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Characterizations of the Pareto distribution in the presence of outliers

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Abstract

Here we have given some characterizations for the Pareto distribution in the presence of outliers. It is proved that a necessary and sufficient condition for $f(x)$ to be a Pareto density function in the presence of outliers is that the statistics $X_{(r)}$ and $\frac{X_{(s)}}{X_{(r)}}$ $(1 \leq r < s \leq n)$ are independent. Further, we have derived some another characterizations of the Pareto distribution in the presence of outliers.

Key Words: Pareto distribution, Power distribution, Characterization, Order statistics, Outliers.

1 Introduction

Ahsanullah and Kabir [2] proved that necessary and sufficient condition for $f(x)$ to be a Pareto distribution is that the statistics $X_{(r)}$ and $\frac{X_{(s)}}{X_{(r)}}$ $(1 \leq r < s \leq n)$ are independent. Dallas [3] proved that for a cumulative distribution function (CDF) $G(y)$ $(y \geq \beta)$, if $\mathrm{E}(Y^r | Y > c) = \mathrm{E}\left(\frac{Yc}{\beta}\right)^r$ holds then Y has a Pareto distribution.
In this paper, we assume that the random variables $(X_1, X_2, ..., X_n)$ are such that k of them are distributed with probability density function (pdf)

$$f_2(x; \alpha, \beta, \theta) = \frac{\alpha(\beta\theta)^\alpha}{x^{\alpha+1}}, \qquad 0 < \beta\theta \leq x, \quad \alpha > 0, \ \beta > 1, \ \theta > 0, \tag{1}$$

and remaining $(n-k)$ random variables are distributed as

$$f_1(x; \alpha, \theta) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \qquad 0 < \theta \leq x, \quad \alpha > 0. \tag{2}$$

One should note that two sets of the observation (i.e. k and $n-k$) are independent. But $X_1, X_2, \ldots, X_n$ are not independent because of the model of outliers (for more details see [5, 7, 8]). Also, we may note that our assumptions are based of Dixit's model for

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the outliers problem and it is totaly different than the mixture models which considers $X_1, X_2, \ldots, X_n$ are independent.
Here, we have extended the approaches of Ahsanullah and Kabir [2] and Dallas [3] for the homogenous case of the Pareto distribution and derived some characterizations of the Pareto distribution in the presence of outliers.

2 Prerequisite Results

Assume that $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the order statistics of a random sample of size n such that k out of n are coming from pdf f_2 (or CDF F_2) and the remaining $(n-k)$ follow the pdf f_1 (or CDF F_1). The CDF and pdf of r^{th} $(1 \le r \le n)$ order statistic are

$$H_{X_{(r)}}(x) = \sum_{i=r}^{n} \sum_{j=m_5}^{m_6} C(k,j)[F_2(x)]^j[1-F_2(x)]^{k-j} C(n-k, i-j)[F_1(x)]^{i-j}[1-F_1(x)]^{n-k-i+j}, \quad (3)$$

where $m_5 = max(0, i-n+k)$ and $m_6 = min(k, i)$ and

$$\begin{aligned} h_{X_{(r)}}(x) &= kf_2(x) \sum_{j=m_1}^{m_2} \{C(k-1,j)[F_2(x)]^j[1-F_2(x)]^{k-j-1}C(n-k, r-1-j) \\ &\times [F_1(x)]^{r-j-1}[1-F_1(x)]^{n-k-r+j+1}\} + (n-k)f_1(x) \sum_{j=m_3}^{m_4} \{C(k,j)[F_2(x)]^j \\ &\times [1-F_2(x)]^{k-j}C(n-k-1, r-1-j)[F_1(x)]^{r-j-1}[1-F_1(x)]^{n-k-r+j}\}, \quad (4) \end{aligned}$$

where $m_1 = max(0, k+r-n-1)$, $m_2 = min(k-1, r-1)$, $m_3 = max(0, k+r-n)$, $m_4 = min(k, r-1)$, respectively (for more details see [4, 5, 6, 8]).
Further, the joint CDF and pdf of $(X_{(r)}, X_{(s)})$ $(1 \le r < s \le n)$ are

