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On the piecewise-spectral homotopy analysis method and its convergence: solution of hyperchaotic Lü system

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Abstract — In this paper, a novel modification of the spectral-homotopy analysis method (SHAM) technique for solving highly nonlinear initial value problems that model systems with chaotic and hyper-chaotic behaviour is presented. The proposed method is based on implementing the SHAM on a sequence of multiple intervals thereby increasing it's radius of convergence to yield highly accurate method which is referred to as the piece-wise spectral homotopy analysis method (PSHAM). We investigate the application of the PSHAM to the Lü system [20] which is well known to display periodic, chaotic and hyper-chaotic behaviour under carefully selected values of it's governing parameters. Existence and uniqueness of solution of SHAM that give a guarantee of convergence of SHAM, has been discussed in details. Comparisons are made between PSHAM generated results and results from literature and Runge–Kutta generated results and good agreement is observed.

Keywords: hyperchaotic system, Banach's fixed point theorem, piecewise-spectral homotopy analysis method, spectral collocation

1. Introduction

The study of initial value problems (IVPs) that model chaotic motion continues to be an active area of research. Chaos theory studies the behaviour

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of dynamical systems that are highly sensitive to initial conditions and have complex and highly unpredictable profiles. Chaotic systems can be observed in a wide variety of applications such as mechanics of nonlinear pendula, nonlinear acoustics, crystal growth, cell automata, turbulent flow, nonlinear feedback systems, population dynamics, electrodynamics, optics, and many other areas of physical and life sciences, engineering and economics.

Research into several classic and new or prototype models of chaos is now well documented. Recently, there has been a surge in the interest of hyper-chaotic systems. Hyper-chaotic systems are typically defined by four dimensional IVPs and show chaotic behaviour with at least two positive Lyapunov exponents. The first four-dimensional hyperchaotic system was identified by Rössler [30]. During the last two decades, various hyperchaotic systems have been discovered such as the hyperchaotic Lorenz–Haken system [18], hyperchaotic Chua's circuit [10], hyperchaotic Chen [8,15,16] and hyperchaotic Lu system [19, 20].

The complex nature of chaotic and hyperchaotic systems precludes the possibility of obtaining closed form analytical solutions of the underlying governing equations. Thus, approximate-analytical methods, which are implemented on a sequence of multiple intervals to increase their radius of convergence, are often used to solve IVPs modelling chaotic systems. Examples of multi-stage methods that have been developed recently to solve IVPs for chaotic and non-chaotic systems include the, multi-stage homotopy analysis method [2, 4, 5], piecewise homotopy perturbation methods [9, 29, 33], multi-stage Adomian decomposition method [1,27], multi-stage differential transformation method, [3, 13, 28], multi-stage variational iteration method [12, 26]. Other multistage methods which use numerical integration techniques have also been proposed such as the piecewise iteration method [11] which uses a spectral collocation method to perform the integration process. Accurate solutions of highly chaotic and hyper-chaotic systems requires resolution over many small intervals. Thus, seeking analytical solutions over the numerous intervals may be impractical or computationally expensive if the solution is sought over very long intervals. In this paper we propose a new approach based on the spectral homotopy analysis method for solving the chaotic and hyperchaotic Lu system [19, 20]. The spectral homotopy analysis method (SHAM) was recently proposed in [21, 22] as a flexible numerical implementation of Liao's [17] homotopy analysis method (HAM). The SHAM has previously been applied on boundary value problems (see, i.e., [21–23,31]) and it may not be useful in its standard form as a method for

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solving IVPs. However, when implemented on a sequence of multiple intervals, the resulting extended version of the SHAM, hereinafter referred to as the peace-wise spectral homotopy analysis method (PSHAM), is highly accurate and robust enough to be a suitable for solving IVPs with chaotic and hyperchaotic behaviour [24, 25]. In this work we demonstrate the applicability of the PSHAM on the Lü system which is known to display periodic, chaotic and hyperchaotic profiles under carefully selected values of its governing parameters.

The organization of this paper is as follows. In Section 2, the basic idea of the spectral homotopy analysis method (SHAM) is presented. In Section 3, piece-wise spectral homotopy analysis method is presented. Existence and uniqueness of solution of SHAM that give a guarantee of convergence of SHAM is presented in Section 4. Section 5 presents the results and discussion. In Section 6, some concluding remarks are given.

