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On products in the coarse shape category

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Abstract. In this talk we intend to the study of coarse shape of Cartesian products of topological spaces. We show that the Cartesian product of two compact Hausdorff spaces is a product in the coarse shape category. Finally, we show that the shape groups and the coarse shape groups commute with products under some conditions.

1 Introduction

Since founding shape theory, by K. Borsuk [1], it has been developing in several directions. Recently, N. Koceic Bilan and N. Uglesic have extended the shape theory by constructing a coarse shape category, denoted by $Sh^*_{(\mathcal{T},\mathcal{P})}$, where $\mathcal{P}$ is a pro-reflective subcategory of $\mathcal{T}$, whose objects are all objects of $\mathcal{T}$. Its isomorphisms classify objects of $\mathcal{T}$ strictly coarser than the shape does. This category is functorially related to the shape category $Sh_{(\mathcal{T},\mathcal{P})}$ by a faithful functor and consequently the shape category $Sh_{(\mathcal{T},\mathcal{P})}$ can be considered as a subcategory of $Sh^*_{(\mathcal{T},\mathcal{P})}$. Since the homotopy category of polyhedra, HPol, is pro-reflective (dense) in the homotopy category HTop, the coarse shape category $Sh^*_{(HTop,HPol)}$:=Sh^* is well-defined. The shape and coarse shape coincide on the class of spaces having homotopy type of polyhedra.

For every category $\mathcal{T}$ an essential question is:
Does a category $\mathcal{T}$ admit products?

Although there are several particular results concerning this question for the shape, this problem is still open for this category. It is well-known that for compact Hausdorff spaces X and Y, their Cartesian product $X \times Y$ is a product in this category [3]. On the other side it was shown by Keesling that Cartesian product of two non-compact spaces need not be their product in the shape category. Therefore, the following question naturally has arisen:

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(Q) For which topological spaces the Cartesian product $X \times Y$ along with the induced projections form a product of X and Y in the shape category?

In this talk, we consider the problem (Q) in the coarse shape category. Before studying products in coarse shape category we study products in the category pro*-HTop and we show that every pair of inverse systems $\mathbf{X}$, $\mathbf{Y}$ in this category has a product. As a consequence we show that the Cartesian product of two compact Hausdorff spaces is a product in the coarse shape. Also we show that the shape groups and the coarse shape groups (of compact Hausdorff spaces) commute with the product under some conditions.

2 Products in coarse shape category

As we know the shape category doesn't have the product, in general. We intend to verify the existence of product in the coarse shape category. By a similar argument of [5] we proved that the Cartesian product of two compact Hausdorff spaces is a product in this category. In this section, we prove that if the Cartesian product of two spaces X and Y admits an HPol-expansion, which is the Cartesian product of HPol-expansion of these spaces, then $X \times Y$ is a product in the coarse shape category. Finally, we show that the kth shape groups and the kth coarse shape groups of compact Hausdorff spaces commute with the product for every $k \in \mathbb{N}$.

Now, we intend to prove that $X \times Y$ is a product in the coarse shape category under some conditions. First consider some notations.

Let $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{q} : Y \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be morphisms of pro-HTop. Then $\mathbf{X} \times \mathbf{Y} = (X_\lambda \times Y_\mu, p_{\lambda\lambda'} \times q_{\mu\mu'}, \Lambda \times M)$ is an inverse system and H-mappings $p_\lambda \times q_\mu : X \times Y \to X_\lambda \times Y_\mu$ form a S-morphism $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ in pro-HTop, where $\Lambda \times M$ is directed by putting $(\lambda, \mu) \leq (\lambda', \mu')$ if and only if $\lambda \leq \lambda'$ and $\mu \leq \mu'$.

