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Some derivative-free solvers for numerical solution of SODEs

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Abstract In this paper, some variants of stochastic solvers free from derivatives for Itô stochastic ordinary differential equations (SODEs) are given. The derived strong variants are convergent and explicit. Then, some implicit solvers are also proposed. Numerical results are reported for confirming convergence properties and for comparing the behavior of these methods for pathwise approximation of SODEs.

Keywords Numerical solution · Stochastic differential equations · Stochastic Runge–Kutta methods · Heuristic derivation · Strong sense

Mathematics Subject Classification 60H10 · 65C30

1 Introduction

Physical systems are often modeled by ordinary differential equations (ODEs). These models may represent idealized situations, as they ignore stochastic effects. By incorporating random elements in the differential equation (either in the initial or boundary conditions for the problem, or in the function describing the physical system), a stochastic differential equation (SDE) arises. Although there is a rich theory for designing efficient computational schemes in solving ODEs, the stochastic counterparts are less well developed [3]. Toward this goal, we here investigate some (explicit and implicit) variants of stochastic Runge–Kutta (SRK) methods for solving Itô SODEs in the strong sense.

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There are many application areas where it is important that the trajectories (the sample paths) of the strong numerical approximations be close to the strong solution of the SDE. These direct simulations of the trajectories can provide considerable insight into the qualitative behavior and dynamics of the SDE. In many cases, these direct simulations can be interpreted as stochastic flows [18].

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, scalar Itô SDE

$$dx(t) = a(t, x(t))dt + b(t, x(t))dw(t), \quad t_0 \le t \le T, \tag{1}$$

where a denotes the drift term and b is the diffusion term. The functions a and b are assumed to be defined and measurable in $[t_0, T] \times \mathbb{R}$ and to satisfy both Lipschitz and linear growth bound conditions in x. These assumptions ensure the existence of a unique solution of the SDE (1) with the initial condition $x_{t_0} = x_0$ if x_0 is $\mathcal{F}_{t_0}$-measurable [20].

The existing known results [8] have shown that the following classic Euler-Maruyama (EM) method [16]

$$x_{i+1} = x_i + a(t_i, x_i)\Delta t_i + b(t_i, x_i)\Delta w_i, \tag{2}$$

converges to the true solution in probability space, where $\Delta t_i = t_{i+1} - t_i$, and $\Delta w_i = w_{t_{i+1}} - w_{t_i}$. According to the recent theory of [10], the classic EM method (2) *sometimes* diverge in L_2 sense in finite time to the true solution of (1). Another disadvantage of (2) is that its strong order is at least $\frac{1}{2}$. However, the classic EM method has its great advantage due to its simple algebraic structure and cheap computational cost [13].

We recall that a time discrete approximation $Y = (Y(t))_{t \in I_h}$ converges strongly respectively in the mean square with order p to x as $h \to 0$ (at time $t \in I_h$), if there exist a constant C and a finite constant $\delta_0 > 0$ such that

$$\mathbb{E}(\|Y(t) - x(t)\|) \le Ch^p, \tag{3}$$

respectively

$$\sqrt{\mathbb{E}(\|Y(t) - x(t)\|^2)} \le Ch^p, \tag{4}$$

holds for each $h \in]0, \delta_0[$. In this work, we will consider convergence in the mean square sense. Besides, by Jensen's inequality we have $(\mathbb{E}\|Y(t) - x(t)\|)^2 \le \mathbb{E}\|(Y(t) - x(t))^2\|$, i.e., the mean square convergence implies strong convergence of the same order.

The remaining sections of this paper are organized in what follows. In the next section, we briefly remind some significant strong solvers for finding the solution of (1). Section 3 is devoted to derive some new formulations and variants of the SRK methods for the Itô SDEs. It is also shown that the new approximation methods are convergent. Some implicit derivative-free solvers are also constructed. In Sect. 4, we apply various numerical tests to put on show the applicability and the efficiency of the derived methods. Section 5 will end the paper by providing some concluding remarks.

2 Background

The simulation methods are based on a finite time discretization:

$$t_0 = t_0 \le t_1 \le \ldots \le t_n = T, \tag{5}$$

of the considered time interval $[t_0, T]$, which may be equidistant, but in the general case also a step size control during the simulation of an approximate trajectory is possible [1].

Each approximate trajectory must be recursively computed at the above discretization points. Instead of the diffusion process itself, one uses so-called time discrete approximations of the diffusion, which allow an easy computation of the increments of approximate trajectories.

