0,10 | .363 | .502 | .325 | .501 | .631 | .431 |
| | 10 | 1,1,1,....,1,1,1 | .376 | .505 | .325 | .507 | .637 | .429 |
| | 15 | 5,0,0,...,0,0,0 | .428 | .649 | .536 | .579 | .771 | .644 |
| | 15 | 0,5,0,...,0,0,0 | .434 | .655 | .537 | .586 | .776 | .645 |
| | 15 | 0,...,0,5,0,...,0 | .466 | .694 | .560 | .618 | .806 | .666 |
| | 15 | 0,0,0,...,0,5,0 | .246 | .401 | .386 | .359 | .514 | .487 |
| | 15 | 0,0,0,...,0,0,5 | .403 | .627 | .527 | .545 | .749 | .640 |
| | 15 | 1,1,...,1,...,1,1 | .358 | .570 | .487 | .509 | .713 | .609 |
| | 18 | 2,0,0,...,0,0,0 | .425 | .692 | .641 | .582 | .809 | .736 |
| | 18 | 0,2,0,...,0,0,0 | .431 | .699 | .640 | .585 | .806 | .738 |
| | 18 | 0,...,0,2,0,...,0 | .444 | .705 | .648 | .599 | .822 | .745 |
| | 18 | 0,0,0,...,0,2,0 | .302 | .529 | .531 | .443 | .674 | .656 |
| | 18 | 0,0,0,...,0,0,2 | .410 | .687 | .629 | .562 | .792 | .732 |
| | 18 | 1,0,0,...,0,0,1 | .413 | .683 | .637 | .571 | .798 | .733 |
| 30 | 15 | 15,0,0,...,0,0,0 | .502 | .663 | .420 | .641 | .783 | .534 |
| | 15 | 0,15,0,...,0,0,0 | .521 | .687 | .434 | .677 | .808 | .535 |
| | 15 | 0,...,0,15,0,...,0 | .595 | .747 | .459 | .723 | .841 | .565 |
| | 15 | 0,0,0,...,0,15,0 | .227 | .336 | .281 | .323 | .436 | .371 |
| | 15 | 0,0,0,...,0,0,15 | .426 | .587 | .405 | .564 | .716 | .517 |
| | 15 | 1,1,1,1,...,1,1,1 | .434 | .600 | .408 | .583 | .728 | .519 |
| | 20 | 10,0,0,...,0,0,0 | .503 | .722 | .583 | .652 | .835 | .692 |
| | 20 | 0,10,0,...,0,0,0 | .512 | .737 | .586 | .669 | .843 | .694 |
| | 20 | 0,...,0,10,0,...,0 | .573 | .793 | .625 | .721 | .885 | .722 |
| | 20 | 0,0,0,...,0,10,0 | .233 | .383 | .376 | .340 | .497 | .475 |

*(Continued on next page)*

**Table 1**

Power of the $T_1$ test for monotone decreasing hazard alternatives at 5% and 10% significance levels for several progressively censored schemes when the sample sizes are $n = 10, 20,$ and 30 *(Continued)*

| n | m | schemes $(R_1, \cdots, R_m)$ | 5% significance level G(0.5) | W(0.5) | LN(2) | 10% significance level G(0.5) | W(0.5) | LN(2) |
|---|---|---|---|---|---|---|---|---|
| | 20 | 0,0,0,...,0,0,10 | .455 | .680 | .558 | .599 | .793 | .673 |
| | 20 | 1,0,1,0,...,0,1,0 | .432 | .667 | .550 | .597 | .786 | .663 |
| | 25 | 5,0,0,...,0,0,0 | .503 | .779 | .720 | .666 | .877 | .805 |
| | 25 | 0,5,0,...,0,0,0 | .507 | .785 | .716 | .663 | .881 | .808 |
| | 25 | 0,...,0,5,0,...,0 | .534 | .812 | .728 | .691 | .901 | .820 |
| | 25 | 0,0,0,...,0,5,0 | .277 | .498 | .523 | .419 | .648 | .638 |
| | 25 | 0,0,0,...,0,0,5 | .469 | .759 | .700 | .621 | .853 | .792 |
| | 25 | 1,1,...,1,...,1,1 | .435 | .713 | .670 | .597 | .832 | .774 |

where $\alpha_{i:m:n} = E(U_{i:m:n})$ is the expected value of the $i$th Type-II progressively censored order statistics from the Uniform (0, 1) distribution, given by Balakrishnan and Sandhu (1995), and

$$\alpha_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \left\{ \frac{j + R_{m-j+1} + \ldots + R_m}{j + 1 + R_{m-j+1} + \ldots + R_m} \right\}.$$

