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Journal of Taibah University for Science 8 (2014) 199–205

Journal of Taibah University for Science

www.elsevier.com/locate/jtusci

Numerical solution of weakly singular Fredholm integral equations via generalization of the Euler–Maclaurin summation formula

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Received 14 September 2013; received in revised form 17 November 2013; accepted 21 November 2013
Available online 5 December 2013

Abstract

Since the classical Euler–Maclaurin summation formula may not be applied for approximating singular integrals, we can not use this formula for obtaining the numerical solution of weakly singular Fredholm integral equations. In this article, we use a generalization of the Euler–Maclaurin summation formula for solving weakly singular Fredholm integral equations of the second kind. By this idea, the basic equations will be changed into the associated systems of algebraic equations. Some numerical examples are given to show the effectiveness of the method.

MSC: 65D30

Keywords: Generalized Euler–Maclaurin summation formula; Bernoulli functions; Product integration rule; Weakly singular Fredholm integral equations

1. Introduction

Weakly singular Fredholm integral equations (WSFIEs) have many applications in mathematical physics. These equations arise in the heat conduction problem posed by mixed boundary conditions, potential problems, the Dirichlet problem, and radiative equilibrium [9]. Furthermore, some important applications of WSFIEs in the fields of fracture mechanics, elastic contact problems, the theory of porous filtering, combined infrared radiation and molecular conduction were provided in Ref. [3]. It is difficult to solve these equations analytically. Hence, numerical schemes are required for dealing with these equations in a proper manner.

In recent years, numerical methods for these equations have been developed by many researchers. For instance, one can refer to the methods that were proposed in Refs. [10,11]. Lifanov [10] introduced hyper singular integral equations with their applications and then introduce some new numerical algorithms for solving them. Also, reproducing kernel

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Peer review under responsibility of Taibah University

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http://dx.doi.org/10.1016/j.jtusci.2013.11.001

method has been presented by Du and his coworkers [7] for solving Fredholm integro-differential equations with weak singularity. Moreover, in Ref. [12], discrete Galerkin method has been proposed and analyzed for obtaining the numerical solution of these equations.

Among the numerical integration schemes, the trapezoidal method is a classic formula which was applied for approximating the integrals in many text books such as [2,5,6]. There are also generalizations of this method for approximating of singular integrals. Such methods have already been investigated in the literature. A generalized Euler–Maclaurin summation formula based on piecewise Lagrangian interpolation is established in Ref. [8]. High-order schemes of product integration methods are developed in Refs. [1,15]. Also, Rzadkowski and Epkowski proposed a generalization of the Euler–Maclaurin summation formula in Ref. [13] and then applied this idea for computing the Fermi-Dirac integrals numerically. Then, Rzadkowski use the previous idea for approximating more general singular integrals in [14]. This method can be seen as a direct generalization of the trapezoidal rule.

In this paper, we establish a new method by using the basic idea in Ref. [14], to find the numerical solution of weakly singular Fredholm integral equations

$$u(t) = f(t) + \int_0^1 K(x, t)u(x)dx, \qquad 0 \le t \le 1, \tag{1}$$

where the singularity of kernel may be stated in the forms $K(x, t) = 1/(x - t)^\alpha$ or $K(x, t) = 1/(1 - x)^\alpha$ with the assumption $0 < \alpha < 1$. It should be noted that the integral $\int_0^1 K(x, t)u(x)dx$, where $K(x, t) = 1/(1 - x)^\alpha$, can not be approximated by the Euler–Maclaurin summation formula, since all the derivatives of $K(x, t)$ at the end-point $x = 1$ are infinity. Therefore, we need to improve the classical Euler–Maclaurin summation formula and then use this improved quadrature for obtaining the numerical solution of Eq. (1). For this purpose, we first introduce Bernoulli functions and then use them for approximating the term $\int_0^1 K(x, t)u(x)dx$ in Eq. (1).

The remainder of this paper is organized as follows. In the next section, we define Bernoulli functions of the first and second order as a generalization of Bernoulli polynomials. The basic idea of this paper will be provided in Section 3 by using the above-mentioned Bernoulli functions. Some numerical examples are given in Section 4 for confirming the efficiency of the proposed method. Section 5 contains conclusions of the paper.