Integration schemes for SDEs can be derived in a manner analogous to that used for ODEs (e.g., via stochastic Itô-Taylor series [15]). Both Milstein method [17]

$$x_{i+1} = x_i + a(t_i, x_i)\Delta t_i + b(t_i, x_i)\Delta w_i + \frac{1}{2} b(t_i, x_i)\frac{\partial b}{\partial x}(t_i, x_i)\left(\Delta w_i^2 - \Delta t_i\right), \quad (6)$$

and Heun's scheme [19]

$$x_{i+1} = x_i + \frac{1}{2}\left[a(t_i, x_i) + a(t_i, x_{i+1}^*)\right]\Delta t_i + \frac{1}{2}\left[b(t_i, x_i) + b(t_i, x_{i+1}^*)\right]\Delta w_i, \quad (7)$$

use a Euler-type step (2) for the deterministic part of the dynamical equation because integration of the stochastic part rarely achieves accuracy with a global error of order higher than one. Note that the Milstein method is an Itô-Taylor method, meaning that it is derived from a truncation of the stochastic Itô-Taylor expansion of the solution.

In some cases, the partial derivative appears in the approximation method (6) or similar methods, which should be provided explicitly by the user, is a disadvantage. To counter this problem, Runge–Kutta methods were developed for SDEs [9]. We recall the following s-stages SRK method for the SDE (1) defined by $x_0 = x_{t_0}$ and

$$x_{n+1} = x_n + \sum_{i=1}^{s} \alpha_i h_n a(t_n + c_i h_n, H_i) + \sum_{i=1}^{s}\left(\beta_i^{(1)} I_{(1),n} + \beta_i^{(2)}\frac{I_{(1,1),n}}{\sqrt{h_n}} + \beta_i^{(3)}\sqrt{h_n}\right) b(t_n + c_i h_n, H_i), \quad (8)$$

making use of the notation $x_n = x_{t_n}$ and with the stages

$$H_i = x_n + \sum_{j=1}^{s} A_{ij} h_n a(t_n + c_j h_n, H_j) + \sum_{j=1}^{s}\left(B_{ij}^{(1)} I_{(1),n} + B_{ij}^{(2)}\frac{I_{(1,1),n}}{\sqrt{h_n}} + B_{ij}^{(3)}\sqrt{h_n}\right) b(t_n + c_j h_n, H_j), \quad (9)$$

for $i = 1, \ldots, s$. For some independent $N(0, 1)$-distributed random variables ξ_n, the random variables $I_{(1),n}$ and the iterated stochastic integrals $I_{(1,1),n}$ can be calculated by $I_{(1),n} = \sqrt{h_n} \cdot \xi_n$ and $I_{(1,1),n} = \frac{1}{2}\left(I_{(1),n}^2 - h_n\right)$, [14]. The coefficients of the SRK method (8) are presented by an extended Butcher tableau (Table 1). Applying the colored rooted tree theory for Itô SDEs, order conditions for the coefficients of the SRK method (8) can be calculated, making use of the vector $e = (1, \ldots, 1)^T \in \mathbb{R}^s$, [14].

Table 1 Extended Butcher tableau

c	A	$B^{(1)}$	$B^{(1)}$	$B^{(1)}$
	α^T	$\beta^{(1)^T}$	$\beta^{(2)^T}$	$\beta^{(3)^T}$

3 Derivative-free variants

In the SDE context, the same trade as in the ODEs can be made with the Milstein method, resulting in a strong order 1 method that requires evaluation of $b(x)$ at two places on each step. A heuristic derivation can be carried out by making the replacement

$$b_x(x_i) \approx \frac{b(x_i + b(x_i)\sqrt{\Delta t_i}) - b(x_i)}{b(x_i)\sqrt{\Delta t_i}}, \tag{10}$$

in (6), which leads to the explicit SRK method as follows

$$\begin{aligned} x_{i+1} &= x_i + a(t_i, x_i)\Delta t_i + b(t_i, x_i)\Delta w_i \\ &+ \frac{1}{2\sqrt{\Delta t_i}}\left[b(t_i, x_i + b(t_i, x_i)\sqrt{\Delta t_i}) - b(t_i, x_i)\right]\left(\Delta w_i^2 - \Delta t_i\right). \end{aligned} \tag{11}$$

An important general family of SRK methods are given in [14], which also contains the scheme (11) as one of its special members. To be more precise, (11) can be written as (8)–(9) with $s = 2$ and $\alpha = (1, 0)$, $\beta^1 = (1, 0)$, $\beta^2 = (-1, 1)$, $\beta^3 = (0, 0)$, $A_{ij} = 0$, $B_{ij}^1 = 0$, $B_{ij}^2 = 0$, $B_{ij}^3 = (0, 0; 1, 0)$.