Then, censored CRKL can be written as

$$\begin{aligned} CRKL(F_{m:n} : F_\theta) &= \int_o^{x_{m;m:n}} (1 - F_{m:n}(x)) \ln \frac{1 - F_{m:n}(x)}{\exp(-\frac{x}{\theta})} \mathrm{d}x \\ &\quad - \int_o^{x_{m;m:n}} (1 - F_{m:n}(x))\, \mathrm{d}x + \int_o^{x_{m;m:n}} \exp(-\frac{x}{\theta}) \mathrm{d}x \\ &= \sum_{i=0}^{m-1} (1 - \alpha_{i:m:n}) \ln(1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n}) \\ &\quad - \sum_{i=0}^{m-1} (1 - \alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n}) - \theta\,(\exp(-x_{i:m:m}) - 1) \\ &\quad + \frac{1}{2\theta} \sum_{i=0}^{m-1} (1 - \alpha_{i:m:n}) \left( x_{i+1:m:n}^2 - x_{i:m:n}^2 \right). \end{aligned} \tag{1}$$

Substituting $\theta$ by its maximum likelihood estimation

$$\hat{\theta} = \frac{1}{m} \sum_{i=1}^{m} (R_i + 1) x_{i:m:n}$$

**Table 2**
Power of the $T_2$ test for monotone increasing hazard alternatives at 5% and 10% significance levels for several progressively censored schemes when the sample sizes are $n = 10, 20,$ and 30

| | | schemes | 5% significance level | | | 10% significance level | | |
|---|---|---|---|---|---|---|---|---|
| n | m | $(R_1, \cdots, R_m)$ | G(2) | W(2) | B(2,1) | G(2) | W(2) | B(2,1) |
| 10 | 5 | 5,0,0,0,0 | .265 | .398 | .496 | .436 | .585 | .674 |
| | 5 | 0,5,0,0,0 | .265 | .398 | .492 | .431 | .582 | .668 |
| | 5 | 0,0,5,0,0 | .264 | .403 | .494 | .433 | .587 | .673 |
| | 5 | 0,0,0,5,0 | .267 | .407 | .507 | .436 | .596 | .681 |
| | 5 | 0,0,0,0,5 | .272 | .408 | .515 | .438 | .599 | .692 |
| | 5 | 1,1,1,1,1 | .267 | .404 | .513 | .434 | .592 | .680 |
| | 7 | 3,0,0,0,0,0,0 | .375 | .560 | .679 | .544 | .721 | .806 |
| | 7 | 0,3,0,0,0,0,0 | .369 | .558 | .677 | .547 | .716 | .803 |
| | 7 | 0,0,0,3,0,0,0 | .376 | .564 | .687 | .552 | .726 | .807 |
| | 7 | 0,0,0,0,0,3,0 | .374 | .5722 | .707 | .553 | .731 | .821 |
| | 7 | 0,0,0,0,0,0,3 | .376 | .579 | .709 | .555 | .738 | .826 |
| | 7 | 1,0,0,1,0,0,1 | .375 | .575 | .698 | .551 | .728 | .817 |
| 20 | 10 | 10,0,0,...,0,0,0 | .544 | .705 | .769 | .698 | .820 | .865 |
| | 10 | 0,10,0,...,0,0,0 | .544 | .705 | .768 | .697 | .818 | .862 |
| | 10 | 0,...,0,10,0,...,0 | .556 | .721 | .791 | .709 | .831 | .875 |
| | 10 | 0,0,0,...,0,10,0 | .564 | .738 | .815 | .715 | .843 | .893 |
| | 10 | 0,0,0,...,0,0,10 | .567 | .735 | .815 | .714 | .842 | .893 |
| | 10 | 1,1,1,...,1,1,1 | .559 | .734 | .805 | .712 | .838 | .887 |
| | 15 | 5,0,0,...,0,0,0 | .657 | .816 | .875 | .783 | .888 | .926 |
| | 15 | 0,5,0,...,0,0,0 | .658 | .815 | .873 | .783 | .887 | .924 |
| | 15 | 0,...,0,5,0,...,0 | .666 | .824 | .879 | .790 | .893 | .931 |
| | 15 | 0,0,0,...,0,5,0 | .684 | .844 | .904 | .797 | .905 | .943 |
| | 15 | 0,0,0,...,0,0,5 | .677 | .840 | .903 | .792 | .901 | .942 |
| | 15 | 1,1,...,1,...,1,1 | .676 | .833 | .893 | .796 | .898 | .938 |
| | 18 | 2,0,0,...,0,0,0 | .702 | .847 | .905 | .809 | .912 | .944 |
| | 18 | 0,2,0,...,0,0,0 | .703 | .847 | .904 | .809 | .912 | .944 |
| | 18 | 0,...,0,2,0,...,0 | .710 | .852 | .906 | .813 | .915 | .946 |
| | 18 | 0,0,0,...,0,2,0 | .715 | .865 | .920 | .821 | .919 | .952 |
| | 18 | 0,0,0,...,0,0,2 | .717 | .861 | .919 | .817 | .916 | .952 |
| | 18 | 1,0,0,...,0,0,1 | .707 | .855 | .912 | .816 | .913 | .951 |
| 30 | 15 | 15,0,0,...,0,0,0 | .691 | .814 | .859 | .806 | .889 | .917 |
| | 15 | 0,15,0,...,0,0,0 | .692 | .815 | .857 | .805 | .889 | .915 |
| | 15 | 0,...,0,15,0,...,0 | .708 | .832 | .874 | .819 | .901 | .928 |
| | 15 | 0,0,0,...,0,15,0 | .720 | .852 | .903 | .828 | .914 | .944 |
| | 15 | 0,0,0,...,0,0,15 | .717 | .850 | .900 | .825 | .912 | .942 |
| | 15 | 1,1,1,1,...,1,1,1 | .717 | .845 | .893 | .826 | .910 | .938 |
| | 20 | 10,0,0,...,0,0,0 | .754 | .862 | .902 | .847 | .921 | .946 |
| | 20 | 0,10,0,...,0,0,0 | .753 | .867 | .906 | .847 | .920 | .945 |
| | 20 | 0,...,0,10,0,...,0 | .769 | .879 | .918 | .857 | .929 | .952 |
| | 20 | 0,0,0,...,0,10,0 | .779 | .896 | .936 | .863 | .938 | .968 |

