A CHARACTERIZATION OF HIGHER DERIVATIONS ON BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra and let every module-valued derivation from $\mathcal{A}$ to any Banach $\mathcal{A}$-bimodule be continuous. We show that if $\{d_m\}$ is a higher derivation from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ with continuous d_0, then there exist a continuous left $\mathcal{A}$-module homomorphism $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \to \mathcal{B}$ and a sequence $\{D_m\}$ of module-valued derivations from $\mathcal{A}$ into $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_m = U \circ D_m$ $(m \geq 1)$, and as a consequence $\{d_m\}$ is automatically continuous. We also obtain a partial result concerning innerness of higher derivations on W^*-algebras.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A family of linear mappings $\{d_m\}_{m=0}^{k}$ (k might be ∞) from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher derivation of rank k* if

$$d_m(ab) = \sum_{j=0}^{m} d_j(a)d_{m-j}(b) \quad (a, b \in \mathcal{A}, \quad m = 0, 1, 2, ..., k).$$

If there is no cause of ambiguity, a higher derivation will be simply denoted by $\{d_m\}$. It is obvious that for a higher derivation $\{d_m\}$, d_0 is a homomorphism and d_1 is a d_0-derivation that is, $d_1(ab) = d_0(a)d_1(b) + d_1(a)d_0(b)$. A standard example of a higher derivation of rank k is $\{\frac{D^m}{m!}\}_{m=0}^{k}$, where $D : \mathcal{A} \longrightarrow \mathcal{A}$ is a derivation. A higher derivation $\{d_m\}$ is said to be continuous if d_m is continuous for all $m \geq 0$.

Higher derivations were introduced by Hasse and Schmidt [5] and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find more about the algebraic properties of higher derivations in [1], [4], [5], [15], [18], [21], [19], [6], [12]. Loy [11], Jewell [9] and Villena [20] proved the automatic continuity of higher derivations in certain cases. In [7] and [8], the authors proved some results concerning higher derivations on JB^*-algebras and Banach algebras. If $\{d_m\}$ is a higher derivation from $\mathcal{A}$ to $\mathcal{A}$ such that d_0 is the identity map on $\mathcal{A}$, then d_1 is a derivation and $\{d_m\}$ is called a *strong* higher derivation. In [10] Jun and Lee proved the Singer-Wrermer theorem for strong higher derivations. Mirzavaziri in [13] gives a characterization of a strong higher derivation defined on an algebra.

Let $\{d_m\}$ be a higher derivation from a Banach algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$. Define

$$a.x = d_0(a)x,\ x.a = xd_0(a) \quad (a \in \mathcal{A}, x \in \mathcal{B}). \tag{1.1}$$

Since d_0 is a homomorphism, $\mathcal{B}$ is an $\mathcal{A}$-bimodule with respect to the mappings

$$(a, x) \to a.x,\ (a, x) \to x.a,\ \mathcal{A} \times \mathcal{B} \to \mathcal{B}.$$

It is easy to see that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule provided that d_0 is continuous. In section 2 we give a characterization for higher derivations on certain Banach algebras. We show that if every module-valued derivation on a Banach algebra $\mathcal{A}$ is continuous, then each higher derivation $\{d_m\}$ from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ with continuous d_0, is of the form $d_m = U \circ D_m$ ($m \geq 1$), where U is a continuous left $\mathcal{A}$-module homomorphism and each D_m is a module-valued derivation. Therefore $\{d_m\}$ is continuous. As a consequence every higher derivation from a C^*-algebra, with continuous d_0, is continuous. In section 3 we define an inner higher derivation. We show that if $\mathcal{A}$ is a commutative W^*-subalgebra of a W^*-algebra $\mathfrak{M}$, then each strong higher derivation from $\mathcal{A}$ to $\mathfrak{M}$ is inner.

2. Characterization

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule. A linear map $S : \mathcal{A} \longrightarrow \mathcal{X}$ is said to be left-intertwining if the map

$$b \longmapsto aS(b) - S(ab),\ \mathcal{A} \longrightarrow \mathcal{X},$$

is continuous for each $a \in \mathcal{A}$, and right-intertwining if the map

$$a \longmapsto S(a)b - S(ab),\ \mathcal{A} \longrightarrow \mathcal{X},$$

is continuous for all $b \in \mathcal{A}$. A linear map $S : \mathcal{A} \longrightarrow \mathcal{X}$ is intertwining if it is both left- and right-intertwining. For more about these objects see [2, Section 2.7].

Remark 2.1

(i) Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Suppose that $\{d_m\}$ is a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ for which d_0 is continuous. Consider $\mathcal{B}$ as a Banach $\mathcal{A}$-bimodule as in (1.1). Then it is easy to see that for every integer $m \geq 1$, $d_m : \mathcal{A} \longrightarrow \mathcal{B}$ is an intertwining map whenever $d_0, ..., d_{m-1}$ are continuous.