The essence of the concept of finite difference approximations such as (10), is embodied in the standard definition of the derivative $b_x(x_i) = \lim_{h\to 0} \frac{b(x_i+h)-b(x_i)}{h}$, where instead of passing to the limit as h approaches zero, the finite spacing to the next adjacent point, $x_{i+1} = x_i + h$, is used so that you get an approximation

$$b_x(x_i) = \frac{b(x_{i+1}) - b(x_i)}{h} - \frac{h}{2} b_{xx}(\zeta_i). \tag{12}$$

An important aspect of this formula is that ζ_i must lie between x_i and x_{i+1} so that the error is local to the interval enclosing the sampling points. It is generally true for finite difference formulas that the error is local to the stencil, or the set of sample points. Typically, for convergence and other analysis, the error is expressed in asymptotic form: $b_x(x_i) = \frac{b(x_{i+1})-b(x_i)}{h} + O(h)$. Clearly, choosing $h = b(x_i)\sqrt{\Delta t_i}$ yields to (10) and subsequently the SRK scheme (11).

We also state that derivative-free schemes of order one can be obtained by discretizing the Wong-Zakai approximation of the SDE by (almost) any derivative-free ODE method, see for more [5].

Now, we apply a different backward approximation as follows

$$b_x(x_i) \approx \frac{b(x_i) - b(x_i - b(x_i)\sqrt{\Delta t_i})}{b(x_i)\sqrt{\Delta t_i}}, \tag{13}$$

which results in a new explicit variant of SRK in what follows

$$\begin{aligned} x_{i+1} &= x_i + a(t_i, x_i)\Delta t_i + b(t_i, x_i)\Delta w_i \\ &+ \frac{1}{2\sqrt{\Delta t_i}}\left[b(t_i, x_i) - b(t_i, x_i - b(t_i, x_i)\sqrt{\Delta t_i})\right]\left(\Delta w_i^2 - \Delta t_i\right). \end{aligned} \tag{14}$$

Another different approximation which is known as the second-order centered difference formula for the first derivative is known as

$$b_x(x_i) \approx \frac{b(x_i + b(x_i)\sqrt{\Delta t_i}) - b(x_i - b(x_i)\sqrt{\Delta t_i})}{2b(x_i)\sqrt{\Delta t_i}}, \tag{15}$$

which ends in another variant of SRK method for solving (1) as follows

$$\begin{aligned} x_{i+1} &= x_i + a(t_i, x_i)\Delta t_i + b(t_i, x_i)\Delta w_i \\ &+ \frac{1}{4\sqrt{\Delta t_i}}\left[b(t_i, x_i + b(t_i, x_i)\sqrt{\Delta t_i}) - b(t_i, x_i - b(t_i, x_i)\sqrt{\Delta t_i})\right]\left(\Delta w_i^2 - \Delta t_i\right). \end{aligned} \tag{16}$$

This implementation allows to achieve first strong order of convergence. Note that this is the highest strong order obtained with a Runge–Kutta approach that keeps a *simple* structure.

Theorem 3.1 *Let $\overline{x}$ and $\overline{\overline{x}}$ be two approximations using* (14) *of the stochastic process $x : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, satisfying the SDE* (1) *with time steps $\{\overline{t}_n\}_{n=0}^{\overline{N}}$, $\overline{t}_0 = t_0, \overline{t}_n = T$ and $\{\overline{\overline{t}}_m\}_{m=0}^{M}$, $\overline{\overline{t}}_0 = t_0$, $\overline{\overline{t}}_M = T$ respectively, and $\Delta t_{max} = \max[\max_{0 \le n \le \overline{N}-1} \overline{t}_{n+1} - \overline{t}_n, \max_{0 \le m \le M-1} \overline{\overline{t}}_{m+1} - \overline{\overline{t}}_m]$. Furthermore suppose that there exists a positive constant C such that the initial data and the given functions $a, b : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy*

$$\mathbb{E}\left[|\overline{x}(0)|^2 + |\overline{\overline{x}}(0)|^2\right] \le C, \tag{17}$$

$$\mathbb{E}\left[(\overline{x}(0) + \overline{\overline{x}}(0))^2\right] \le C\Delta t_{max}^2, \tag{18}$$

and $|a(t, x) - a(t, y)| < C|x - y|$, $|b(t, x) - b(t, y)| < C|x - y|$, $|a(t, x) - a(s, x)| + |b(t, x) - b(s, x)| < C(1 + |x|)\sqrt{|t - s|}$. Also we assume that the diffusion coefficient is sufficiently differentiable, which is necessary to obtain an order one method. Then, there is a constant K such that

$$\max\left\{\mathbb{E}[\overline{x}^2(t, \cdot)], \mathbb{E}[\overline{\overline{x}}^2(t, \cdot)]\right\} \le KT, \quad t < T, \tag{19}$$

and subsequently

$$\mathbb{E}\left[(\overline{x}(t, \cdot) - \overline{\overline{x}}(t, \cdot))^2\right] \le K\Delta t_{max}^2, \quad t < T. \tag{20}$$

