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Expansion of the Exp-function method for solving systems of two-dimensional Navier–Stokes equations

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Abstract

Navier–Stokes equations are the most important equations in fluid dynamics for finding the velocity and pressure functions. The main purpose of this paper is to consider the method for the solution of incompressible Navier–Stokes equations by the Exp-function method.

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1. Introduction

Navier–Stokes equations are the most important equations in fluid dynamics for finding the velocity and pressure functions. In considering the flow of a fluid, viscosity plays an important role. Viscosity is a characteristic of a fluid that becomes of that, it shows resistance to shear stress (tar is an example of these kind of fluids with high viscosity and water and air are examples of fluids with low viscosity).

Movement equations for a fluid can be obtained by considering effective forces on a small portion of fluid that contains shear stresses resulting from fluid movement and viscosity. These equations are known as Navier–Stokes equations.

Analyzing methods for engineering problems are so widespread, and by advances in computer application in engineering, major, faster and more exact methods are presenting day by day. The most reliable method for analyzing problems is exact solution method. The investigation of exact traveling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. The wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibers, etc.

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In the recent years, seeking exact solutions of nonlinear partial differential equations (NLPDEs) is of great significance, and the nonlinear complex physical phenomena related to the NLPDEs are involved in many fields from physics, mechanics, chemistry and engineering. As mathematical models of the phenomena, the investigation of exact solutions of LPDEs (Linear partial differential equations) will help one to understand the mechanism that governs these physical models or to better provide knowledge the physical problem and possible applications. Some new powerful solving methods have been developed such as homotopy perturbation [1], tanh-function method [2], trial function method [3], F-expansion method [4], variational iterations method [5], and so on. Recently, He and Wu [6] proposed a straightforward and concise method, called Exp-function method, to obtain generalized solitary solutions and periodic solutions of NLEEs (nonlinear evolution equations). The solution procedure of this method, with the help of Maple or Mathematica, is of utter simplicity and this method can be easily extended to all kinds of NLEEs. The aim of this paper is to use the novel identical reforming of differential equation combined with the Exp-function method to solve the Navier–Stokes equations which can be written in the following basic form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}\right), \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial^2 x}+\frac{\partial^2 v}{\partial^2 y}\right), \tag{3}$$

where u and v speed components in direction to x and y, p is pressure, ρ is fluid density, and ν is kinematics of fluid coefficient that is positive constant.

2. Solution of the two-dimensional Navier–Stokes equations by the Exp-function method

According to the Exp-function method, we introduce a complex variable ζ, where

$$\zeta = kx+wt+sy,$$

so, Eqs. (1)–(3) turn into the following system of ordinary differential equations

$$ku'+sv'=0, \tag{4}$$

$$wu'+kuu'+svu'+\frac{k}{\rho}p'-\nu(k^2u''+s^2u'')=0, \tag{5}$$

$$wv'+kuu'+svv'+\frac{s}{\rho}p'-\nu(k^2v''+s^2v'')=0, \tag{6}$$

where the primes denote differentiations with respect to ζ. We assume that the solutions can be expressed in the following forms

$$u(\zeta)=\frac{a_c\exp(c\zeta)+\cdots+a_{-d}\exp(-d\zeta)}{b_p\exp(p\zeta)+\cdots+b_{-q}\exp(-q\zeta)}=\frac{\sum_{k=-d}^{c}a_k\exp(k\zeta)}{\sum_{k=-q}^{p}b_k\exp(k\zeta)}, \tag{7}$$

$$v(\zeta)=\frac{a'_c\exp(m\zeta)+\cdots+a'_{-n}\exp(-n\zeta)}{b'_l\exp(l\zeta)+\cdots+b'_{-r}\exp(-r\zeta)}=\frac{\sum_{k=-n}^{m}a'_k\exp(k\zeta)}{\sum_{k=-r}^{l}b'_k\exp(k\zeta)}, \tag{8}$$

$$p(\zeta)=\frac{a''_i\exp(i\zeta)+\cdots+a''_{-h}\exp(-h\zeta)}{b''_j\exp(j\zeta)+\cdots+b''_{-f}\exp(-f\zeta)}=\frac{\sum_{k=-h}^{i}a''_k\exp(k\zeta)}{\sum_{k=-f}^{j}b''_k\exp(k\zeta)}, \tag{9}$$

where c, d, p, q, m, n, l, r, i, h, j and f are positive integers and a_k, b_k, a'_k, b'_k, a''_k and b''_k are unknown constants to be determined. To determine the values of parameters c, p, m, l, k and j, we balance the linear terms of the highest order with the highest order nonlinear terms in (1)–(3).