2. Bernoulli polynomials and functions

Bernoulli polynomials have a great popularity in science and engineering [4]. They form a semi orthogonal base in $L^2[0, 1]$. Also, applications of these polynomials were found in many problems such as Pantograph delay differential equations [17], two dimensional hyperbolic partial differential equations [18], boundary value problems which arise from calculus of variation [19] and high order linear complex differential equations [20]. Bernoulli polynomials can be defined in many ways. One of the famous ways (with the assumption $B_0(x) = 1$) is that

$$\begin{cases} B_n'(x) = nB_{n-1}(x), & n = 1, 2, \ldots \\ \int_0^1 B_n(x)dx = 0, & n = 1, 2, \ldots \end{cases} \tag{2}$$

The above-mentioned recurrence relation tells us $B_n(x)$ is the anti-derivative of $B_{n-1}(x)$. For instance $B_1(x) - B_1(0) = \int_0^1 B_0(t)dt$. It should be noted that Bernoulli polynomials play a crucial rule in the classical Euler–Maclaurin summation formula and have the famous property $B_m(1) = B_m(0)$ for $m \ge 2$. Similar to the Bernoulli polynomials, Rzadkowski [14] defined the Bernoulli functions which we will review this process in the following lines. Assume that $F_0(x)$ is an arbitrary integrable function. We assume that $F_0(x)$ is the Bernoulli function of the zero order. Our aim is to construct the Bernoulli function of the first order in the interval $[x_{k-1}, x_k]$, which is denoted by $F_{1(k)}(x)$, such that $F_{1(k)}(x)$ be the anti-derivative of the function $F_0(x)$ in the interval $[x_{k-1}, x_k]$ and $\int_{x_{k-1}}^{x_k} F_{1(k)}(x)dx = 0$. For evaluating $F_{1(k)}(x)$, one can subtract mean value of $F_0(x)$ in the interval $[x_{k-1}, x_k]$ from its anti-derivative as follows

$$\begin{aligned} F_{1(k)}(x) &= \int_{x_{k-1}}^{x} F_0(t)dt - \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} dt \int_{x_{k-1}}^{t} F_0(u)du \\ &= \int_{x_{k-1}}^{x} F_0(t)dt - \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} (x_k - t)F_0(t)dt. \end{aligned} \tag{3}$$

Trivially we have

$$F_{1(k)}(x_{k-1}) = -\frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - t)F_0(t)dt,$$
$$F_{1(k)}(x_k) = \frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(t - x_{k-1})F_0(t)dt.$$

In a similar way, the function $F_{2(k)}(x)$ is defined in the interval $[x_{k-1}, x_k]$ such that (see Ref. [13])

$$F'_{2(k)}(x) = F_{1(k)}(x), \qquad \int_{x_{k-1}}^{x_k} F_{2(k)}(x)dx = 0.$$

Therefore

$$\begin{aligned} F_{2(k)}(x) &= \int_{x_{k-1}}^{x} F_{1(k)}(t)dt - \frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - t)F_{1(k)}(t)dt \\ &= \int_{x_{k-1}}^{x}(x - t)F_0(t)dt - \frac{x - x_{k-1}}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - t)F_0(t)dt - \frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - t)\int_{x_{k-1}}^{t}F_0(u)dudt + \frac{1}{2}\int_{x_{k-1}}^{x_k}(x_k - t)F_0(t)dt \\ &= \int_{x_{k-1}}^{x}(x - t)F_0(t)dt - \frac{x - x_{k-1}}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - t)F_0(t)dt - \frac{1}{2(x_k - x_{k-1})}\int_{x_{k-1}}^{x_k}(x_k - t)^2F_0(t)dt + \frac{1}{2}\int_{x_{k-1}}^{x_k}(x_k - t)F_0(t)dt \\ &= \int_{x_{k-1}}^{x}(x - t)F_0(t)dt - \frac{1}{2(x_k - x_{k-1})}\int_{x_{k-1}}^{x_k}(x_k - t)(2x - x_{k-1} - t)F_0(t)dt. \end{aligned}$$

$F_{2(k)}(x)$ has an interesting property (at the end-points) in the following form

$$F_{2(k)}(x_k) = F_{2(k)}(x_{k-1}) = \frac{1}{2(x_k - x_{k-1})}\int_{x_{k-1}}^{x_k}(x_k - t)(t - x_{k-1})F_0(t)dt. \tag{4}$$

We find from Theorem 2 in [13] that the nth order Bernoulli function $F_{n(k)}(x)$ in the interval $[x_{k-1}, x_k]$ has the following form

$$F_{n(k)}(x) = \frac{(x_k - x_{k-1})^{n-1}}{n!}\left[B_n\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)\int_{x_{k-1}}^{x_k}F_0(t)dt - \int_{x_{k-1}}^{x_k}F_0(t)B_n^*\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)dt\right], \tag{5}$$

where $B^*(x) = B_n(x - [x])$ is introduced in Ref. [4]. It should be noted that, in our method we just use $F_{1(k)}(x)$ for clarity of presentation. Higher order methods which are based on higher order Bernoulli functions can be used in a similar way.