$$\begin{aligned} H_{X_{(r)},X_{(s)}}(x,y) &= \sum_{j=s}^{n} \sum_{i=r}^{j} \sum_{m=w_9}^{w_{10}} \sum_{l=t_9}^{t_{10}} \{C(k,m)C(k-m,l)[F_2(x)]^m[F_2(y)-F_2(x)]^l[1-F_2(y)]^{k-m-l} \\ &\times C(n-k, i-m)C(n-k-i+m, j-i-l)[F_1(x)]^{i-m}[F_1(y)-F_1(x)]^{j-i-l} \\ &\times [1-F_1(y)]^{n-k-j+m+l}\}, \quad (5) \end{aligned}$$

where $w_9 = max(0, i-n+k)$, $w_{10} = min(k, i)$, $t_9 = max(0, j-n+k-m)$ and $t_{10} = min(k-m, j-i)$ and

$$\begin{aligned} h_{X_{(r)},X_{(s)}}(x,y) &= k(k-1)f_2(x)f_2(y) \sum_{j=w_1}^{w_2} \sum_{i=t_1}^{t_2} \{C(k-2,j)C(n-k, r-1-j)[F_2(x)]^j \\ &\times [F_1(x)]^{r-1-j}C(k-2-j, i)C(n-k-r+j+1, n-s-i)[1-F_2(y)]^i \\ &\times [1-F_1(y)]^{n-s-i}[F_2(y)-F_2(x)]^{k-j-i-2}[F_1(y)-F_1(x)]^{s-r-k+i+j+1}\} \\ &+ (n-k)(n-k-1)f_1(x)f_1(y) \sum_{j=w_3}^{w_4} \sum_{i=t_3}^{t_4} \{C(n-k-2,j)C(k, r-1-j) \end{aligned}$$

$$
\begin{aligned}
&\times\ [F_1(x)]^j[F_2(x)]^{r-1-j}C(n-k-2-j,i)C(k-r+j+1,n-s-i)[1-F_1(y)]^i \\
&\times\ [1-F_2(y)]^{n-s-i}[F_1(y)-F_1(x)]^{n-k-2-i-j}[F_2(y)-F_2(x)]^{s-r+k-n+i+j+1}\} \\
&+\ k(n-k)f_1(x)f_2(y)\sum_{j=w_5}^{w_6}\sum_{i=t_5}^{t_6}\{C(n-k-1,j)C(k-1,r-1-j) \\
&\times\ [F_1(x)]^j[F_2(x)]^{r-1-j}C(n-k-j-1,i)C(k-r+j,n-s-i)[1-F_1(y)]^i \\
&\times\ [1-F_2(y)]^{n-s-i}[F_1(y)-F_1(x)]^{n-k-i-j-1}[F_2(y)-F_2(x)]^{s-r-n+k+i+j}\} \\
&+\ k(n-k)f_2(x)f_1(y)\sum_{j=w_7}^{w_8}\sum_{i=t_7}^{t_8}\{C(k-1,j)C(n-k-1,r-1-j) \\
&\times\ [F_2(x)]^j[F_1(x)]^{r-1-j}C(k-j-1,i)C(n-k-r+j,n-s-i)[1-F_2(y)]^i \\
&\times\ [1-F_1(y)]^{n-s-i}[F_2(y)-F_2(x)]^{k-i-j-1}[F_1(y)-F_1(x)]^{s-r-k+i+j}\}, \qquad (6)
\end{aligned}
$$

where $w_1 = max(0, r-n+k-1)$, $w_2 = min(k-2, r-1)$, $t_1 = max(0, k-s+r-j-1)$ and $t_2 = min(k-j-2, n-s)$, $w_3 = max(0, r-k-1)$, $w_4 = min(n-k-2, r-1)$, $t_3 = max(0, n-s-k+r-j-1)$ and $t_4 = min(n-k-j-2, n-s)$, $w_5 = max(0, r-k)$, $w_6 = min(n-k-1, r-1)$, $t_5 = max(0, n-s-k+r-j-1)$, $t_6 = min(n-k-j-1, n-s)$, $w_7 = max(0, r-n+k)$, $w_8 = min(k-1, r-1)$, $t_7 = max(0, k-s+r-j-1)$, $t_8 = min(k-j-1, n-s)$, respectively.
One should note that if $k = 1$ the joint pdf of $(X_{(r)}, X_{(s)})$ is given in Sinha [10]. Also, if we put $f_1 = f_2$ and $F_1 = F_2$ then all pdfs and CDFs are reduced to homogeneous cases.
The following equations are named as Pexider's equations.