Similarly, to determine the values of parameters d, q, n, r, h and f, we balance the linear terms of the lowest order with the lowest order nonlinear terms in (1)–(3). By simple manipulations in the first equations (4)–(6), we have

$$uu' = \frac{c_1 \exp[(p+2c)\zeta] + \cdots}{c_2 \exp(3p\zeta) + \cdots} = \frac{c_1 \exp[(p+2c+2j)\zeta] + \cdots}{c_2 \exp[(3p+2j)\zeta] + \cdots}, \tag{10}$$

$$uu' = \frac{c_3 \exp[(p+c+m)\zeta] + \cdots}{c_4 \exp[(2p+l)\zeta] + \cdots}, \tag{11}$$

and

$$\begin{aligned} u' &= \frac{c_7 \exp[(p+c)\zeta] + \cdots}{c_8 \exp[(2p\zeta) + \cdots} \\ &= \frac{c_7 \exp[(p+c+l)\zeta] + \cdots}{c_8 \exp[(2p+l)\zeta] + \cdots} \\ &= \frac{c_1 \exp[(2p+c)\zeta] + \cdots}{c_2 \exp[(3p\zeta) + \cdots}, \end{aligned} \tag{12}$$

by balancing the highest order of the Exp-function in (10)–(12), we derive

$$p+2c = 2p+c, \quad p+c+m = p+c+l, \quad p+2c+2j = k+j+3p,$$

which leads to the following result

$$p = c, \quad m = l, \quad j = k,$$

by the same way, we get

$$d = q, \quad n = r, \quad h = f,$$

the values of c, d, m, n, k and h can be chosen arbitrary. For simplicity, we set $p=c=1$, $d=q=1$, $m=l=1$, $n=r=1$, $k=j=1$, and $h=f=1$ so (7)–(9) reduce to

$$u(\zeta) = \frac{a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta)}{\exp(\zeta) + b_0 + b_{-1} \exp(-\zeta)}, \tag{13}$$

$$v(\zeta) = \frac{a'_1 \exp(\zeta) + a'_0 + a'_{-1} \exp(-\zeta)}{\exp(\zeta) + b'_0 + b'_{-1} \exp(-\zeta)}, \tag{14}$$

$$p(\zeta) = \frac{a''_1 \exp(\zeta) + a''_0 + a''_{-1} \exp(-\zeta)}{\exp(\zeta) + b''_0 + b''_{-1} \exp(-\zeta)}, \tag{15}$$

by substituting (13)–(15) in (4)–(6), and by using Maple 12, clearing the denominator and setting the coefficients of power terms like $\exp(j\zeta)$, $j=1, 2, \ldots$, to zero yields a system of algebraic equations, we obtain the following exact solutions:

$$\text{Case 1:} \quad \begin{aligned} a'_{-1} &= b_{-1}a'_1 + i(a_1 b_{-1} - a_{-1}), \\ a'_0 &= a'_1 b''_0, \\ a''_0 &= a''_1 b''_0, \\ b'_{-1} &= b_{-1}, \\ b''_{-1} &= b_{-1}, \\ k &= is, \\ a''_{-1} &= \frac{s a''_1 b_{-1} - s\rho a_1 a_{-1} + s\rho a_1^2 b_{-1} + i(-s a_1 a'_1 \rho b_{-1} + a'_1 \rho s a_{-1} - w\rho a_1 b_{-1} + w\rho a_{-1})}{s}, \end{aligned} \tag{16}$$

where $i^2 = -1$.

Substituting these results into (13)–(15), the following exact solution can be obtained

$$u(\zeta)=\frac{a_1\exp(\zeta)+a_1b_0''+a_{-1}\exp(\zeta)}{\exp(\zeta)+b_0''+b_{-1}\exp(-\zeta)}, \tag{17}$$

$$v(\zeta)=\frac{a_1'\exp(\zeta)+a_1'b_0''+b_{-1}a_1'+i(a_1b_{-1}-a_{-1})\exp(-\zeta)}{\exp(\zeta)+b_0''+b_{-1}\exp(-\zeta)}, \tag{18}$$

$$p(\zeta)=\frac{1}{s(\exp(\zeta)+b_0''+b_{-1}\exp(\zeta))}[a_1''\exp(\zeta)+sa_1''b_0''+((sa_1''b_{-1}-s\rho a_1a_{-1}+s\rho a_1^2b_{-1}) \\ +i(-sa_1a_1'\rho b_{-1}+a_1'\rho sa_{-1}-\omega\rho a_1b_{-1}+\omega\rho a_{-1}))\exp(-\zeta)], \tag{19}$$

where $\zeta = isx + sy + \omega t$.