To define the canonical projection $\pi^*_{\mathbf{X}}$ fix an index $\mu \in M$. Let ${}^\mu\pi^*_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ be the S^*-morphism which consists of the index function $f^\mu : \Lambda \to \Lambda \times M$ defined by $f^\mu(\lambda) = (\lambda, \mu)$ and of the set of homotopy classes of projections ${}^\mu\pi^n_{X_\lambda} : X_\lambda \times Y_\mu \to X_\lambda$, $\lambda \in \Lambda$, $n \in \mathbb{N}$. If μ' is another index in M, then ${}^\mu\pi^*_{\mathbf{X}}$ and ${}^{\mu'}\pi^*_{\mathbf{X}}$ are equivalent S^*-morphisms. Hence they induce the same morphism of pro*-HTop which is denoted by $\pi^*_{\mathbf{X}}$. In the following theorem we show that every pair of inverse systems $\mathbf{X}$ and $\mathbf{Y}$ has a product in the category of pro*-HTop.

Theorem 1. Let $\mathbf{X}$ and $\mathbf{Y}$ be inverse systems of spaces. Let $\mathbf{f}^* : \mathbf{Z} \to \mathbf{X}$ and $\mathbf{g}^* : \mathbf{Z} \to \mathbf{Y}$ be morphisms of pro*-HTop, then there exists a unique morphism $\mathbf{h}^* : \mathbf{Z} \to \mathbf{X} \times \mathbf{Y}$ in pro*-HTop such that $\pi^*_{\mathbf{X}}\mathbf{h}^* = \mathbf{f}^*$ and $\pi^*_{\mathbf{Y}}\mathbf{h}^* = \mathbf{g}^*$.

Theorem 2. If X and Y admit HPol-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$, respectively such that $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is an HPol-expansion, then $X \times Y$ along with the coarse shape morphisms $S^*(\pi_X)$ and $S^*(\pi_Y)$, induced by ordinary projections, is a product in the coarse shape category.

Mardesic showed that if $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ are HPol-expansions of compact Hausdorff spaces X and Y, respectively, then $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is also an HPol-expansion of $X \times Y$ [5, Lemma 2 and Theorem 4]. Therefore we have the following result from Theorem 2.

Corollary 1. If X and Y are compact Hausdorff spaces, then $X \times Y$ together with the coarse shape morphisms $S^*(\pi_X)$ and $S^*(\pi_Y)$ is a product in the coarse shape category Sh*.

N. Koceic Bilan [3] introduced the kth coarse shape group $\check{\pi}^*_k(X, x)$, $k \in \mathbb{N}$, as the set of all coarse shape morphisms $F^* : (S^k, *) \to (X, x)$ with the following multiplication which makes it a group.

$$F^* + G^* = < \mathbf{f}^* > + < \mathbf{g}^* > = < \mathbf{f}^* + \mathbf{g}^* > = < [(f^n_\lambda)] + [(g^n_\lambda)] > = < [(f^n_\lambda + g^n_\lambda)] >,$$

where coarse shape morphisms F^* and G^* are represented by morphisms $\mathbf{f}^* = [(f, f^n_\lambda)]$ and $\mathbf{g}^* = [(g, g^n_\lambda)] : (S^k, *) \to (\mathbf{X}, \mathbf{x})$ in pro*-HPol$_*$, respectively. In follow, we show that the shape groups and the coarse shape groups of coarse shape path connected, compact and Hausdorff spaces commute with the product. N. Koceic Bilan proved that if X is a coarse shape path connected space, then $\check{\pi}_k^*(X, x_0) \cong \check{\pi}_k^*(X, x_1)$ for any two points $x_0, x_1 \in X$ and every $k \in \mathbb{N}_0$ [4, Corollary 1].

Theorem 3. If X and Y are coarse shape path connected spaces which admit HPol-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$, respectively such that $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is an HPol-expansion, then $\check{\pi}_k^*(X \times Y) \cong \check{\pi}_k^*(X) \times \check{\pi}_k^*(Y)$, for every $k \in \mathbb{N}$.

Remark 1. (i) With the assumption of Theorem 3 and similar to its proof we can show that $\check{\pi}_k(X \times Y) \cong \check{\pi}_k(X) \times \check{\pi}_k(Y)$, for every $k \in \mathbb{N}$.
(ii) If X and Y are coarse shape path connected, compact and Hausdorff spaces, then by a result of [5, Theorem 10] $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is an HPol-expansion, where $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ are HPol-expansions of X and Y, respectively. Therefore in this case the $\check{\pi}_k$ and $\check{\pi}_k^*$ commute with finite products, for every $k \in \mathbb{N}$, by Theorem 3.

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