(ii) Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule. Consider $\mathcal{A}_1 = \mathbb{C} \oplus A$ to be the Banach algebra unitization of $\mathcal{A}$. Even if $\mathcal{A}$ is unital, $\mathcal{A}_1 \neq \mathcal{A}$ and $\mathcal{A}_1$ is a unital Banach algebra containing $\mathcal{A}$ as a closed ideal. The identity $(1, 0)$ of $\mathcal{A}_1$ will be denoted by 1. Set $\mathcal{F} = \mathfrak{B}(\mathcal{A}_1, \mathcal{X})$, the Banach space of all bounded linear operators from $\mathcal{A}_1$ to $\mathcal{X}$. For $a \in \mathcal{A}$ and $T \in \mathcal{F}$, define

$$(a.T)(b) = aT(b), \qquad (T.a)(b) = T(ab) \quad (b \in \mathcal{A}_1).$$

Then $\mathcal{F}$ is an $\mathcal{A}$-bimodule with respect to the maps

$$(a, T) \longrightarrow a.T, \ (a, T) \longrightarrow T.a, \ \mathcal{A} \times \mathcal{F} \longrightarrow \mathcal{F}.$$

Now, the map

$$U : T \longmapsto T(1), \ \mathcal{F} \longrightarrow \mathcal{X}$$

is a continuous linear operator and clearly

$$U(a.T) = aU(T) \quad (a \in \mathcal{A}, \ S \in \mathcal{F}),$$

so that U is a left $\mathcal{A}$-module homomorphism.

Dales and Villena in [3, Theorem 2.1] proved that $\mathcal{F}$ is a Banach $\mathcal{A}$-bimodule. Also in the same theorem it has been shown that each left-intertwining map $S : \mathcal{A} \longrightarrow \mathcal{X}$ is of the form $S = U \circ D$, where U is defined as above and $D : \mathcal{A} \longrightarrow \mathcal{F} = \mathfrak{B}(\mathcal{A}_1, \mathcal{X})$ is a derivation defined by

$$D(a)(\beta, b) = S(\beta a + ab) - a.S(b) \qquad (\beta \in \mathbb{C}, \ a, b \in \mathcal{A}).$$

Theorem 2.2 *Let $\mathcal{A}$ be a Banach algebra for which every derivation from $\mathcal{A}$ into an arbitrary Banach $\mathcal{A}$-bimodule is continuous. Suppose that $\{d_m\}$ is a higher derivation from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ with a continuous d_0. Then there exists a sequence $\{D_m\}_{m\geq 1}$ of derivations from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_m = U \circ D_m \quad (m \geq 1)$, where $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \to \mathcal{B}$ is the continuous left $\mathcal{A}$-module homomorphism defined by $U(T) = T(1)$ for all $T \in \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$. Moreover, $\{d_m\}$ is automatically continuous.*

Proof. By continuity of d_0, $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule with module operations defined in (1.1). Therefore d_1 is a module-valued derivation to a Banach $\mathcal{A}$-bimodule and also an intertwining by Remark 2.1 (i). Now, by Remark 2.1 (ii) there exists a derivation $D_1 : \mathcal{A} \to \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_1 = U \circ D_1$, where

$U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \to \mathcal{B}$ is defined by $U(T) = T(1)$ $(T \in \mathfrak{B}(\mathcal{A}_1, \mathcal{B}))$ which is a continuous left $\mathcal{A}$-module homomorphism. Continuity of d_1 is obvious by the assumption. Now by induction assume that for $i = 1, ..., m-1$, $d_i = U \circ D_i$, where each D_i is a derivation from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ which is continuous by the hypothesis and $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \to \mathcal{B}$ is as before. Now we have $d_0, d_1, ..., d_{m-1}$ are continuous and hence d_m is an intertwining map and again by Remark 2.1 (ii) it is of the form $U \circ D_m$, where U is as before and $D_m : \mathcal{A} \to \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ is a derivation. Since by the assumption each D_m is continuous, the last assertion follows easily. ∎

Corollary 2.3 *Every higher derivation $\{d_m\}$ from a C^*-algebra to a Banach algebra, with continuous d_0, is continuous.*

Proof. Since every module-valued derivation from a C^*-algebra is continuous [14], then by Theorem 2.2 we have the result. ∎

We recall that a derivation δ from a Banach algebra $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is said to be inner if there exists $x \in \mathcal{X}$ such that $\delta(a) = ax - xa$ $(a \in \mathcal{A})$. A Banach algebra $\mathcal{A}$ for which every bounded module-valued derivation to an arbitrary Banach $\mathcal{A}$-bimodule is inner is called *super-amenable* [16].