Case 2:

$$a_0=\frac{a_{-1}+a_1b_0^2}{b_0},\quad b_{-1}=0,$$
$$a_0'=\frac{a_1'b_0b_0'-ia_{-1}}{b_0},\quad a_{-1}'=-\frac{ib_0'a_{-1}}{b_0},\quad a_{-1}''=0,$$
$$b_{-1}'=0,\quad b_{-1}''=0,\quad b_0''=0,$$
$$a_0''=-\frac{a_{-1}\rho(-wi-a_1'si+sa_1)}{sb_0},\quad k=is. \tag{20}$$

Substituting these results into (13)–(15), the following exact solution can be found

$$u(\zeta)=\frac{a_1b_0\exp(\zeta)+a_{-1}+a_1b_0^2+a_{-1}b_0\exp(-\zeta)}{b_0(\exp(\zeta)+b_0)}, \tag{21}$$

$$v(\zeta)=\frac{a_1'b_0\exp(\zeta)+a_1'b_0b_0'-ia_{-1}-ib_0'a_{-1}\exp(-\zeta)}{b_0(\exp(\zeta)+b_0')}, \tag{22}$$

$$p(\zeta)=\frac{a_1''sb_0\exp(\zeta)+a_{-1}\rho(wi+a_1'si-sa_1)}{sb_0\exp(\zeta)}, \tag{23}$$

where $\zeta = isx + sy + wt$.

Case 3:

$$a_0=\frac{a_{-1}+a_1b_0^2}{b_0},\quad b_{-1}=0,\quad a_0'=\frac{b_0^2-ia_{-1}}{b_0},\quad a_{-1}'=-ia_{-1},$$
$$b_0'=b_0,\quad b_{-1}'=0,\quad b_{-1}''=0,\quad k=is,$$
$$a_0''=-\frac{-sa_0''b_0b_0''-i(w\rho a_{-1}-a_1'\rho sa_{-1})}{sb_0},$$
$$a_{-1}''=-\frac{a_{-1}\rho b_0''(-wi-a_1'si+sa_1)}{sb_0}. \tag{24}$$

Substituting these results into (13)–(15), we can obtain the following exact solution

$$u(\zeta)=\frac{a_1b_0\exp(\zeta)+a_{-1}+a_1b_0^2+a_{-1}b_0\exp(-\zeta)}{b_0(\exp(\zeta)+b_0)}, \tag{25}$$

$$v(\zeta)=\frac{a_1'b_0\exp(\zeta)+a_1'b_0^2-ia_{-1}-ib_0a_{-1}\exp(-\zeta)}{b_0(\exp(\zeta)+b_0')}, \tag{26}$$

$$p(\zeta)=\frac{1}{sb_0(\exp(\zeta)+b_0'')}[a_1''sb_0\exp(\zeta)+sa_1''b_0b_0''+iw\rho a_{-1}+ia_1'\rho sa_{-1}-sa_1\rho a_{-1} \\ +(ia_{-1}\rho b_0''w+ia_{-1}\rho b_0''a_1's-sa_1\rho a_{-1}b_0'')\exp(-\zeta)], \tag{27}$$

where $\zeta = isx + sy + wt$.

Case 4:
$$a_0 = \frac{a_{-1} + a_1 b_0^2}{b_0}, \quad b_{-1} = -2b_0^2, \quad a'_{-1} = 0, \quad a'_0 = \frac{2b_0^2 a'_1 + 2ia_1 b_0^2 + ia_{-1}}{b_0},$$
$$a''_0 = sa''_1 b''_0, \quad a''_{-1} = a''_1 b''_{-1}, \quad b'_0 = 2b_0, \quad b'_{-1} = 0, \quad w = -sa'_1 - isa_1. \tag{28}$$

Substituting these results into (13)–(15), we can obtain the following exact solution

$$u(\zeta) = \frac{-a_1 b_0 \exp(\zeta) + a_{-1} + a_1 b_0^2 + a_{-1} b_0 \exp(-\zeta)}{b_0(-\exp(\zeta) - b_0 + 2b_0^2 \exp(-\zeta))}, \tag{29}$$

$$v(\zeta) = \frac{a'_1 b_0 \exp(\zeta) + 2b_0^2 a'_1 + 2ia_1 b_0^2 + ia_{-1}}{b_0(\exp(\zeta) + 2b_0)}, \tag{30}$$

$$p(\zeta) = a''_1, \tag{31}$$

where $\zeta = isx + sy + (-sa'_1 - isa_1)t$.

3. Conclusions

This paper utilized the Exp-function method to study the two-dimensional Navier–Stokes equations. As a result, some traveling wave solutions of the Navier–Stokes equations have been obtained. The results show that the Exp-function method is a powerful tool for obtaining a periodic solution and traveling wave solution.

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