3. Basic idea

In this section, we will give an interesting idea for solving Eq. (1) numerically. For this reason, the uniform mesh $0 = x_0 < x_1 < \cdots < x_n = 1$ $(x_i = i/n,\ i = 0, 1, \ldots, n)$ is considered. Our aim is to approximate the integral term $\int_0^1 K(x, t)u(x)dx$ by the aid of the first order Bernoulli functions $F_{1(k)}(x)$ $(k = 1, 2, \ldots, n)$. On the other hand

$$\int_0^1 K(x, t)u(x)dx = \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}K(x, t)u(x)dx.$$

Now assume that t is fixed and $u(x)$ is twice continuously differentiable in the interval $[x_{k-1}, x_k]$. Also suppose that $F_{1(k)}(x, t)$ is the Bernoulli function of the first order associated with the function $F_0(x, t) := K(x, t)$ (note that t is fixed and $K(x, t) = (\partial/\partial x)F_{1(k)}(x, t)$) and is constructed similar to the formula (3). In other words

$$F_{1(k)}(x, t) = \int_{x_{k-1}}^{x}K(\tau, t)d\tau - \frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - \tau)K(\tau, t)d\tau, \quad 1 \le k \le n. \tag{6}$$

Integrating twice by parts in the interval $[x_{k-1}, x_k]$ yields

$$\begin{aligned}\int_{x_{k-1}}^{x_k} K(x,t)u(x)dx &= F_{1(k)}(x,t)u(x)|_{x=x_{k-1}}^{x=x_k} - \int_{x_{k-1}}^{x_k} F_{1(k)}(x,t)u'(x)dx \\ &= F_{1(k)}(x,t)u(x)|_{x=x_{k-1}}^{x=x_k} - F_{2(k)}(x,t)u'(x)|_{x=x_{k-1}}^{x=x_k} + \int_{x_{k-1}}^{x_k} F_{2(k)}(x,t)u''(x)dx.\end{aligned} \tag{7}$$

By summing up Eq. (7) over $k = 1, 2, \ldots, n$, we have (see [14, Theorem 2])

$$\int_0^1 K(x,t)u(x)dx = F_{1(n)}(1,t)u(1) - F_{1(1)}(0,t)u(0) + \sum_{k=1}^{n-1}\left(F_{1(k)}(x_k,t) - F_{1(k+1)}(x_k,t)\right)u(x_k) + O\left(\frac{1}{n^2}\right).$$

In other words, we use the term $F_{1(n)}(1,t)u(1) - F_{1(1)}(0,t)u(0) + \sum_{k=1}^{n-1}\left(F_{1(k)}(x_k,t) - F_{1(k+1)}(x_k,t)\right)u(x_k)$ as an approximation of $\int_0^1 K(x,t)u(x)dx$ in our next computations. Therefore the basic Eq. (1) would be reduced to the following form

$$u(t) \approx f(t) + F_{1(n)}(1,t)u(1) - F_{1(1)}(0,t)u(0) + \sum_{k=1}^{n-1}\left(F_{1(k)}(x_k,t) - F_{1(k+1)}(x_k,t)\right)u(x_k). \tag{8}$$

We now assume that u_i is an approximation of $u(x_i)$ and $f_i = f(t_i)$, $i = 0, 1, \ldots, n$. Collocating the above equation at the points $t_i = i/n$ yields

$$\begin{aligned}u_0 &= f_0 - F_{1(1)}(0,t_0)u_0 + \sum_{k=1}^{n-1}\left(F_{1(k)}(x_k,t_0) - F_{1(k+1)}(x_k,t_0)\right)u_k + F_{1(n)}(1,t_0)u_n,\\ u_1 &= f_1 - F_{1(1)}(0,t_1)u_0 + \sum_{k=1}^{n-1}\left(F_{1(k)}(x_k,t_1) - F_{1(k+1)}(x_k,t_1)\right)u_k + F_{1(n)}(1,t_1)u_n,\\ &\vdots\\ u_n &= f_n - F_{1(1)}(0,t_n)u_0 + \sum_{k=1}^{n-1}\left(F_{1(k)}(x_k,t_n) - F_{1(k+1)}(x_k,t_n)\right)u_k + F_{1(n)}(1,t_n)u_n.\end{aligned}$$