*(Continued on next page)*

**Table 2**

Power of the $T_2$ test for monotone increasing hazard alternatives at 5% and 10% significance levels for several progressively censored schemes when the sample sizes are $n = 10, 20$, and 30 *(Continued)*

| n | m | schemes $(R_1, \cdots, R_m)$ | 5% significance level G(2) | W(2) | B(2,1) | 10% significance level G(2) | W(2) | B(2,1) |
|---|---|---|---|---|---|---|---|---|
| | 20 | 0,0,0,...,0,0,10 | .775 | .892 | .936 | .860 | .935 | .963 |
| | 20 | 1,0,1,0,...,0,1,0 | .774 | .886 | .923 | .861 | .933 | .958 |
| | 25 | 5,0,0,...,0,0,0 | .794 | .894 | .931 | .873 | .938 | .961 |
| | 25 | 0,5,0,...,0,0,0 | .791 | .896 | .931 | .873 | .938 | .960 |
| | 25 | 0,...,0,5,0,...,0 | .802 | .902 | .936 | .878 | .942 | .964 |
| | 25 | 0,0,0,...,0,5,0 | .813 | .915 | .951 | .884 | .949 | .972 |
| | 25 | 0,0,0,...,0,0,5 | .809 | .912 | .950 | .881 | .947 | .971 |
| | 25 | 1,1,...,1,...,1,1 | .806 | .908 | .943 | .881 | .945 | .968 |

in (1) and dividing to $\int_o^{x_{m:m:n}} 1 - F_{m:n}(x)\mathrm{d}x$, the test statistic is as follows:

$$T_1 = A + \frac{B}{2\hat{\theta}} - \hat{\theta}C - 1, \tag{2}$$

where

$$A = \frac{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})\ln(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})},$$

$$B = \frac{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}^2 - x_{i:m:n}^2)}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})},$$

$$C = \frac{\exp(-\frac{x_{m:m:n}}{\hat{\theta}})}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})}.$$

Similarly, based on the CKL, the test statistic is as follows

$$T_2 = D - E + \hat{\theta}F + 1, \tag{3}$$

where

$$D = \frac{\sum_{i=1}^{m-1}\alpha_{i:m:n}\ln(\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})},$$

$$E = \frac{\sum_{i=1}^{m-1}\alpha_{i:m:n}\int_{x_{i:m:n}}^{x_{i+1:m:n}}\ln(1-\exp(-\frac{x}{\hat{\theta}}))\mathrm{d}x}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})},$$

$$F = \frac{\exp(-\frac{x_{m:m:n}}{\hat{\theta}}) - 1}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})}.$$

It is obvious that $T_1$ and $T_2$ are scale invariant and are appropriate for goodness-of-fit testing.