Proof Consider the grid (5) defined by the set of nodes $\{\overline{t}_n\}_{n=0}^{\overline{N}}$, $\overline{t}_0 = t_0, \overline{t}_n = T$ and define the discrete stochastic process $\overline{x}$ by (14) as follows:

$$\begin{aligned} \overline{x}(\overline{t}_{n+1}) - \overline{x}(\overline{t}_n) &= a(\overline{t}_n, \overline{x}(\overline{t}_n))[\overline{t}_{n+1} - \overline{t}_n] + b(\overline{t}_n, \overline{x}(\overline{t}_n))[w(\overline{t}_{n+1}) - w(\overline{t}_n)] \\ &+ \frac{1}{2\sqrt{\Delta t_i}}[b(\overline{t}_n, \overline{x}(\overline{t}_n)) - b((\overline{t}_n, \overline{x}(\overline{t}_n)) - b(\overline{t}_n, \overline{x}(\overline{t}_n)))\sqrt{\Delta t_i})] \\ &\times([w(\overline{t}_{n+1}) - w(\overline{t}_n)])^2 \\ &- \frac{1}{2\sqrt{\Delta t_i}}[b(\overline{t}_n, \overline{x}(\overline{t}_n) - b((\overline{t}_n, \overline{x}(\overline{t}_n)) - b(\overline{t}_n, \overline{x}(\overline{t}_n)))\sqrt{\Delta t_i})][\overline{t}_{n+1} - \overline{t}_n], \end{aligned} \tag{21}$$

for $n = 0, 1, \ldots, \overline{N} - 1$. Now extend $\overline{x}$ continuously for theoretical purposes only, to all values of t by

$$\overline{x}(t) = \overline{x}(t_n) + \int_{\overline{t}_n}^{t} a(\overline{t}_n, \overline{x}(\overline{t}_n))ds + \int_{\overline{t}_n}^{t} b(\overline{t}_n, \overline{x}(\overline{t}_n))dw(s). \tag{22}$$

In other words, the process $\overline{x} : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, satisfies the stochastic differential equation

$$d\overline{x}(s) = \overline{a}(s, \overline{x})ds + \overline{b}(s, \overline{x})dw(s), \quad \overline{t}_n \le s \le \overline{t}_{n+1}, \tag{23}$$

wherein $\bar{a}(s,\bar{x}) \equiv a(\bar{t}_n, \bar{x}(\bar{t}_n))$, $\bar{b}(s,\bar{x}) \equiv b(\bar{t}_n, \bar{x}(\bar{t}_n))$, for $\bar{t}_n \le s \le \bar{t}_{n+1}$, and the nodal values of the process $\bar{x}$ is defined by (21).

We also now consider (19) holds. Subtracting the two approximate solutions $\bar{x}$ and $\bar{\bar{x}}$, we arrive at

$$\bar{x}(s) - \bar{\bar{x}}(s) = \bar{x}(0) - \bar{\bar{x}}(0) + \int_0^s (\bar{a} - \bar{\bar{a}})(t)dt + \int_0^s (\bar{b} - \bar{\bar{b}})(t)dw(t), \tag{24}$$

wherein $\Delta a(t) = (\bar{a} - \bar{\bar{a}})(t)$, $\Delta b(t) = (\bar{b} - \bar{\bar{b}})(t)$. The definition of the discretized solutions implies that

$$\begin{aligned}\Delta a(t) &= a(\bar{t}_n, \bar{x}(\bar{t}_n)) - a(\bar{\bar{t}}_m, \bar{\bar{x}}(\bar{\bar{t}}_m)) \\ &= a(\bar{t}_n, \bar{x}(\bar{t}_n)) - a(t, \bar{x}(t)) \\ &\quad + a(t, \bar{x}(t)) - a(t, \bar{\bar{x}}(t)) + a(t, \bar{\bar{x}}(t)) - a(\bar{\bar{t}}_m, \bar{\bar{x}}(\bar{\bar{t}}_m)),\end{aligned} \tag{25}$$

wherein $t \in [\bar{\bar{t}}_m, \bar{\bar{t}}_{m+1}) \cap [\bar{t}_n, \bar{t}_{n+1})$.

The assumptions in the theorems show further that

$$\begin{aligned}|\bar{x}(\bar{t}_n) - \bar{x}(t)| &= |a(\bar{t}_n, \bar{x}(\bar{t}_n))(t - \bar{t}_n) + b(\bar{t}_n, \bar{x}(\bar{t}_n))(w(t) - w(\bar{t}_n))| \\ &\le C(1 + |\bar{x}(\bar{t}_n)|)((t - \bar{t}_n) + |w(t) - w(\bar{t}_n)|).\end{aligned} \tag{26}$$