Corollary 2.4 *Let $\mathcal{A}$ be a super-amenable Banach algebra satisfying the hypothesis of Theorem* 2.2. *Then for every higher derivation $\{d_m\}$ from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$, with continuous d_0, we have $d_m = U \circ \delta_m$ $(m \geq 1)$, where each δ_m is an inner derivation from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ and U is defined as in Remark* 2.1 (ii).

3. Inner higher derivations

We recall the definition of an inner higher derivation from [15].

Definition 3.1 Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\{d_m\}$ be a higher derivation from $\mathcal{A}$ into $\mathcal{B}$. Then $\{d_m\}$ is called *inner* if for each $m \in \mathbb{N}$, there are $u_1, \ldots, u_m \in \mathcal{B}$ such that

$$d_m(a) = d_0(a)u_m - \sum_{i=0}^{m-1} u_{m-i} d_i(a) \quad (a \in \mathcal{A},\ m \in \mathbb{N}).$$

Note that if d_0 is continuous, then the inner higher derivation $\{d_m\}$ is also continuous.

Example 3.2 If $\{d_m\}$ is a higher derivation from a unital Banach algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ such that $d_0(\mathcal{A})\mathcal{B} = 0$, then $\{d_m\}$ is inner. To see this, suppose $a \in \mathcal{A}$ and let e be the identity element of $\mathcal{A}$. Then

$$d_m(a) = d_m(ea) = \sum_{i=0}^{m} d_i(e) d_{m-i}(a) = d_1(e)d_{m-1}(a) + \ldots + d_{m-1}(e)d_1(a) + d_m(e)d_0(a).$$

For $1 \le i \le m$ put $u_i = -d_i(e)$, then we have

$$d_m(a) = d_0(a)u_m - u_m d_0(a) - u_{m-1}d_1(a) - \ldots - u_1 d_{m-1}(a).$$

Therefore $\{d_m\}$ is inner.

It is well known that every derivation from a W^*-algebra $\mathfrak{M}$ to itself is inner [17, Theorem 2.5.3]. Also by [17, Corollary 2.5.4], if $\mathcal{A}$ is a C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$ where $\mathfrak{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, then every derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is inner when we consider it as a derivation from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{H})$. More precisely, there exists $x \in \mathcal{B}(\mathcal{H})$ such that $\delta(a) = ax - xa$ $(a \in \mathcal{A})$.

Proposition 3.3 *Let $\mathfrak{M}$ be a commutative W^*-algebra and $\{d_m\}$ a strong higher derivation from $\mathfrak{M}$ to $\mathfrak{M}$. Then each d_m $(m \ge 1)$ is zero.*

Proof. By [17, Theorem 2.5.3] the result is obvious. ∎

We are far from a proof of Sakai's result [17, Theorem 2.5.3] for higher derivations, but we can prove some partial results concerning higher derivations on W^*-algebras.

We recall the well known Markov-Kakutani theorem.

Theorem 3.4 *Let K be a non-empty convex compact subset of a locally convex space and let $\mathcal{S}$ be a commutative semigroup of continuous affine maps on K. Then $\mathcal{S}$ has a fixed point.*

Theorem 3.5 *Let $\mathfrak{M}$ be a W^*-algebra with identity element e and $\{d_m\}$ a strong higher derivation from $\mathfrak{M}$ to $\mathfrak{M}$. Let $\mathcal{A}$ be a commutative W^*-subalgebra of $\mathfrak{M}$ containing e. Then for each $m \in \mathbb{N}$ there are $u_0 = e, u_1, \ldots, u_m$ in $\mathfrak{M}$ such that $d_m(a) = au_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a)$ for all $a \in \mathcal{A}$ and*

$$\|u_m\| \le \|d_m\| + \|u_{m-1}\|\|d_1\| + \ldots + \|u_1\|\|d_{m-1}\|.$$

Proof. By [17, Lemma 2.5.1] the result holds for $m = 1$. Now suppose that for each $j \in \{1, ..., m-1\}$ there exist $u_0 = e, u_1, \ldots, u_j$ in $\mathfrak{M}$ such that, $d_j(a) = au_j - \sum_{i=0}^{j} u_{j-i}d_i(a)$ for all $a \in \mathcal{A}$ and

$$\|u_{m-1}\| \le \|d_{m-1}\| + \|u_{m-2}\|\|d_1\| + \ldots + \|u_1\|\|d_{m-2}\|.$$

Let $\mathcal{A}^u$ be the group of all unitary elements in $\mathcal{A}$. Since each element of $\mathcal{A}$ is a finite linear combination of elements in $\mathcal{A}^u$, so it is enough to show that the result holds for $\mathcal{A}^u$. For $a \in \mathcal{A}^u$, define $T_a(x) = [ax - d_m(a) - u_{m-1}d_1(a) - \ldots - u_1 d_{m-1}(a)]a^{-1}$ $(x \in \mathfrak{M})$. Then each T_a is an affine map. If $a, b \in \mathcal{A}^u$, then we have