2. Basic idea behind the spectral homotopy analysis method

In this section, we give a brief description of the basic idea behind the standard spectral homotopy analysis method that was initially proposed in Motsa et. al. [21,22] for solving nonlinear boundary value problems. At first, we take into account the following properties of shifted Legendre polynomials.

2.1. Properties of shifted Legendre polynomials

The well-known Legendre polynomials are defined on the interval $(-1,1)$ and can be determined with the aid of the following recurrence formula:

$$\begin{gathered} L_0(x)=1, \quad L_1(x)=x \\ L_{j+1}(x)=\frac{2j+1}{j+1}xL_j(x)-\frac{j}{j+1}L_{j-1}(x), \quad j\geqslant 1. \end{gathered} \tag{2.1}$$

In order to use these polynomials on the interval $x\in(0,T)$ we defined the so-called shifted Legendre polynomials by introducing the change of variable $x=2t/T-1$. Let the shifted Legendre polynomials $L_j(2t/T-1)$ be denoted by $L_{T,j}(t)$. Then $L_{T,j}(t)$ can be generated by using the following recurrence relation:

$$L_{T,j+1}(t)=\frac{2j+1}{j+1}\left(\frac{2t}{T}-1\right)L_{T,j}(t)-\frac{j}{j+1}L_{T,j-1}(t), \quad j=1,2,\dots \tag{2.2}$$

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where $L_{T,0}(t) = 1$ and $L_{T,1}(t) = 2t/T - 1$. The orthogonality condition is

$$\int_0^T L_{T,i}(t)L_{T,k}(t)\,\mathrm{d}t = \frac{T}{2i+1}\delta_{i,k} \tag{2.3}$$

where $\delta_{i,k}$ is the Kronecker function. Any function $u(t)$, square integrable in $(0,T)$, may be expressed in terms of shifted Legendre polynomials as

$$u(t) = \sum_{j=0}^{\infty} a_j L_{T,j}(t) \tag{2.4}$$

where the coefficients a_j are given by

$$a_j = \frac{2l+1}{T}\int_0^T y(t)L_{T,j}(t)\,\mathrm{d}t, \quad j = 0,1,2,\ldots. \tag{2.5}$$

In practice, only the first $(N+1)$-terms shifted Legendre polynomials are considered. Hence we can write

$$u(t) = \sum_{j=0}^{N} a_j L_{T,j}(t). \tag{2.6}$$

Now, we turn to Legendre–Gauss interpolation. We denote by t_j^N, $0 \leqslant j \leqslant N$, the nodes of the standard Legendre–Gauss interpolation on the interval $(-1,1)$. The corresponding Christoffel numbers are ω_j^N, $0 \leqslant j \leqslant N$. The nodes of the shifted Legendre–Gauss interpolation on the interval $(0,T)$ are the zeros of $L_{T,N+1}(t)$, which are denoted by $t_{T,j}^N$, $0 \leqslant j \leqslant N$. Clearly $t_{T,j}^N = T(t_j^N+1)/2$. The corresponding Christoffel numbers are $\omega_{T,j}^N = T\omega_j^N/2$. Let $\mathscr{P}_N(0,T)$ be the set of all polynomials of degree at most N. Due to the property of the standard Legendre–Gauss quadrature, it follows that for any $\Phi \in \mathscr{P}_{2N+1}(0,T)$:

$$\begin{aligned}\int_0^T \Phi(t)\,\mathrm{d}t &= \frac{T}{2}\int_{-1}^{1}\Phi\left(\frac{T}{2}(t+1)\right)\mathrm{d}t \\ &= \frac{T}{2}\sum_{j=0}^{N}\omega_j^N\Phi\left(\frac{T}{2}(t_j^N+1)\right) = \sum_{j=0}^{N}\omega_{T,j}^N\Phi(t_{T,j}^N).\end{aligned} \tag{2.7}$$

Definition 2.1. Let $(u,v)_T$ and $\|v\|_T$ be the inner product and the norm of space $L^2(0,T)$, respectively. We introduce the following discrete inner product and norm,

$$(u,v)_{T,N} = \sum_{j=0}^{N} u(t_{T,j}^N)v(t_{T,j}^N)\omega_{T,j}^N, \qquad \|v\|_{T,N} = (v,v)_{T,N}^{1/2}. \tag{2.8}$$

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From (2.7), for any $\Phi\psi \in \mathscr{P}_{2N+\lambda}(0,T)$,

$$(\Phi,\psi)_T = (\Phi,\psi)_{T,N} \tag{2.9}$$

where $\lambda = -1,0,1$ for the Legendre Gauss interpolation, the Legendre Gauss–Radau interpolation and the Legendre Gauss–Lobatto integration respectively.