Therefore, we have

$$\begin{aligned}|\Delta a(t)|^2 \le C(&|\bar{x}(t) - \bar{\bar{x}}(t)|^2 \\ &+ (1 + |\bar{x}(\bar{t}_n)|^2)(|t - \bar{t}_n|^2 + |w(t) - w(\bar{t}_n)|^2) \\ &+ (1 + |\bar{\bar{x}}(\bar{\bar{t}}_m)|^2)(|t - \bar{\bar{t}}_m|^2 + |w(t) - w(\bar{\bar{t}}_m)|^2)).\end{aligned} \tag{27}$$

Note that $\mathbb{E}[(w(t) - w(s))^2] = t - s,\ s < t$. This together with the assumption (19) leads to

$$\begin{aligned}\mathbb{E}[|\Delta a(t)|^2] &\le C\left(\mathbb{E}[|\bar{x}(t) - \bar{\bar{x}}(t)|^2] + (1 + \mathbb{E}[|\bar{x}(\bar{t}_n)|^2] + \mathbb{E}[|\bar{\bar{x}}(\bar{\bar{t}}_m)|^2])\Delta t_{max}^2\right) \\ &\le C\left(\mathbb{E}[|\bar{x}(t) - \bar{\bar{x}}(t)|^2] + \Delta t_{max}^2\right).\end{aligned} \tag{28}$$

If we similarly obtain

$$\mathbb{E}[|\Delta b(t)|^2] \le C'(\mathbb{E}[|\bar{x}(t) - \bar{\bar{x}}(t)|^2] + \Delta t_{max}^2). \tag{29}$$

Now we introduce a refined grid $\{t_h\}_{h=0}^N$ by the union $\{t_h\} \equiv \{\bar{t}_n\} \cup \{\bar{\bar{t}}_m\}$. Observe that both the functions $\Delta a(t)$ and $\Delta b(t)$ are adapted and piecewise constant on the refined grid. Hence, using (24), we have

$$\begin{aligned}\mathbb{E}[|\bar{x}(s) - \bar{\bar{x}}(s)|^2] &\le \mathbb{E}\left[\left(\bar{x}(0) - \bar{\bar{x}}(0) + \int_0^s (\bar{a} - \bar{\bar{a}})(t)dt + \int_0^s (\bar{b} - \bar{\bar{b}})(t)dw(t)\right)^2\right] \\ &\le 3\mathbb{E}[|\bar{x}(0) - \bar{\bar{x}}(0)|^2] + 3\mathbb{E}\left[\left(\int_0^s (\bar{a} - \bar{\bar{a}})dt\right)^2\right] \\ &\quad + 3\mathbb{E}\left[\left(\int_0^s (\bar{b} - \bar{\bar{b}})(t)dw_t\right)^2\right] \\ &\le 3\left(C\Delta t_{max}^2 + \int_0^s \mathbb{E}[\Delta a(t)]^2 dt + \int_0^s \mathbb{E}[\Delta b(t)]^2 dt\right).\end{aligned} \tag{30}$$

Inequalities (27)–(30) results in

$$\mathbb{E}[|\overline{x}(s)-\overline{\overline{x}}(s)|^2] \leq C\left(\int_0^s \mathbb{E}[|\overline{x}(s)-\overline{\overline{x}}(s)|^2]dt + \Delta t_{max}^2\right). \tag{31}$$

Finally, applying the Grönwall's lemma [4] yields to the following convergence result:

$$\mathbb{E}[|\overline{x}(s)-\overline{\overline{x}}(s)|^2] \leq \mathcal{O}(\Delta t_{max}^2). \tag{32}$$

Hence, the proof is complete. □

It is straight forward from (32) that (14) has first order of convergence in the strong sense. In the meantime, the analysis of the scheme (16) would be similar to Theorem 3.1.

Implicit (ODE) solvers have been successful and have become the method of choice for a large class of stiff problems. Accordingly, it would be riveting to derive derivative-free implicit solvers for SODEs. In a system for which different components evolve on different time scales, implicit solvers allow us to capture the dynamics of the system on the slow time scale without resolving the transient effects on the fast time scale.

An implicit variant for the derived methods can be obtained using the stochastic theta method [7] and written as follows

$$\begin{aligned} x_{i+1} &= x_i + (1-\theta)a(t_i,x_i)\Delta t_i + \theta a(t_i,x_{i+1})\Delta t_i + b(t_i,x_i)\Delta w_i \\ &+ \frac{1}{2\sqrt{\Delta t_i}}\left[b(t_i,x_i) - b(t_i, x_i - b(t_i,x_i)\sqrt{\Delta t_i})\right]\left(\Delta w_i^2 - \Delta t_i\right), \end{aligned} \tag{33}$$

where $\theta \in [0,1]$ and also

$$\begin{aligned} x_{i+1} &= x_i + (1-\theta)a(t_i,x_i)\Delta t_i + \theta a(t_i,x_{i+1})\Delta t_i + b(t_i,x_i)\Delta w_i \\ &+ \frac{1}{4\sqrt{\Delta t_i}}\left[b(t_i, x_i + b(t_i,x_i)\sqrt{\Delta t_i}) - b(t_i, x_i - b(t_i,x_i)\sqrt{\Delta t_i})\right]\left(\Delta w_i^2 - \Delta t_i\right). \end{aligned} \tag{34}$$

Choosing $\theta = 0$ simplifies (33)–(34) to (14)–(16), while other values of θ yield in implicit schemes.