$$f(xy) = g(x) + h(y), \qquad (7)$$

and

$$f(xy) = g(x)h(y). \qquad (8)$$

For solving these equations, the following Theorem has taken from Aczel [1] (Theorem 4. in p. 144) and Kuczma [9] (Theorem 13.3.4. in p. 358).
Theorem 2.1. The general solutions, with f continuous in a point of (7) and (8), respectively, both supposed for positive x and y, are

$$f(t) = c\ln(\alpha\beta t), \quad g(t) = c\ln(\alpha t), \quad h(t) = c\ln(\beta t), \quad (\alpha > 0,\ \beta > 0,\ t > 0), \qquad (9)$$

and

$$f(t) = abt^c, \quad g(t) = at^c, \quad h(t) = bt^c, \quad (t > 0), \qquad (10)$$

respectively, supplemented with the following trivial solutions in case of (8).

$$\begin{cases} f(t) = 0, \\ g(t) = 0, \\ h(t) \text{ arbitrary}, \end{cases} \quad \text{and} \quad \begin{cases} f(t) = 0, \\ g(t) \text{ arbitrary}, \\ h(t) = 0. \end{cases} \qquad (11)$$

3 Characterization of the Pareto distribution in the presence of outliers

Theorem 3.1. Let X be a random variable having an absolutely continuous CDF $F(x)$. A necessary and sufficient condition that X follows the Pareto distribution in the presence of outliers as given by (1) and (2) is that for some r and s $(1 \leq r < s \leq n)$ the statistics $X_{(r)}$ and $\frac{X_{(s)}}{X_{(r)}}$ are independent.

Proof. Necessity:
From (6) we can get the joint pdf of $X_{(r)}$ and $X_{(s)}$. Substituting $U = X_{(r)}$ and $V = \frac{X_{(s)}}{X_{(r)}}$ in (6), we can obtain the joint pdf of U and V as

$$h_{U,V}(u,v) = u\ h_{X_{(r)},X_{(s)}}(u,uv).$$

Then after some simplification (replace $h_{U,V}(u,v)$ with $h(u,v)$)

$$\begin{aligned}
h(u,v) &= \alpha^2\theta^{\alpha(n-r+1)}\beta^{\alpha k}v^{\alpha(s-n-1)-1}[1-v^{-\alpha}]^{s-r-1} \\
&\times\ u^{\alpha(r-n-1)-1}\Bigg\{k(k-1)\sum_{j=w_1}^{w_2}\sum_{i=t_1}^{t_2}A_1\beta^{-\alpha j}\left[1-\left(\frac{\beta\theta}{u}\right)^{\alpha}\right]^{j}\left[1-\left(\frac{\theta}{u}\right)^{\alpha}\right]^{r-1-j} \\
&+\ (n-k)(n-k-1)\beta^{-\alpha(r-1)}\sum_{j=w_3}^{w_4}\sum_{i=t_3}^{t_4}A_2\beta^{\alpha j}\left[1-\left(\frac{\theta}{u}\right)^{\alpha}\right]^{j}\left[1-\left(\frac{\beta\theta}{u}\right)^{\alpha}\right]^{r-1-j} \\
&+\ k(n-k)\beta^{-\alpha(r-1)}\sum_{j=w_5}^{w_6}\sum_{i=t_5}^{t_6}A_3\beta^{\alpha j}\left[1-\left(\frac{\theta}{u}\right)^{\alpha}\right]^{j}\left[1-\left(\frac{\beta\theta}{u}\right)^{\alpha}\right]^{r-1-j} \\
&+\ k(n-k)\sum_{j=w_7}^{w_8}\sum_{i=t_7}^{t_8}A_4\beta^{-\alpha j}\left[1-\left(\frac{\beta\theta}{u}\right)^{\alpha}\right]^{j}\left[1-\left(\frac{\theta}{u}\right)^{\alpha}\right]^{r-1-j}\Bigg\}, \qquad (12)
\end{aligned}$$