Moreover, for the Legendre Gauss integration and the Legendre Gauss–Radau integration,

$$\|\varphi\|_T = \|\varphi\|_{T,N}, \quad \varphi \in \mathscr{P}_N(0,T). \tag{2.10}$$

For the Legendre Gauss–Lobatto integration, $\|\varphi\|_T \neq \|\varphi\|_{T,N}$ usually. But for mostly used orthogonal systems in $[0,T]$, they are equivalent, namely, for certain positive constants c_l and c_2,

$$c_1\|\varphi\|_T \leqslant \|\varphi\|_{T,N} \leqslant c_2\|\varphi\|_T. \tag{2.11}$$

As a consequence, for Legendre Gauss–Lobatto interpolation and for $\varphi \in \mathscr{P}_N(0,T)$, we have

$$\|\varphi\|_T \leqslant \|\varphi\|_{T,N} \leqslant \sqrt{2+\frac{1}{N}}\|\varphi\|_T. \tag{2.12}$$

2.2. Spectral homotopy analysis method

For convenience of the interested reader, we will first present a brief description of the basic idea behind the standard SHAM [21,22]. This will be followed by a description of the piecewise version of the SHAM algorithm which is suitable for solving initial value problems. To this end, we consider the initial value problem (IVP) of dimension n given as

$$\dot{\mathbf{u}}(t) = \mathbf{f}(t,\mathbf{u}(t)), \qquad \mathbf{u}(t_0) = \mathbf{u}^0 \tag{2.13}$$

$$\mathbf{u}:\mathbb{R}\to\mathbb{R}^n, \quad \mathbf{f}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n \tag{2.14}$$

where the dot denotes differentiation with respect to t . We make the usual assumption that $\mathbf{f}$ is sufficiently smooth for linearization techniques to be valid. If $\mathbf{u} = (u_1,u_2,\ldots,u_n)$ we can apply the SHAM by rewriting equation (2.13) as

$$\dot{u}_r + \sum_{k=1}^{n}\alpha_{r,k}u_k + g_r(u_1,u_2,\ldots,u_n) = 0 \tag{2.15}$$

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subject to the initial conditions

$$u_r(0) = u_r^0 \tag{2.16}$$

where u_r^0 are the given initial conditions, $\alpha_{r,k}$ are known constant parameters and g_r is the nonlinear component of the rth equation.

The SHAM approach imports the conventional ideas of the standard homotopy analysis method (HAM) by defining the following zeroth-order deformation equations

$$(1-q)\mathscr{L}_r[U_r(t;q) - u_{r,0}(t)] = q\hbar_r \mathscr{N}_r[U_r(t;q)] \tag{2.17}$$

where $q \in [0,1]$ is an embedding parameter, $U_r(t;q)$ are unknown functions, $\hbar_r$ is a convergence controlling parameter. The operators $\mathscr{L}_r$ and $\mathscr{N}_r$ are defined as

$$\mathscr{L}_r[U_r(t;q)] = \frac{\partial U_r}{\partial t} + \sum_{k=1}^{n} \alpha_{r,k} U_k \tag{2.18}$$

$$\mathscr{N}_r[U_r(t;q)] = \mathscr{L}_r[U_r(t;q)] + g_r[U_1(t;q), U_2(t;q), \ldots, U_n(t;q)]. \tag{2.19}$$

Using the ideas of the standard HAM approach [17], we differentiate the zeroth-order equations (2.17) m times with respect to q and then set $q = 0$ and finally divide the resulting equations by $m!$ to obtain the following equations, which are referred to as the mth order (or higher order) deformation equations,

$$\mathscr{L}_r[u_{r,m}(t) - \chi_m u_{r,m-1}(t)] = \hbar_r R_{r,m-1} \tag{2.20}$$

subject to

$$u_{r,m}(0) = 0 \tag{2.21}$$

where

$$R_{r,m-1} = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathscr{N}_r[U_r(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{2.22}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1 \\ 1, & m > 1. \end{cases} \tag{2.23}$$