Another type of implicit solvers which have recently been discussed in the literature are split-step methods. Drifting split-step methods for the numerical solution of (1) for a single noise channel have appeared in the literature over the past several years. Ding et al. [6] considered the split-step stochastic θ-method for the solution of (1). Accordingly, we can present several drifting split-step methods based on (14) and (16) free from derivatives as follows:

$$\begin{aligned} y_i &= x_i + (1-\theta)a(t_i,x_i)\Delta t_i + \theta a(t_i,y_i)\Delta t_i, \\ x_{i+1} &= y_i + b(t_i,y_i)\Delta w_i + \frac{1}{2\sqrt{\Delta t_i}}\left[b(t_i,y_i) - b(t_i, y_i - b(t_i,y_i)\sqrt{\Delta t_i})\right]\left(\Delta w_i^2 - \Delta t_i\right), \end{aligned} \tag{35}$$

where $\theta \in [0,1]$ and also

$$\begin{aligned} y_i &= x_i + (1-\theta)a(t_i,x_i)\Delta t_i + \theta a(t_i,y_i)\Delta t_i \\ x_{i+1} &= y_i + b(t_i,y_i)\Delta w_i \\ &+ \frac{1}{4\sqrt{\Delta t_i}}[b(t_i, y_i + b(t_i,y_i)\sqrt{\Delta t_i}) - b(t_i, y_i - b(t_i,y_i)\sqrt{\Delta t_i})]\left(\Delta w_i^2 - \Delta t_i\right). \end{aligned} \tag{36}$$

All of the above-mentioned implicit drifting split-step-type methods possess similar one strong order of convergence for the mean-square approximation of (1).

4 Numerical experiments

This section addresses issues related to the numerical precision of the new SDE solvers, using Mathematica 8 built-in precision, [11]. For numerical comparisons in this section, we have used the methods (2) denoted by "EM", (6) denoted by "Mi", (11) denoted by "SRKF", (14) denoted by "SRKB", (16) denoted by "SRKC", (33) denoted by "ISRKB", and (35) denoted by "ISSB". As the programs were running, we found the running time using the command `AbsoluteTiming[]` to report the elapsed CPU time (in second) for the experiments. The computer specifications are Microsoft Windows XP Intel(R), Pentium(R) 4, CPU 3.20 GHz, with 4 GB of RAM.

Note that the Brownian motion is modeled by the increments Δw_i, which are determined from a normal random number generator. Define $N(0, 1)$ to be the standard random variable that is normally distributed with mean 0 and standard deviation 1. Hence, the random increment Δw_i is computed as $\Delta w_i = z_i\sqrt{\Delta t_i}$.

Example 4.1 Consider the Black-Scholes SDE [2] with a single noise process as follows

$$\begin{aligned} dx(t) &= \sigma x(t)dt + \mu x(t)dw(t), \quad 0 \le t \le T, \\ x(t_0) &= x(0), \end{aligned} \tag{37}$$

with the conditions $\sigma = 2.1$, $\mu = 2.9$, $x(0) = 10$, and $T = 1$.

The true stochastic process of the solution is a set of realizations. One such realization is shown in Figs. 1 and 2, corresponding to a particular realization of $w(t)$. If one fixes a realization of $w(t)$, then the red curve represents the EM method using tiny samples of $w(t)$ for the dw (applying $n = 2^8$, $dt = T/n$ and `SeedRandom[1234]`), in contrast to the exact realization with blue color. The low order of convergence for EM could easily be seen in Fig. 1 which resulted in rough approximations, while the higher order schemes such as SRKB ended in accurate approximations for the sample path in Fig. 2.

Some further examples and simulations have now been done (using [12]) to support the underlying theory given in the previous section. We also report the mean-square error instead of reporting errors for one sample path to clearly show the advantageous of the derived solvers in the strong sense.