The matrix-vector form of the above equations can be written in the following form

$$U = F + AU \tag{9}$$

where $U = [\,u_0 \quad u_1 \quad \cdots \quad u_n\,]^T$, $F = [\,f_0 \quad f_1 \quad \cdots \quad f_n\,]^T$ and

$$A = \begin{bmatrix} -F_{1(1)}(0,t_0) & F_{1(1)}(x_1,t_0) - F_{1(2)}(x_1,t_0) & \cdots & F_{1(n)}(1,t_0) \\ -F_{1(1)}(0,t_1) & F_{1(1)}(x_1,t_1) - F_{1(2)}(x_1,t_1) & \cdots & F_{1(n)}(1,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ -F_{1(1)}(0,t_n) & F_{1(1)}(x_1,t_n) - F_{1(2)}(x_1,t_n) & \cdots & F_{1(n)}(1,t_n) \end{bmatrix}_{(n+1)\times(n+1)}.$$

System (9) can be rewritten in the form $(I - A)U = F$ (where $I_{(n+1)\times(n+1)}$ is the identity matrix) and the unknown solution U may be obtained by the iterative methods. Thus, we can reach a discrete solution of (1).

Remark: For more information about iterative methods one can refer to the book [16]. In this book, several iterative methods have been introduced for solving large sparse linear systems.

4. Numerical examples

In this section, three numerical examples are given to illustrate the accuracy and effectiveness of the proposed methods and all of them are performed on a computer using programs written in MATLAB 2011b. In this regard, we have reported in tables and figures the values of the the maximum absolute error function $e_N(x) = |u(x) - u_N(x)|$

Table 1
Numerical results of Example 1.

n	E_n of the presented method	E_n of the Trapezoidal rule
4	2.623e−02	3.493e-01
16	3.253e−03	2.109e-01
32	1.149e−03	2.006e-01
64	3.968e−04	1.641e-01
256	5.075e−05	1.273e-01

Table 2
Numerical results of Example 2.

n	E_n of the presented method	E_n of the Trapezoidal rule
4	1.8714e−01	6.4935e-01
16	1.2357e−03	4.2543e-01
32	3.1378e−04	4.1023e-01
64	7.9306e−05	3.6025e-01
256	5.0214e−06	2.5231e-01

at any uniform mesh of the given interval (e.g., [0, 1]). It should be noted that in all examples we provide weakly singular Fredholm integral equations in which our results are more accurate with regard to the Trapezoidal rule [2]. Moreover, since the Trapezoidal rule can not solve these singular equations in the interval [0,1], the associated results of this classic scheme have been done in the interval [0.05,0.95]. Before presenting of our numerical examples, we will show that how one can compute the $F_{1(k)}(x, t)$ for any weakly singular kernel. For instance if we assume that $K(x, t) = 1/(\sqrt{1-x})$, then, by using Eq. (6), we have

$$F_{1(k)}(x,t) = \frac{4}{3}x_{k-1}(1-x)^{3/2} + \frac{2}{3}\,\frac{x_{k-1}\sqrt{1-x}(3/2\,x_k{}^2 - 2\,x_k - xx_k)}{x_k - x_{k-1}}, \quad 1 \le k \le n. \tag{10}$$

Example 1. We consider the following weakly singular Fredholm integral equation of the second kind with the exact solution $u(t) = \sqrt{t}$

$$u(t) = \sqrt{t} - \frac{\pi}{2} + \int_0^1 \frac{u(x)}{\sqrt{1-x}}dx, \quad 0 \le t \le 1.$$

For solving this equation, we follow Section 3 in which the basic idea of the paper has been provided. Therefore, some values of n such as 4, 16, 32, 64 and 256 are used and the corresponding vector $U = [u_0\ u_1\ \cdots\ u_n]^T$ has been obtained from solving Eq. (9) by a suitable iterative method. For each values of n, $E_n := \max\limits_{0 \le i \le n} |u(t_i) - u_i|$ denote the maximum values of errors at the points u_i, $i = 0, 1, \ldots, n$. Numerical results of this example are provided in Table 1. It is obviously seen our presented method is more accurate than the classic Trapezoidal rule.