where

$$\begin{cases}
A_1 = C(k-2,j)C(n-k,r-1-j)C(k-2-j,i)C(n-k-r+j+1,n-s-i), \\
A_2 = C(n-k-2,j)C(k,r-1-j)C(n-k-2-j,i)C(k-r+j+1,n-s-i), \\
A_3 = C(n-k-1,j)C(k-1,r-1-j)C(n-k-j-1,i)C(k-r+j,n-s-i), \\
A_4 = C(k-1,j)C(n-k-1,r-1-j)C(k-j-1,i)C(n-k-r+j,n-s-i).
\end{cases} \qquad (13)$$

Therefore, it establishes the independence of U and V.

Sufficiency:
Here we assume that U and V are independent. The joint pdf of U and V is

$$h(u,v) \;=\; k(k-1)uf_2(u)f_2(uv)\sum_{j=w_1}^{w_2}\sum_{i=t_1}^{t_2}\{A_1[F_2(u)]^j[F_1(u)]^{r-1-j}[1-F_2(uv)]^i$$

$$
\begin{aligned}
&\times\ [1-F_1(uv)]^{n-s-i}[F_2(uv)-F_2(u)]^{k-j-i-2}[F_1(uv)-F_1(u)]^{s-r-k+i+j+1}\}\\
&+\ (n-k)(n-k-1)uf_1(u)f_1(uv)\sum_{j=w_3}^{w_4}\sum_{i=t_3}^{t_4}\{A_2[F_1(u)]^j[F_2(u)]^{r-1-j}[1-F_1(uv)]^i\\
&\times\ [1-F_2(uv)]^{n-s-i}[F_1(uv)-F_1(u)]^{n-k-2-i-j}[F_2(uv)-F_2(u)]^{s-r+k-n+i+j+1}\}\\
&+\ k(n-k)uf_1(u)f_2(uv)\sum_{j=w_5}^{w_6}\sum_{i=t_5}^{t_6}\{A_3[F_1(u)]^j[F_2(u)]^{r-1-j}[1-F_1(uv)]^i\\
&\times\ [1-F_2(uv)]^{n-s-i}[F_1(uv)-F_1(u)]^{n-k-i-j-1}[F_2(uv)-F_2(u)]^{s-r-n+k+i+j}\}\\
&+\ k(n-k)uf_2(u)f_1(uv)\sum_{j=w_7}^{w_8}\sum_{i=t_7}^{t_8}\{A_4[F_2(u)]^j[F_1(u)]^{r-1-j}[1-F_2(uv)]^i\\
&\times\ [1-F_1(uv)]^{n-s-i}[F_2(uv)-F_2(u)]^{k-i-j-1}[F_1(uv)-F_1(u)]^{s-r-k+i+j}\}, \qquad (14)
\end{aligned}
$$

where A_1, A_2, A_3 and A_4 are given in (13).
By using some elementary algebra we have