After obtaining solutions for equations (2.20), the approximate solution for each $u_r(t)$ is determined as the series solution

$$u_r(t) = u_{r,0}(t) + u_{r,1}(t) + u_{r,2}(t) + \ldots \tag{2.24}$$

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A SHAM solution is said to be of order M if the above series is truncated at $m = M$, that is, if

$$u_r(t) = \sum_{m=0}^{M} u_{r,m}(t). \tag{2.25}$$

The SHAM was introduced as a possible improvement of the HAM which offers flexibility in choosing the linear operator $\mathscr{L}_r$ and removes some of the perceived limitations of the HAM such as the requirement that the solution must conform to the so called rule of solution expression and coefficient ergodicity. The SHAM specifies a clear criteria for choosing the linear operator as just the linear part of the governing equation. The initial approximation is chosen to the solution of the linear part of the governing equations when solved subject to the given initial conditions. The SHAM higher order deformation equations are reduced into a system of linear algebraic equations by transforming the derivatives using the Legendre spectral collocation method.

The initial approximation is obtained as a solution of the following system of equations

$$\dot{u}_r + \sum_{k=1}^{n} \alpha_{r,k} u_k = 0 \tag{2.26}$$

subject to the initial conditions

$$u_r(0) = u_r^0. \tag{2.27}$$

The solution of equation (2.26) can be obtained analytically for most IVPs. If the analytical solution is not available, numerical methods can be used to estimate the solution. The solution $u_{r,0}(t)$ of equation (2.26) is then substituted in the higher order deformation equation (2.20) which is iteratively solved for $u_{r,m}(t)$ (for $m = 1, 2, \ldots, M$).

In this paper, we use a spectral collocation method with Legendre–Gauss–Lobatto (LGL) points [7, 32] to integrate the algorithm (2.20). We remark that before applying the spectral method, we use the transformation $t = t_F(\tau + 1)/2$ to map the region $[0, t_F]$ to the interval $[-1, 1]$ on which the spectral method is defined.

After the transformation, the interval $\tau \in [-1, 1]$ is discretized using the Legendre–Gauss–Lobatto (LGL) nodes. These points, $\tau_j,\ j = 0, 1, \ldots, N$, are unevenly distributed on $[-1, 1]$ and are defined by $\tau_0 = -1$, $\tau_N = 1$ and for $1 \leqslant j \leqslant N - 1$, τ_j are the zeros of $\dot{L}_N$, the derivative of the Legendre polynomial of degree N, L_N.

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The unknown functions $u(t)$ are approximated by the Nth degree polynomials of the form

$$\mathbf{u}(t) = \sum_{k=0}^{N} \mathbf{u}_k \varphi_k(t) \tag{2.28}$$

where, for $k = 0, 1, \ldots, N$ we have

$$\varphi_k(t) = \frac{1}{N(N+1)L_N(t_k)} \frac{(t^2-1)\dot{L}_N(t)}{t-t_k} \tag{2.29}$$

are the Lagrange polynomials of order N which interpolate the functions at the LGL points. The Legendre spectral differentiation matrix D is used to approximate the derivatives of the unknown variables $u_{r,m}(t)$ at the collocation points as the matrix vector product

$$\frac{\mathrm{d}u_{r,m}}{\mathrm{d}t}(t_j) = \sum_{k=0}^{N} \mathbf{D}_{jk} u_{r,m}(\tau_k) = \mathbf{D}\mathbf{U}_{r,m}, \quad j = 0, 1, \ldots, N \tag{2.30}$$

where $\mathbf{D} = 2D/t_F$ and $\mathbf{U}_{r,m} = [u_{r,m}(\tau_0), u_{r,m}(\tau_1), \ldots, u_{r,m}(\tau_N)]^T$ is the vector function at the collocation points τ_j. The matrix D is of size $(N+1) \times (N+1)$ and its entries are defined [7, 32] as

$$D_{jk} = \begin{cases} -\dfrac{N(N+1)}{4}, & j = k = 0 \\ \dfrac{N(N+1)}{4}, & j = k = N \\ \dfrac{L_N(t_j)}{L_N(t_k)} \dfrac{1}{t_j - t_k}, & j \neq k \\ 0, & \text{otherwise.} \end{cases} \tag{2.31}$$

Applying the the Legendre spectral collocation method in equations (2.20)–(2.21) gives

$$\mathbf{A}\left[\mathbf{W}_m - \chi_m \mathbf{W}_{m-1}\right] = \bar{h}_r \mathbf{R}_{m-1}, \quad \mathbf{W}_m(\tau_N) = 0 \tag{2.32}$$

where $\mathbf{R}_{m-1}$ is an $(N+1)n \times 1$ vector corresponding to $R_{r,m-1}$ when evaluated at the collocation points and $\mathbf{W}_m = [\mathbf{U}_{1,m}; \mathbf{U}_{2,m}; \ldots; \mathbf{U}_{n,m}]$.