**Table 3**
Power of the $T_2$ test for non-monotone hazard alternatives at 5% and 10% significance levels for several progressively censored schemes when the sample sizes are $n = 10, 20,$ and 30

| | | schemes | 5% significance level | | | 10% significance level | | |
|---|---|---|---|---|---|---|---|---|
| n | m | $(R_1, \cdots, R_m)$ | LN(1) | LL(.5) | LN(1.2) | LN(1) | LL(.5) | LN(1.2) |
| 10 | 5 | 5,0,0,0,0 | .251 | .306 | .145 | .420 | .484 | .265 |
| | 5 | 0,5,0,0,0 | .250 | .302 | .144 | .418 | .479 | .264 |
| | 5 | 0,0,5,0,0 | .250 | .305 | .144 | .419 | .482 | .264 |
| | 5 | 0,0,0,5,0 | .251 | .305 | .145 | .424 | .483 | .264 |
| | 5 | 0,0,0,0,5 | .249 | .304 | .144 | .421 | .485 | .263 |
| | 5 | 1,1,1,1,1 | .249 | .305 | .144 | .420 | .484 | .263 |
| | 7 | 3,0,0,0,0,0,0 | .314 | .402 | .155 | .501 | .572 | .281 |
| | 7 | 0,3,0,0,0,0,0 | .313 | .398 | .155 | .500 | .580 | .281 |
| | 7 | 0,0,0,3,0,0,0 | .314 | .403 | .154 | .500 | .573 | .282 |
| | 7 | 0,0,0,0,0,3,0 | .311 | .400 | .153 | .494 | .584 | .280 |
| | 7 | 0,0,0,0,0,0,3 | .309 | .398 | .153 | .491 | .582 | .279 |
| | 7 | 1,0,0,1,0,0,1 | .317 | .400 | .153 | .487 | .583 | .281 |
| 20 | 10 | 10,0,0,...,0,0,0 | .613 | .605 | .338 | .781 | .742 | .511 |
| | 10 | 0,10,0,...,0,0,0 | .612 | .603 | .338 | .780 | .743 | .510 |
| | 10 | 0,...,0,10,0,...,0 | .613 | .608 | .334 | .782 | .752 | .515 |
| | 10 | 0,0,0,...,0,10,0 | .604 | .617 | .330 | .775 | .752 | .503 |
| | 10 | 0,0,0,...,0,0,10 | .599 | .619 | .325 | .769 | .750 | .500 |
| | 10 | 1,1,1,...,1,1,1 | .613 | .614 | .330 | .774 | .754 | .506 |
| | 15 | 5,0,0,...,0,0,0 | .694 | .686 | .358 | .838 | .801 | .536 |
| | 15 | 0,5,0,...,0,0,0 | .702 | .691 | .358 | .840 | .802 | .536 |
| | 15 | 0,...,0,5,0,...,0 | .702 | .695 | .359 | .840 | .806 | .528 |
| | 15 | 0,0,0,...,0,5,0 | .680 | .688 | .342 | .823 | .801 | .511 |
| | 15 | 0,0,0,...,0,0,5 | .676 | .686 | .341 | .824 | .796 | .515 |
| | 15 | 1,1,...,1,...,1,1 | .689 | .688 | .348 | .831 | .800 | .521 |
| | 18 | 2,0,0,...,0,0,0 | .716 | .709 | .354 | .849 | .813 | .538 |
| | 18 | 0,2,0,...,0,0,0 | .718 | .711 | .354 | .850 | .813 | .534 |
| | 18 | 0,...,0,2,0,...,0 | .725 | .711 | .348 | .854 | .813 | .530 |
| | 18 | 0,0,0,...,0,2,0 | .702 | .705 | .332 | .840 | .810 | .504 |
| | 18 | 0,0,0,...,0,0,2 | .703 | .704 | .342 | .841 | .808 | .512 |
| | 18 | 1,0,0,...,0,0,1 | .714 | .704 | .347 | .848 | .809 | .517 |
| 30 | 15 | 15,0,0,...,0,0,0 | .828 | .742 | .526 | .921 | .840 | .698 |
| | 15 | 0,15,0,...,0,0,0 | .828 | .740 | .526 | .921 | .840 | .698 |
| | 15 | 0,...,0,15,0,...,0 | .834 | .756 | .527 | .925 | .852 | .699 |
| | 15 | 0,0,0,...,0,15,0 | .814 | .756 | .499 | .915 | .852 | .678 |
| | 15 | 0,0,0,...,0,0,15 | .809 | .754 | .495 | .911 | .850 | .671 |
| | 15 | 1,1,1,1,...,1,1,1 | .809 | .758 | .511 | .911 | .853 | .683 |
| | 20 | 10,0,0,...,0,0,0 | .867 | .779 | .549 | .941 | .869 | .711 |
| | 20 | 0,10,0,...,0,0,0 | .867 | .785 | .549 | .941 | .867 | .711 |
| | 20 | 0,...,0,10,0,...,0 | .871 | .793 | .547 | .943 | .875 | .709 |
| | 20 | 0,0,0,...,0,10,0 | .846 | .796 | .507 | .930 | .875 | .674 |