$T_aT_b(x) = T_a\big[\big(bx - d_m(b) - u_{m-1}d_1(b) - \ldots - u_1d_{m-1}(b)\big)b^{-1}\big]$
$= \big(abxb^{-1} - ad_m(b)b^{-1} - au_{m-1}d_1(b)b^{-1} - \ldots - au_1d_{m-1}(b)b^{-1}$
$- d_m(a) - u_{m-1}d_1(a) - \ldots - u_1d_{m-1}(a)\big)a^{-1}$
$= \big(abx - d_m(ab) + d_m(a)b + d_{m-1}(a)d_1(b) + \ldots + d_1(a)d_{m-1}(b)\big)(ab)^{-1}$
$- au_{m-1}d_1(b)(ab)^{-1} - au_{m-2}d_2(b)(ab)^{-1} - \ldots - au_1d_{m-1}(b)(ab)^{-1}$
$- d_m(a)a^{-1} - u_{m-1}d_1(a)a^{-1} - u_{m-2}d_2(a)a^{-1} - \ldots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1}.$

Now, by the induction hypothesis, we have

$T_aT_b(x) = abx(ab)^{-1} - d_m(ab)(ab)^{-1} - \big(au_{m-1} - d_{m-1}(a)\big)d_1(b)(ab)^{-1}$
$- \ldots - \big(au_1 - d_1(a)\big)d_{m-1}(b)(ab)^{-1} - u_{m-1}d_1(a)a^{-1} - u_{m-2}d_2(a)a^{-1}$
$- \ldots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1} = abx(ab)^{-1} - d_m(ab)(ab)^{-1}$
$- \big(au_{m-1} - au_{m-1} + u_{m-1}a + u_{m-2}d_1(a) + \ldots + u_1d_{m-2}(a)\big)d_1(b)(ab)^{-1}$
$- \ldots - \big(au_1 - u_1a + u_1a\big)d_{m-1}(b)(ab)^{-1} - u_{m-1}d_1(a)a^{-1}$
$- u_{m-2}d_2(a)a^{-1} - \ldots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1}$
$= abx(ab)^{-1} - d_m(ab)(ab)^{-1} - u_{m-1}\big(ad_1(b) + d_1(a)b\big)(ab)^{-1}$
$- u_{m-2}\big(d_1(a)d_1(b) + d_2(a)b + ad_2(b)\big)(ab)^{-1}$
$- \ldots - u_1\big(ad_{m-1}(b) + d_1(a)d_{m-2}(b) + \ldots + d_{m-2}(a)d_1(b) + d_{m-1}(a)b\big)(ab)^{-1}$
$= \big(abx - d_m(ab) - u_{m-1}d_1(ab) - u_{m-2}d_2(ab) - \ldots - u_1d_{m-1}(ab)\big)(ab)^{-1}$
$= T_{ab}(x) = T_{ba}(x) = T_bT_a(x).$

Let σ denote the weak operator topology on $\mathfrak{M}$ and let K_m be the σ-closed convex hull of $\{T_a(0) : a \in \mathcal{A}^u\}$. Since each T_a is σ-continuous and $T_aT_b(x) = T_{ab}(x)$ $(a, b \in \mathcal{A}^u, x \in \mathfrak{M})$, it is easy to show that $T_a(K_m) \subseteq K_m$. On the other hand

$$\begin{aligned}\|T_a(0)\| &= \|\big(-d_m(a) - u_{m-1}d_1(a) - \ldots - u_1d_{m-1}(a)\big)a^{-1}\| \\ &\leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \ldots + \|u_1\|\|d_{m-1}\|,\end{aligned}$$

and it follows that

$$\sup_{x\in K_m} \|x\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \ldots + \|u_1\|\|d_{m-1}\|.$$

Therefore, K_m is σ-compact and $\{T_a : a \in \mathcal{A}^u\}$ is a commutative semigroup of σ-continuous affine maps on K_m. Thus by Theorem 3.4 there is an element $u_m \in \mathfrak{M}$ such that

$$T_a(u_m) = u_m \quad (a \in \mathcal{A}^u).$$

Therefore

$$d_m(a) = au_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \quad (a \in \mathcal{A}^u),$$

and clearly

$$\|u_m\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \ldots + \|u_1\|\|d_{m-1}\|.$$

■

Corollary 3.6 *If $\mathcal{A}$ is a commutative W^*-subalgebra of a W^*-algebra $\mathfrak{M}$ containing the identity element, then each strong higher derivation from $\mathcal{A}$ to $\mathfrak{M}$ is inner.*

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