The matrix $\mathbf{A}$ is an $(N+1)n \times (N+1)n$ matrix that is derived from transforming the linear operator $\mathscr{L}_r$ using the derivative matrix D and is defined as

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}, \qquad \mathbf{A}_{pq} = \begin{cases} \mathbf{D} + \sigma_{pq}\mathbf{I}, & p = q \\ \sigma_{pq}\mathbf{I}, & p \neq q \end{cases} \tag{2.33}$$

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where **I** is an identity matrix of order $N+1$.

Thus, starting from the initial approximation obtained as the solution of equation (2.26), the recurrence formula (2.32) can be used to obtain the solution $u_r(t)$.

3. Piece-wise spectral homotopy analysis method

It is worth noting that the SHAM method described above is ideally suited for boundary value problems whose solutions don't rapidly change in behaviour or oscillate over small regions of the domain of the governing problem. The SHAM solution can thus be considered to be local in nature and may not be suitable for initial value problems at very large values of the independent variable t. A simple way of ensuring the validity of the approximations for large t is to determine the solution in a sequence of equal intervals, which are subject to continuity conditions at the end points of each interval. To extend this solution over the interval $\Lambda = [t^0, t^F]$, we divide the interval Λ into sub-intervals $\Lambda_i = [t^{i-1}, t^i]$, $i = 1, 2, 3, \ldots, F$ where $t^0 \leqslant t^1 \leqslant \cdots \leqslant t^F$. We solve (2.18) in each subinterval Λ_i. Let $u_r^1(t)$ be the solution of (2.15) in the first subinterval $[t^0, t^1]$ and $u_r^i(t)$ be the solutions in the subintervals Λ_i for $2 \leqslant i \leqslant F$. The initial conditions used in obtaining the solutions in the subinterval $\Lambda_i (2 \leqslant i \leqslant F)$ are obtained from the initial conditions of the subinterval Λ_{i-1}. Thus, we solve

$$\mathscr{L}_r[u^i_{r,m}(t) - \chi_m u^i_{r,m-1}(t)] = \hbar_r R^i_{r,m-1}, \quad t \in [t^{i-1}, t^i] \tag{3.1}$$

subject to

$$u^i_{r,m}(t^{i-1}) = 0. \tag{3.2}$$

The initial approximations for solving equation (3.1) are obtained as solutions of the following equations

$$\dot{u}^i_{r,0} + \sum_{k=1}^{n} \alpha_{r,k} u^i_{k,0} = 0, \quad t \in [t^{i-1}, t^i] \tag{3.3}$$

subject to the initial conditions

$$u^i_{r,0}(t^{i-1}) = u^{i-1}_r(t^{i-1}). \tag{3.4}$$

After transforming the interval $[t^{i-1}, t^i]$ into $[-1, 1]$, the Legendre spectral collocation method is then applied to solve equations (3.1)–(3.2) on each

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interval $[t^{i-1}, t^i]$. This results in the following recursive formula for $m \geqslant 1$:

$$\mathbf{W}_m^i = \chi_m \mathbf{W}_{m-1}^i + h_r \mathbf{A}^{-1} \mathbf{R}_{m-1}^i \tag{3.5}$$

for $t \in [t^{i-1}, t^i]$. The initial approximation for the iterative formula (3.5) is is obtained as a solution of (3.3 - 3.4). The solution approximating $u_r(t)$ in the entire interval $[t^0, t^F]$ is given by

$$u_r(t) = \begin{cases} u_r^1(t), & t \in [t^0, t^1] \\ u_r^2(t), & t \in [t^1, t^2] \\ \vdots \\ u_r^F(t), & t \in [t^{F-1}, t^F]. \end{cases} \tag{3.6}$$

It should be noted that when $F = 1$, the proposed piecewise spectral homotopy analysis method (PSHAM) becomes equivalent to the original SHAM algorithm.

4. Existence and uniqueness of solution of SHAM

We consider the initial value problem (IVP) of dimension n (2.13) that is rewritten as

$$\mathscr{L}[u(t)] + \mathscr{N}[u(t)] = \varphi(t) \tag{4.1}$$

where $\mathscr{L}$ is a linear operator which is derived from the entire the linear part of (2.13) and $\mathscr{N}$ is the remaining nonlinear component.