*(Continued on next page)*

**Table 3**
Power of the $T_2$ test for non-monotone hazard alternatives at 5% and 10% significance levels for several progressively censored schemes when the sample sizes are $n = 10, 20$, and 30 *(Continued)*

| | | schemes | 5% significance level | | | 10% significance level | | |
|---|---|---|---|---|---|---|---|---|
| n | m | $(R_1, \cdots, R_m)$ | LN(1) | LL(.5) | LN(1.2) | LN(1) | LL(.5) | LN(1.2) |
| | 20 | 0,0,0,...,0,0,10 | .846 | .786 | .510 | .930 | .868 | .689 |
| | 20 | 1,0,1,0,...,0,1,0 | .846 | .794 | .532 | .930 | .873 | .694 |
| | 25 | 5,0,0,...,0,0,0 | .884 | .802 | .544 | .949 | .879 | .710 |
| | 25 | 0,5,0,...,0,0,0 | .884 | .807 | .544 | .949 | .880 | .709 |
| | 25 | 0,...,0,5,0,...,0 | .886 | .812 | .542 | .950 | .884 | .707 |
| | 25 | 0,0,0,...,0,5,0 | .862 | .812 | .502 | .936 | .884 | .669 |
| | 25 | 0,0,0,...,0,0,5 | .869 | .803 | .507 | .942 | .878 | .691 |
| | 25 | 1,1,...,1,...,1,1 | .869 | .805 | .520 | .942 | .879 | .688 |

## 3. Monte Carlo Study

For $T_1$ and $T_2$ testes, the null hypothesis will be rejected, when the test statistics are more than the corresponding critical values at a designed significance level $\alpha$. Because the null distribution of the test statistics $T_1$ and $T_2$ are not available, we proceed the Mont Carlo simulation to determine critical values of the test statistics. A total of 100,000 random samples were generated from the standard exponential distribution by Balakrishnan and Sandhu algorithm (see, Balakrishnan and Sandhu, 1995) and then progressively Type-II censored samples are generated for $n = 10$, 20, and 30 and some different $m$ and schemes $(R_1, R_2, ..., R_m)$. For each sample, the test statistics $T_1$ and $T_2$ as defined in (2) and (3), respectively, were calculated. The values were then used to determine the critical values $T_{1,0.95}$ and $T_{1,0.90}$, and $T_{2,0.95}$ and $T_{2,0.90}$. For power study, we consider the alternatives according to the type of hazard function as follows:

(a) For $T_1$, Monotone decreasing hazard: Gamma: G(0.5, 1) and Weibull: W(0.5, 1) and Lognormal: LN(0, 2).
(b) For $T_2$, Monotone increasing hazard: Gamma: G(2, 1), Weibull: W(2, 1) and Beta: B(2, 1).
(c) For $T_2$, Non-monotone hazard: Lognormal: LN(0, 1), Lognormal: LN(0, 1.2) and Loglogistic: LL(0.5, 1).

We use 100,000 Monte Carlo simulations for $n = 10, 20,$ and 30 to estimate the power of our proposed tests. The simulation results for different censoring schemes are summarized in Tables 1–3 and Figs. 1–3.