$$
\begin{aligned}
h(u,v) &= k(k-1)uf_2(u)f_2(uv)[F_1(u)]^{r-1}[1-F_1(uv)]^{n-s}[F_1(uv)-F_1(u)]^{s-r-k+1}\\
&\times\ [F_2(uv)-F_2(u)]^{k-2}\sum_{j=w_1}^{w_2}\sum_{i=t_1}^{t_2}\{A_1\left[\frac{F_2(u)}{F_1(u)}\right]^j\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^j\left[\frac{1-F_2(uv)}{F_2(uv)-F_2(u)}\right]^i\\
&\times\ \left[\frac{F_1(uv)-F_1(u)}{1-F_1(uv)}\right]^i\left[\frac{F_1(uv)-F_1(u)}{1-F_1(u)}\right]^j\left[\frac{1-F_2(u)}{F_2(uv)-F_2(u)}\right]^j\}\\
&+\ (n-k)(n-k-1)uf_1(u)f_1(uv)[F_2(u)]^{r-1}[1-F_2(uv)]^{n-s}[F_1(uv)-F_1(u)]^{n-k-2}\\
&\times\ [F_2(uv)-F_2(u)]^{s-r+k-n+1}\sum_{j=w_3}^{w_4}\sum_{i=t_3}^{t_4}\{A_2\left[\frac{F_1(u)}{F_2(u)}\right]^j\left[\frac{1-F_2(u)}{1-F_1(u)}\right]^j\left[\frac{1-F_1(uv)}{F_1(uv)-F_1(u)}\right]^i\\
&\times\ \left[\frac{F_2(uv)-F_2(u)}{1-F_2(uv)}\right]^i\left[\frac{1-F_1(u)}{F_1(uv)-F_1(u)}\right]^j\left[\frac{F_2(uv)-F_2(u)}{1-F_2(u)}\right]^j\}\\
&+\ k(n-k)uf_1(u)f_2(uv)[F_2(u)]^{r-1}[1-F_2(uv)]^{n-s}[F_1(uv)-F_1(u)]^{n-k-1}\\
&\times\ [F_2(uv)-F_2(u)]^{s-r+k-n}\sum_{j=w_5}^{w_6}\sum_{i=t_5}^{t_6}\{A_3\left[\frac{F_1(u)}{F_2(u)}\right]^j\left[\frac{1-F_2(u)}{1-F_1(u)}\right]^j\left[\frac{1-F_1(uv)}{F_1(uv)-F_1(u)}\right]^i\\
&\times\ \left[\frac{F_2(uv)-F_2(u)}{1-F_2(uv)}\right]^i\left[\frac{F_2(uv)-F_2(u)}{1-F_2(u)}\right]^j\left[\frac{1-F_1(u)}{F_1(uv)-F_1(u)}\right]^j\}\\
&+\ k(n-k)uf_2(u)f_1(uv)[F_1(u)]^{r-1}[1-F_1(uv)]^{n-s}[F_2(uv)-F_2(u)]^{k-1}\\
&\times\ [F_1(uv)-F_1(u)]^{s-r-k}\sum_{j=w_7}^{w_8}\sum_{i=t_7}^{t_8}\{A_4\left[\frac{F_2(u)}{F_1(u)}\right]^j\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^j\left[\frac{1-F_2(uv)}{F_2(uv)-F_2(u)}\right]^i\\
&\times\ \left[\frac{F_1(uv)-F_1(u)}{1-F_1(uv)}\right]^i\left[\frac{F_1(uv)-F_1(u)}{1-F_1(u)}\right]^j\left[\frac{1-F_2(u)}{F_2(uv)-F_2(u)}\right]^j\}. \qquad (15)
\end{aligned}
$$

Also from (4) and after some simplification, the marginal pdf of $U=X_{(r)}$ is as $h_1(u)$.

$$h_1(u)=[1-F_2(u)]^{k-1}[F_1(u)]^{r-1}[1-F_1(u)]^{n-k-r+1}D, \qquad (16)$$

where

$$\begin{aligned}
D &= kf_2(u)\sum_{j=m_1}^{m_2} B_1\left[\frac{F_2(u)}{F_1(u)}\right]^j\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^j \\
&+ (n-k)f_1(u)\left[\frac{1-F_2(u)}{1-F_1(u)}\right]\sum_{j=m_3}^{m_4} B_2\left[\frac{F_2(u)}{F_1(u)}\right]^j\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^j,
\end{aligned}$$

and

$$\begin{cases} B_1 = C(k-1,j)C(n-k,r-1-j), \\ B_2 = C(k,j)C(n-k-1,r-1-j). \end{cases} \tag{17}$$