Let us define the nonlinear operator $\mathscr{N}$ and the sequence $\{U_m\}_{m=0}^{\infty}$ as,

$$\mathscr{N}[u(t)] = \sum_{k=0}^{\infty} N_k(u_0, u_1, \ldots, u_k) \tag{4.2}$$

$$\begin{cases} U_0 = u_0 \\ U_1 = u_0 + u_1 \\ \vdots \\ U_m = u_0 + u_1 + u_2 + \cdots + u_m. \end{cases} \tag{4.3}$$

The SHAM gives the following equation, which is referred to as the mth order (or higher order) deformation equation,

$$\mathscr{L}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t) R_m[u_{m-1}(t)] \tag{4.4}$$

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subject to the initial condition

$$u_m(0) = 0 \tag{4.5}$$

where

$$R_m(\vec{u}_{m-1}) = \mathscr{L}[u_{m-1}] + \mathscr{N}_{m-1}[u_0, u_1, \ldots, u_{m-1}] - (1-\chi_m)\varphi(t). \tag{4.6}$$

Therefore,

$$\begin{aligned}
&\mathscr{L}[u_1(t)] = \hbar H(t)\{\mathscr{L}[u_0] + \mathscr{N}_0 - \varphi(t)\}\\
&\mathscr{L}[u_2(t) - u_1(t)] = \hbar H(t)\{\mathscr{L}[u_1] + \mathscr{N}_1\}\\
&\mathscr{L}[u_3(t) - u_2(t)] = \hbar H(t)\{\mathscr{L}[u_2] + \mathscr{N}_2\}\\
&\vdots\\
&\mathscr{L}[u_m(t) - u_{m-1}(t)] = \hbar H(t)\{\mathscr{L}[u_{m-1}] + \mathscr{N}_{m-1}\}
\end{aligned}$$

after summing this equations, we have

$$\mathscr{L}[u_m(t)] = \hbar H(t)\left\{\sum_{k=0}^{m-1}\mathscr{L}[u_k] + \sum_{k=0}^{m-1}\mathscr{N}_k - \varphi(t)\right\} \tag{4.7}$$

from (4.3) we have

$$\mathscr{L}[U_m(t) - U_{m-1}(t)] = \hbar H(t)\{\mathscr{L}[U_{m-1}] + \mathscr{N}[U_{m-1}] - \varphi(t)\} \tag{4.8}$$

subject to the initial condition

$$U_m(0) = 0. \tag{4.9}$$

Consequently, the collocation method is based on a solution $U^N(t) \in \mathscr{P}_{N+1}(0,T)$, for (4.8) such that

$$\begin{aligned}
\mathscr{L}[U_m^N(t_{T,k}^N) - U_{m-1}^N(t_{T,k}^N)] \;=\; &\hbar H^N(t_{T,k}^N)\{\mathscr{L}[U_{m-1}^N(t_{T,k}^N)] + \mathscr{N}[U_{m-1}^N(t_{T,k}^N)]\\
&- \varphi^N(t_{T,k}^N)\}
\end{aligned} \tag{4.10}$$

subject to the initial condition

$$U_m^N(0) = 0. \tag{4.11}$$

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Definition 4.1. A mapping f of space $L^2(0,T)$, into itself is said to satisfy a Lipschitz condition with Lipschitz constant γ if for any z and z^*,

$$|f(z,t) - f(z^*,t)| \leqslant \gamma|z - z^*|. \tag{4.12}$$

If this conditions is satisfied with a Lipschitz constant γ such that $0 \leqslant \gamma < 1$ then f is called a contraction mapping.

Theorem 4.1 (Banach's fixed point theorem [6]). *Assume that K is a non-empty closed set in a Banach space V, and further, that $T : K \to K$ is a contractive mapping with contractivity constant γ, $0 \leqslant \gamma < 1$. Then there exists a unique $U \in K$ such that $U = T(U)$.*

Theorem 4.2 (existence and uniqueness of the solution). *Assume that $\mathbf{f}(t,\mathbf{u}(t))$ in the initial value problem (IVP)* (2.13) *satisfies condition of* (4.12), *then* (4.10) *has a unique solution.*