Balakrishnan et al. (2007) constructed a goodness-of-fit test statistic for exponentiality based on Kullback-Leibler information with progressively Type-II censored data and compared its powers against several alternatives under different progressive censoring schemes. Comparing with their proposed test, for the alternatives with monotone decreasing hazard function; and non-monotone hazard function, $T_1$ and $T_2$ tests, respectively, have very higher powers (Tables 1 and 3). But, by Table 2, for the alternatives with monotone increasing hazard function, the power differences between $T_2$ test and their proposed test are not so

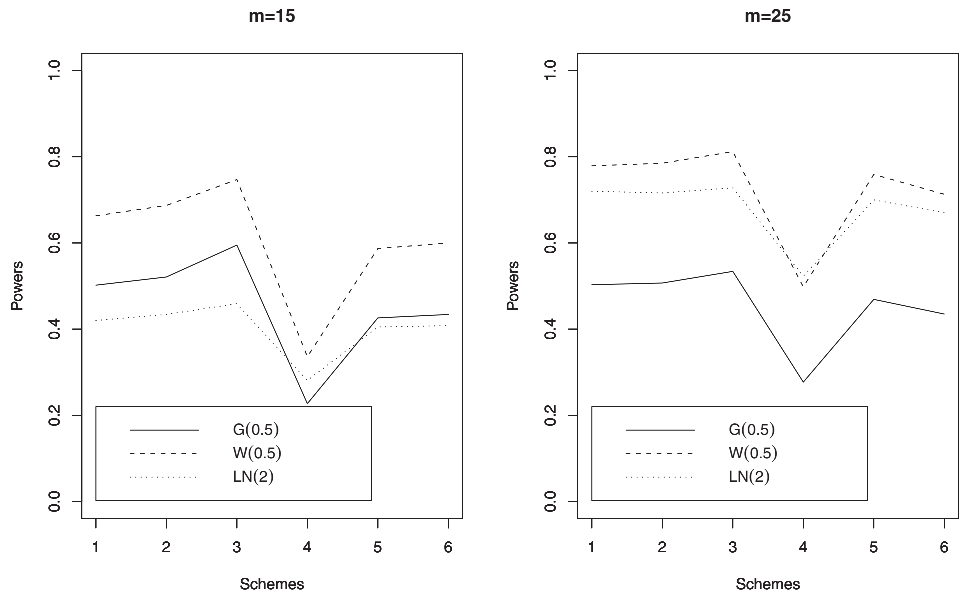


**Figure 1.** Comparing different schemes for monotone decreasing hazard alternatives for the $T_1$ test at 5% significance levels when $n = 30$ and $m = 15$ and 25.

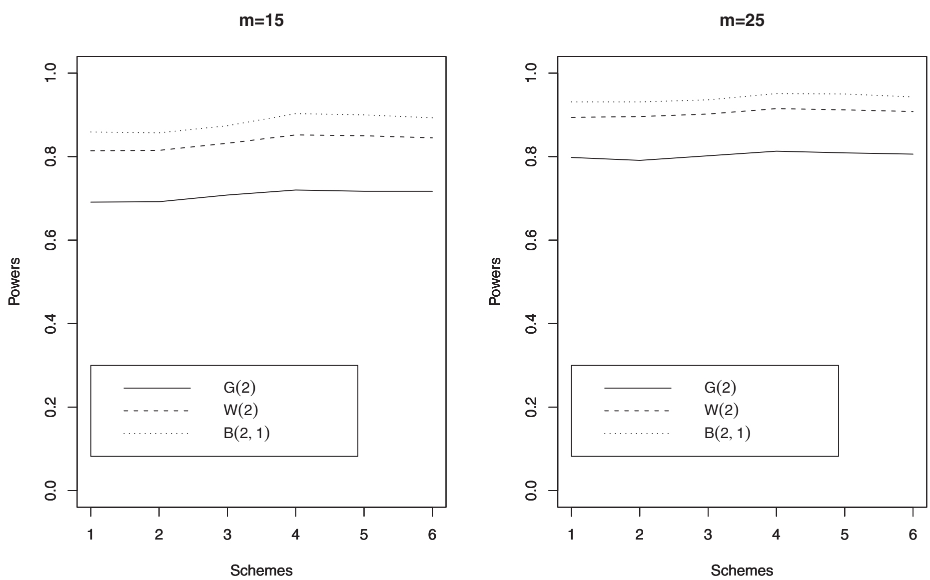


**Figure 2.** Comparing different schemes for monotone increasing hazard alternatives for the $T_2$ test at 5% significance levels when $n = 30$ and $m = 15$ and 25.

remarkable. Also, for $T_1$ test, from Table 1 and Fig. 1, we conclude that when the surviving units are withdrawn only from the middle, the power is higher. But, from Tables 2–3 and Figs. 2–3, we observe that the powers of $T_2$ test do not depend on different schemes.