Therefore from independency of U and V, we can write

$$h_2(v) = \frac{h(u,v)}{h_1(u)}, \tag{18}$$

where $h_2(v)$ is pdf of V.
Letting $p = p(u,v) = \frac{1-F_1(uv)}{1-F_1(u)}$ and $q = q(u,v) = \frac{1-F_2(uv)}{1-F_2(u)}$, we obtain

$$\begin{aligned}
h_2(v) &= -\{k(k-1)p^{n-s}[1-p]^{s-k-r+1}[1-q]^{k-2}\sum_{j=w_1}^{w_2}\sum_{i=t_1}^{t_2} A_1\left[\frac{F_2(u)}{F_1(u)}\right]^j\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^j \\
&\times p^{-i}[1-p]^{i+j}q^i[1-q]^{-i-j}f_2(u)\frac{\partial q}{\partial v} \\
&+ (n-k)(n-k-1)\left[\frac{F_2(u)}{F_1(u)}\right]^{r-1}\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^{r-2}[1-p]^{n-k-2}q^{n-s}[1-q]^{k+s-r-n+1} \\
&\times \sum_{j=w_3}^{w_4}\sum_{i=t_3}^{t_4} A_2\left[\frac{F_1(u)}{F_2(u)}\right]^j\left[\frac{1-F_2(u)}{1-F_1(u)}\right]^j q^{-i}[1-q]^{i+j}p^i[1-p]^{-i-j}f_1(u)\frac{\partial p}{\partial v} \\
&+ k(n-k)\left[\frac{F_2(u)}{F_1(u)}\right]^{r-1}\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^{r-2} q^{n-s}[1-q]^{k+s-r-n}[1-p]^{n-k-1} \\
&\times \sum_{j=w_5}^{w_6}\sum_{i=t_5}^{t_6} A_3\left[\frac{F_1(u)}{F_2(u)}\right]^j\left[\frac{1-F_2(u)}{1-F_1(u)}\right]^j p^i[1-p]^{-i-j}q^{-i}[1-q]^{i+j}f_1(u)\frac{\partial q}{\partial v} \\
&+ k(n-k)p^{n-s}[1-p]^{s-k-r}[1-q]^{k-1}\sum_{j=w_7}^{w_8}\sum_{i=t_7}^{t_8} A_4\left[\frac{F_2(u)}{F_1(u)}\right]^j\left[\frac{1-F_1(u)}{1-F_2(u)}\right]^j \\
&\times p^{-i}[1-p]^{i+j}q^i[1-q]^{-i-j}f_2(u)\frac{\partial p}{\partial v}\}D^{-1}.
\end{aligned} \tag{19}$$

From the assumption, we know that U and V are independent. So $h_2(v)$ is independent of u and by using the lemma in Ahsanullah and Kabir [2] $p = p(u,v) = g_1(v)$ and

$q = q(u,v) = g_2(v)$ (we say functions of v only) and the remaining parts should be constant. Therefore

$$\begin{cases} 1 - F_1(uv) = [1 - F_1(u)]g_1(v), & \theta \le u,\ 1 < v,\ \theta > 0, \\ 1 - F_2(uv) = [1 - F_2(u)]g_2(v), & \beta\theta \le u,\ 1 < v,\ \theta > 0,\ \beta > 1. \end{cases} \tag{20}$$

It is clear that these are version of Pexider's equation. So from Theorem 2.1 we can solve them. Since $F_1(x)$ and $F_2(x)$ are CDFs continuous for all $x \in [\theta, \infty)$ and $x \in [\beta\theta, \infty)$, respectively. We may conclude that

$$\begin{cases} 1 - F_1(x) = c_1 x^{-\alpha}, & \theta \le x,\ \theta > 0, \\ 1 - F_2(x) = c_2 x^{-\alpha}, & \beta\theta \le x,\ \theta > 0,\ \beta > 1, \end{cases} \tag{21}$$

where c_1, c_2 and α are constant.
After replacing these solutions in (19) and using some simplification we get

$$h_2(v) \quad = \quad \alpha v^{-\alpha(n-s+1)-1}[1 - v^{-\alpha}]^{s-r-1} H[c_2 D]^{-1}, \tag{22}$$