For the proof of the theorems we should consider the following. From (4.10) we have

$$\begin{aligned}\mathscr{L}[U_m^N(t_{T,k}^N)] = (1+\hbar H^N(t_{T,k}^N))\mathscr{L}[U_{m-1}^N(t_{T,k}^N)] + \hbar H^N(t_{T,k}^N)&\\ \times\{\mathscr{N}[U_{m-1}^N(t_{T,k}^N)] - \varphi^N(t_{T,k}^N)\}, \quad 0 \leqslant k \leqslant N, \ \ m \geqslant 1&\\ U_m^N(0) = 0, \quad m \geqslant 0.&\end{aligned} \tag{4.13}$$

Since $f(t,z)$ satisfies the Lipschitz-continuous condition, then there exists a constant $\gamma \geqslant 0$ such that

$$|f(t,z) - f(t,z^*)| \leqslant \gamma|z - z^*| \tag{4.14}$$

for all $t \in [0,T]$, and all z and z^*.

Now, for the problem (4.1), we choose $L[U(t)] = \mathrm{d}U/\mathrm{d}t + \alpha(t)U$, $N[U(t)] = -\alpha(t)U - f(t,U)$ and $\varphi(t) \equiv 0$, where $\alpha(t)$ is an arbitrary analytic function.

Let $\tilde{U}_m^N(t) = U_m^N(t) - U_{m-1}^N(t)$, then we have from (4.13) that

$$\begin{aligned}\mathscr{L}[\tilde{U}_m^N(t_{T,k}^N)] \ = \ & (1+\hbar H(t_{T,k}^N))\mathscr{L}[U_{m-1}^N(t_{T,k}^N) - U_{m-2}^N(t_{T,k}^N)]\\ & + \hbar H(t_{T,k}^N)\{\mathscr{N}[U_{m-1}^N(t_{T,k}^N)] - \mathscr{N}[U_{m-2}^N(t_{T,k}^N)]\}\\ & 0 \leqslant k \leqslant N, \quad m \geqslant 1\end{aligned} \tag{4.15}$$

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or according to the definitions of $L[U(t)]$ and $N[U(t)]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\tilde{U}_m^N(t_{T,k}^N)] + \alpha(t_{T,k}^N)\tilde{U}_m^N &= (1+\hbar H(t_{T,k}^N))\\ &\quad\times\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\tilde{U}_{m-1}^N(t_{T,k}^N)] + \alpha(t_{T,k}^N)\tilde{U}_{m-1}^N\right\} - \hbar H(t_{T,k}^N)\\ &\quad\times\{f(t_{T,k}^N, U_{m-1}^N(t_{T,k}^N)) + \alpha(t_{T,k}^N)U_{m-1}^N - f(t_{T,k}^N, U_{m-2}^N(t_{T,k}^N))\\ &\quad-\alpha(t_{T,k}^N)U_{m-2}^N\}, \quad 0 \leqslant k \leqslant N, \quad m \geqslant 1 \end{aligned} \tag{4.16}$$

from where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\tilde{U}_m^N(t_{T,k}^N)] =\ & (1+\hbar H(t_{T,k}^N))\frac{\mathrm{d}}{\mathrm{d}t}[\tilde{U}_{m-1}^N(t_{T,k}^N)] + \alpha(t_{T,k}^N)\tilde{U}_{m-1}^N - \alpha(t_{T,k}^N)\tilde{U}_m^N\\ &-\hbar H(t_{T,k}^N)\{f(t_{T,k}^N, U_{m-1}^N(t_{T,k}^N)) - f(t_{T,k}^N, U_{m-2}^N(t_{T,k}^N))\}\\ &\qquad 0 \leqslant k \leqslant N, \quad m \geqslant 1.\end{aligned} \tag{4.17}$$

It is obvious that, $\tilde{U}_m^N(0) = 0$. Clearly,

$$\begin{aligned}\left(\tilde{U}_m^N(T)\right)^2 &= 2\left(\tilde{U}_m^N, \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{U}_m^N)\right)_T\\ &\leqslant 2\|\tilde{U}_m^N\|_T \|\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{U}_m^N)\|_T.\end{aligned} \tag{4.18}$$

Furthermore, for any $t \in [0, T]$,

$$\begin{aligned}\left(\tilde{U}_m^N(t)\right)^2 &= \left(\tilde{U}_m^N(T)\right)^2 - \int_t^T \frac{\mathrm{d}}{\mathrm{d}x}\left(\tilde{U}_m^N(x)\right)^2 \mathrm{d}x\\ &\leqslant \left(\tilde{U}_m^N(T)\right)^2 + 2\|\tilde{U}_m^N\|_T \left\|\frac{\mathrm{d}}{\mathrm{d}x}\tilde{U}_m^N\right\|_T.\end{aligned}$$