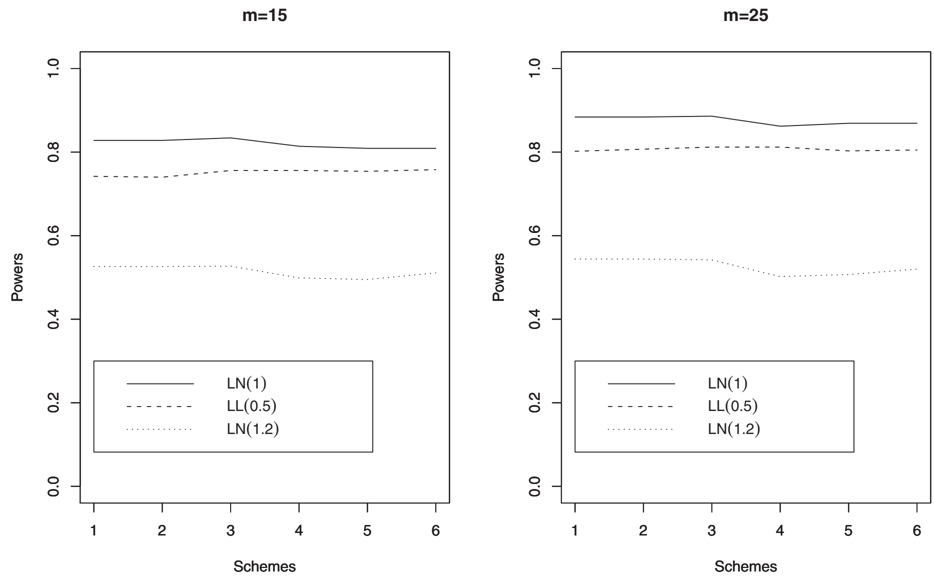


**Figure 3.** Comparing different schemes for non-monotone hazard alternatives for the $T_2$ test at 5% significance levels when $n = 30$ and $m = 15$ and 25.

**Table 4**
Progressively censored sample generated from the times to breakdown data on insulating fluids tested at 34 kilovolts, given by Viveros and Balakrishnan (1994)

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $x_{i:8:19}$ | 0.19 | 0.78 | 0.96 | 1.31 | 2.078 | 4.85 | 6.50 | 7.35 |
| $R_i$ | 0 | 0 | 3 | 0 | 3 | 0 | 0 | 5 |

## 4. Illustrative Example

In this section, we present an example to illustrate the use of the test statistics $T_1$ and $T_2$ for testing the validity of the exponential distribution for an observed progressively Type-II censored sample.

Nelson (1982) reported data on times to breakdown of an insulating fluid in an accelerated test conducted at various test voltages. Let us consider this progressively Type-II censored sample of size $m = 8$ generated from the $n = 19$ observations recorded at 34 kilovolts, as given in Table 4.

The computed test statistics are $T_1 = 0.0074$ and $T_2 = 0.1836$, and the $p$-values are then computed as 0.615 and 0.458, respectively, which provide very strong evidence that the observed progressively Type-II censored sample is from an exponential distribution.

## 5. Conclusions

In this article, we have considered inference for the exponential distribution when the data is progressively Type-II censored. We have constructed two goodness-of-fit tests $T_1$ and $T_2$ based on the CRE and CE, respectively. Comparing with the test proposed by Balakrishnan et al. (2007), for the alternatives with monotone decreasing hazard function; and non-monotone hazard function, $T_1$ and $T_2$ tests, respectively, have higher powers. Also, for the alternatives with monotone increasing hazard function, the power differences between $T_2$ and their proposed test are not so remarkable. Although this article focuses on exponential lifetime distribution, similar inferential procedures can be developed for other lifetime distributions such as the Weibull, Pareto, and Burr Type-XII distributions, etc.

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