Example 2. We consider the following Fredholm integral equation with the exact solution $u(t) = t^2$

$$u(t) = t^2 - \frac{16}{15} + \int_0^1 \frac{u(x)}{\sqrt{1-x}}dx, \quad 0 \le t \le 1.$$

By a similar procedure, we solve this equation numerically and the associated numerical results of this example has been provided in Table 2. However, even if the conventional trapezoidal rule seems to be an appropriate method to numerically compute the solution of integral equations, at least in the presented examples, our method gives lower values of the error. This fact is shown in Tables 1 and 2.

Example 3. We consider the following Fredholm integral equation with the exact solution $u(t) = e^t$

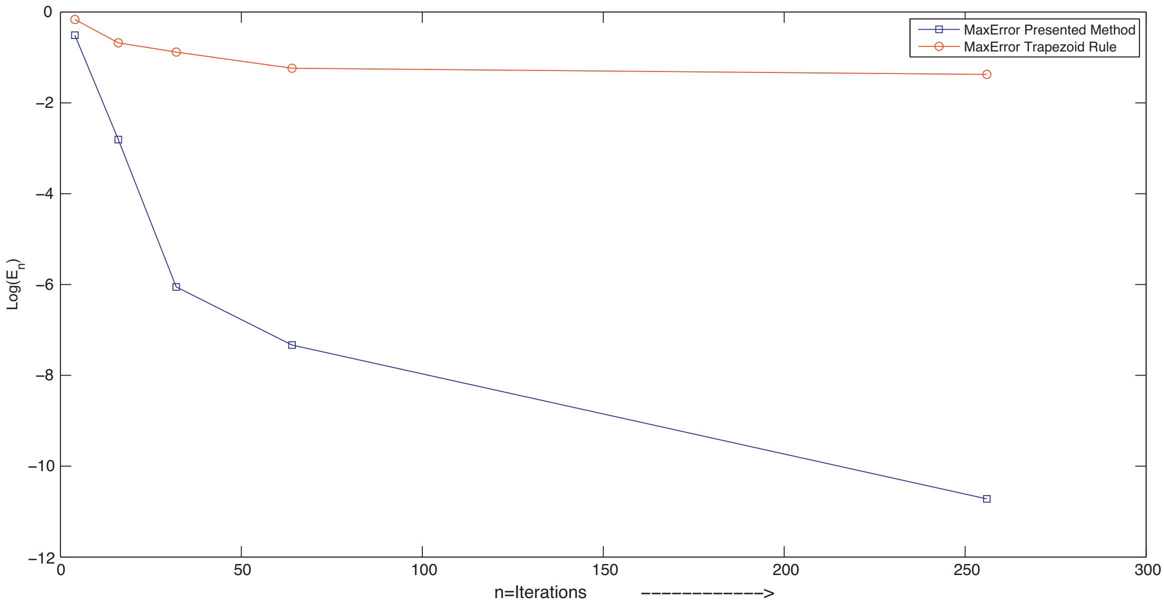

Fig. 1. Comparison of E_n history associated to the PM and TR.

$$u(t) = e^t - 4.0602 + \int_0^1 \frac{u(x)}{\sqrt{1-x}}dx, \quad 0 \leq t \leq 1.$$

Similar to the previous examples, we solve this example numerically and make a comparison with the Trapezoidal rule. In Fig. 1 we see that history of E_n corresponding to the presented method (PM) are lower than those of the Trapezoidal rule (TR).

5. Conclusions and future works

A generalization of the Euler–Maclaurin summation formula has been applied for solving a class of weakly singular Fredholm integral equations. The classical Euler–Maclaurin summation formula needs to compute all the required derivative values of the integrand at the end-points of the computational interval, meanwhile in practice we may have some integrands such that they do not have derivative at the end-points. Therefore, some modifications have been done by using Bernoulli functions to construct such generalized quadratures. The efficiency and applicability of the proposed method with respect to the Trapezoidal rule is established in our numerical examples. Our idea can be applied for weakly singular Volterra integral equations and also systems of weakly singular integral equations, but some modifications should be done.

Conflict of interest

The authors declare that they do not have any conflict of interest in their submitted manuscript.

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