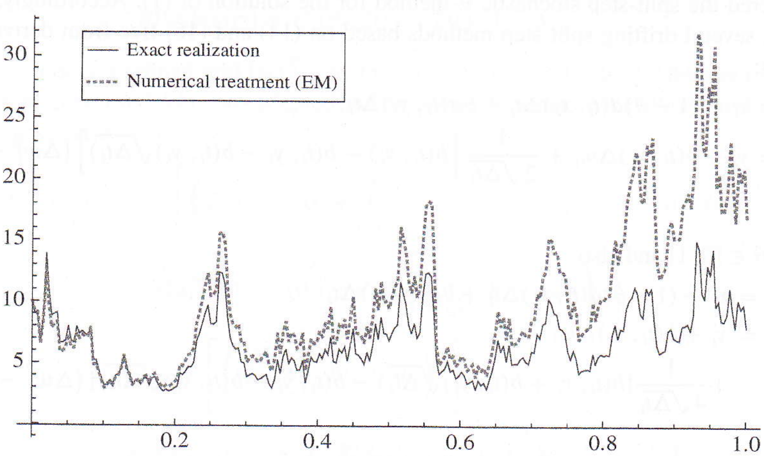

Fig. 1 The comparison of an exact realization (*blue*) and the approximate solution (*red*) of EM method in Example 4.1 (color figure online)

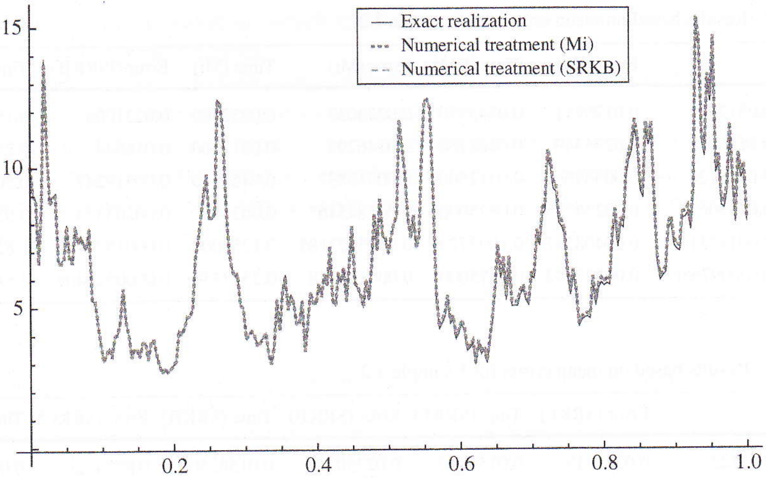

Fig. 2 The comparison of an exact realization (*blue*) and the approximate solution (*red*) of Mi and SRKB (*brown*) methods in Example 4.1 (color figure online)

Example 4.2 The following nonlinear test has an interesting cautionary property. We can find an explicit solution, but it exists only for a finite time span.

$$\begin{aligned} dx(t) &= -2e^{-2x(t)}dt + 2e^{-x(t)}dw(t), \quad 0 \le t \le T = 4. \\ x(t_0) &= 2. \end{aligned} \tag{38}$$

Using Itô formula, one may show that $x(t) = \ln(2w(t) + e^{x_0})$ is a solution, as long as the quantity inside the logarithm is positive. At the first time t when the Brownian motion realization causes $2w(t) + e^{x_0}$ to be negative, the solution stops existing.

In this test, we have considered the mean error based on 1,000 trajectories, and $n = 2^{12}$ (in the simulation of the Brownian motion), for different equi-distant step sizes (Δt) as described in Tables 2 and 3. In theses tables, the values represent the error $e(T) = |x(T) - x_{\Delta t}(T)|$ at $T = 4$ over 1,000 realizations (mean errors). Clearly, the computational complexity of SRKC is higher than the two other explicit forms SRKF and SRKB, although we expect to obtain somehow better accuracies from SRKC.

Example 4.3 Consider the nonlinear stochastic differential equation

$$\begin{aligned} dx(t) &= \left(\tfrac{1}{3}x(t)^{\frac{1}{3}} + 6x(t)^{\frac{2}{3}}\right)dt + x(t)^{\frac{2}{3}}dw(t), \\ x(t_0) &= 1, \end{aligned} \tag{39}$$

with the exact solution $x(t) = \left(2t + 1 + \frac{1}{3}w(t)\right)^3$, wherein $0 \le t \le 1$.

We compare the behavior of different methods discussed in Sections 1–3 for 500 trajectories and $n = 2^{16}$ in our Mathematica codes. The results of comparisons are provided in Tables 4 and 5. In theses tables, the values represents the error $e(T) = |x(T) - x_{\Delta t}(T)|$ at $T = 1$ over 500 realizations. The step sizes $(\Delta t = 2^{-16})$ are enough small for the nonlinear SDE. As could be observed from the numerical results, the new variant of SRK methods have similar or better accuracy in contrast to the existing methods EM, Mi, and SRKF.