where

$$\begin{aligned} H \quad = \quad & k(k-1)c_2 \sum_{j=w_1}^{w_2} \sum_{i=t_1}^{t_2} A_1 \left[\frac{1 - c_2 u^{-\alpha}}{1 - c_1 u^{-\alpha}}\right]^j \left(\frac{c_1}{c_2}\right)^j \\ + \quad & (n-k)(n-k-1)c_1 \sum_{j=w_3}^{w_4} \sum_{i=t_3}^{t_4} A_2 \left[\frac{1 - c_2 u^{-\alpha}}{1 - c_1 u^{-\alpha}}\right]^{r-1-j} \left(\frac{c_1}{c_2}\right)^{r-2-j} \\ + \quad & k(n-k)c_1 \sum_{j=w_5}^{w_6} \sum_{i=t_5}^{t_6} A_3 \left[\frac{1 - c_2 u^{-\alpha}}{1 - c_1 u^{-\alpha}}\right]^{r-1-j} \left(\frac{c_1}{c_2}\right)^{r-2-j} \\ + \quad & k(n-k)c_2 \sum_{i=t_7}^{t_8} A_4 \left[\frac{1 - c_2 u^{-\alpha}}{1 - c_1 u^{-\alpha}}\right]^j \left(\frac{c_1}{c_2}\right)^j . \end{aligned}$$

We know that $C(n,j) = 0$ if $j > n$, then by using some elementary algebra $H[c_2 D]^{-1} = (n-r)C(n-r-1, n-s)$ and the right side of (22) is only depend on v and it is pdf of V. Finally, from the property of CDF, $\alpha > 0$, $c_1 = \theta^\alpha$ and $c_2 = (\beta\theta)^\alpha$. Thus sufficiency is established and the proof is complete.

Theorem 3.2. Let X be a random variable with CDF $F(x) = bF_2(x) + \bar{b}F_1(x)$ such that $F_1(x)$ $(x \ge \theta)$ and $F_2(x)$ $(x \ge \beta\theta)$ are CDFs, where $b = \frac{k}{n}$, $\bar{b} = 1 - b$, $\theta > 0$ and $\beta > 1$. If

$$\mathrm{E}(X^\alpha | X > c) = b\mathrm{E}_2\left(\frac{Xc}{\beta\theta}\right)^\alpha + \bar{b}\mathrm{E}_1\left(\frac{Xc}{\theta}\right)^\alpha, \tag{23}$$

holds for some $\alpha > 0$ then $F(x)$ is the Pareto distribution in the presence of outliers. We assume that $\mathrm{E}(X^\alpha) < \infty$.
Proof. Proof is similar as given in Dallas [3]. In the process to prove the theorem,

we should note that the solution of the differential equation $cP'(c) = -\gamma P(c)$ $(P(c) = 1 - F(c))$ is $P(c) = Ac^{-\alpha}$, where A is a constant, $\gamma = \alpha\delta/(\delta - 1) > 0$ and

$$\delta = b\int_{\beta\theta}^{\infty}\left(\frac{X}{\beta\theta}\right)^{\alpha} dF_2(x) + \bar{b}\int_{\theta}^{\infty}\left(\frac{X}{\theta}\right)^{\alpha} dF_1(x). \tag{24}$$

Comparing the solution with the assumption imply that $A = b(\beta\theta)^{\alpha} + \bar{b}\theta^{\alpha}$ and the proof is complete.

4 An actual example

Here, we have given an example of motor insurance company from Dixit and Jabbari Nooghabi [7]. From the example, we know that the data follow the Preto distribution in the presence of outliers. So by using Theorem 3.1, we can check the sufficiency. Assuming $r = 3$ and $s = 12$, we have $x_{(r)}$=63000, and $\frac{x_{(s)}}{x_{(r)}}$=2.857. So, using the copula method and independent test by package 'copula' in **R**, the result is as follows:

Global Cramer-von Mises statistic: 0.03125 with p-value 0.9950495
Combined p-values from the Mobius decomposition:
0.9950495 from Fisher's rule,
0.9950495 from Tippett's rule.

Therefore, $X_{(r)}$ and $\frac{X_{(s)}}{X_{(r)}}$ are independent, because of the p-value is grater than 0.05, as significant level of the test. So, we can conclude that the data follow the Pareto distribution in the presence of outliers.

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