Also note that the importance of the higher order schemes is obvious here since by reducing the step sizes more than $\Delta t = 2^{-16}$, almost all methods will face with un-stability due to

Table 2 Results based on mean errors for Example 4.2

Δt	Error (EM)	Time (EM)	Error (Mi)	Time (Mi)	Error (ISRKB)	Time (ISRKB)
$2^{-5} \approx 0.03125$	0.0179711	0.0343750	0.0224037	0.0352500	0.0231008	0.1562500
$2^{-6} \approx 0.015625$	0.0275349	0.0468750	0.0146293	0.0312500	0.016544	0.2500000
$2^{-7} \approx 0.0078125$	0.00569661	0.0312500	0.00318837	0.0468750	0.00319247	0.5000000
$2^{-8} \approx 0.00390625$	0.00234795	0.0625000	0.00184546	0.0625000	0.00203333	0.9375000
$2^{-9} \approx 0.00195313$	0.00408547	0.1093750	0.0005827185	0.1250000	0.0001524163	1.8281250
$2^{-10} \approx 0.000976563$	0.00323342	0.1875000	0.000103748	0.2343750	0.0000524409	3.5468750

Table 3 Results based on mean errors for Example 4.2

Δt	Error (SRKF)	Time (SRKF)	Error (SRKB)	Time (SRKB)	Error (SRKC)	Time (SRKC)
$2^{-5} \approx 0.03125$	0.0221819	0.0156250	0.0225077	0.0156250	0.0224826	0.0156250
$2^{-6} \approx 0.015625$	0.0125281	0.0312500	0.0168671	0.0312500	0.0147089	0.0312500
$2^{-7} \approx 0.0078125$	0.00300313	0.0468750	0.00538018	0.0468750	0.00519326	0.0625000
$2^{-8} \approx 0.00390625$	0.00154449	0.1093750	0.00315535	0.0781250	0.00385011	0.0781250
$2^{-9} \approx 0.00195313$	0.000171299	0.1718750	0.0002834118	0.1406250	0.000181388	0.1562500
$2^{-10} \approx 0.000976563$	0.0000546799	0.2656250	0.000153606	0.2656250	0.00010416	0.2968750

Table 4 Results based on mean errors for Example 4.3

Δt	Error (EM)	Time (EM)	Error (Mi)	Time (Mi)	Error (ISRKB)	Time (ISRKB)
2^{-16}	0.00770806	3.9531250	0.000522505	4.9218750	0.000841329	18.6406250

Table 5 Results based on mean errors for Example 4.3

Δt	Error (SRKF)	Time (SRKF)	Error (SRKB)	Time (SRKB)	Error (SRKC)	Time (SRKC)
2^{-16}	0.000525561	5.3281250	0.000519441	5.3125000	0.000522501	5.7812500

the round-off errors. Although the implicit solvers are costly in terms of computational time, they are very useful for stiff problems or cases at which not a too small time step size is chosen.

5 Concluding remarks

SDEs are becoming more and more important due to their applications for modeling stochastic phenomena in different fields. Unfortunately, in many cases analytic solutions of these equations are not available and we are forced to use computational schemes to solve them. Roughly speaking, there are two basic ways to achieve these approximations. When sample paths of the solutions need to be approximated, mean-square convergence is used and the derived methods are called strong solvers. When we are only interested in the moments or

other functionals of the solution, wh
methods are called weak schemes.
In this work, we have derived son
approximations of the derivatives with
the new variants. It has been shown tha
Our main methods were furnished as
have been performed from different
proposed variants.

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Mi)	Time (Mi)	Error (ISRKB)	Time (ISRKB)
037	0.0352500	0.0231008	0.1562500
293	0.0312500	0.016544	0.2500000
8837	0.0468750	0.00319247	0.5000000
4546	0.0625000	0.00203333	0.9375000
827185	0.1250000	0.0001524163	1.8281250
03748	0.2343750	0.0000524409	3.5468750

SRKB)	Time (SRKB)	Error (SRKC)	Time (SRKC)
5077	0.0156250	0.0224826	0.0156250
8671	0.0312500	0.0147089	0.0312500
38018	0.0468750	0.00519326	0.0625000
15535	0.0781250	0.00385011	0.0781250
2834118	0.1406250	0.000181388	0.1562500
153606	0.2656250	0.00010416	0.2968750

Time (Mi)	Error (ISRKB)	Time (ISRKB)
4.9218750	0.000841329	18.6406250

Time (SRKB)	Error (SRKC)	Time (SRKC)
5.3125000	0.000522501	5.7812500

rs are costly in terms of computational time,
at which not a too small time step size is

e to their applications for modeling stochas-
, in many cases analytic solutions of these
use computational schemes to solve them.
chieve these approximations. When sample
mean-square convergence is used and the
n we are only interested in the moments or

other functionals of the solution, which implies a much weaker form of convergence, the methods are called weak schemes.

In this work, we have derived some variants of strong SRK methods. Finite difference approximations of the derivatives with a proper increments has been done in Sect. 3 to derive the new variants. It has been shown that the variants are convergent in quadratic mean as well. Our main methods were furnished as several implicit solvers. Several numerical examples have been performed from different application areas to support the applicability of the proposed variants.

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