Integrating the above with respect to t, yields that

$$\|\tilde{U}_m^N\|_T^2 \leqslant T\left(\tilde{U}_m^N(T)\right)^2 + 2T\|\tilde{U}_m^N\|_T \left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_m^N\right\|_T \tag{4.19}$$

from where

$$\left(\tilde{U}_m^N(T)\right)^2 \geqslant \frac{1}{T}\|\tilde{U}_m^N\|_T^2 - 2\|\tilde{U}_m^N\|_T \left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_m^N\right\|_T. \tag{4.20}$$

Using (4.20) and (4.18) we have

$$\left\|\tilde{U}_m^N\right\|_T \leqslant 4T\left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_m^N\right\|_T, \quad m \geqslant 0. \tag{4.21}$$

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Let $\alpha = \max_{t\in[0,T]} |\alpha(t)|$ and $H = \max_{t\in[0,T]} |H(t)|$. Therefore, a combination (2.12), (4.12), and (4.18) leads to that

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_m^N\right\|_T \leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_m^N\right\|_{T,N} &\leqslant (1+|\hbar|H)\left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_{m-1}^N\right\|_{T,N} \\ &\quad + c\gamma|\hbar|H\left\|\tilde{U}_{m-1}^N\right\|_{T,N} + \alpha\|\tilde{U}_m^N\|_{T,N} + \alpha\|\tilde{U}_{m-1}^N\|_{T,N} \\ &\leqslant \sqrt{2+\frac{1}{N}}(1+|\hbar|H)\left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_{m-1}^N\right\|_T + \sqrt{2+\frac{1}{N}}C\gamma|\hbar|H\|\tilde{U}_{m-1}^N\|_T \\ &\quad + \sqrt{2+\frac{1}{N}}\alpha\|\tilde{U}_m^N\|_T + \sqrt{2+\frac{1}{N}}\alpha\|\tilde{U}_{m-1}^N\|_T. \end{aligned} \tag{4.22}$$

Using (4.21) and (4.22) results in

$$\begin{aligned}\left\|\tilde{U}_m^N\right\|_T \leqslant\ & \frac{4\sqrt{2+\frac{1}{N}}T\,(1+|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_{m-1}^N\right\|_T \\ & + \frac{4\sqrt{2+\frac{1}{N}}T\,(\alpha+c\gamma|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T}\left\|\tilde{U}_{m-1}^N\right\|_T \end{aligned} \tag{4.23}$$

where c is a positive constant. Then, by (4.21) with $m-1$, instead of m and multiplying the resulting inequality by

$$\frac{\sqrt{2+\frac{1}{N}}(1+|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T} \geqslant 0$$

we have

$$\frac{\sqrt{2+\frac{1}{N}}(1+|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T}\left\|\tilde{U}_{m-1}^N\right\|_T \leqslant \frac{4\sqrt{2+\frac{1}{N}}T\,(1+|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}_{m-1}^N\right\|_T. \tag{4.24}$$

Subtracting (4.24) from (4.23), after simplifying, we obtain

$$\left\|\tilde{U}_m^N\right\|_T \leqslant \frac{\sqrt{2+\frac{1}{N}}\,(1+|\hbar|H) + 4\sqrt{2+\frac{1}{N}}T\,(\alpha+c\gamma|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T}\left\|\tilde{U}_{m-1}^N\right\|_T. \tag{4.25}$$

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Therefore, if

$$\frac{\sqrt{2+\frac{1}{N}}\,(1+|\hbar|H)+4\sqrt{2+\frac{1}{N}}T\,(\alpha+c\,\gamma|\hbar|H)}{1-4\sqrt{2+\frac{1}{N}}\alpha T}\leqslant\beta<1$$

then $\|\tilde{U}_m^N\|_T \to 0$ as $m \to \infty$. According to Theorem 4.2, it implies the existence and uniqueness of solution of (4.10).

5. Results and discussion

To demonstrate the applicability of the proposed piecewise spectral homotopy analysis method (PSHAM) algorithm as an appropriate tool for solving nonlinear IVPs, we apply the proposed algorithm to the Hyperchaotic Lü system, which was presented by Lü et al. [19] as

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t}&=a(u_2-u_1)+u_4\\ \frac{\mathrm{d}u_2}{\mathrm{d}t}&=-u_1u_3+cu_2\\ \frac{\mathrm{d}u_3}{\mathrm{d}t}&=u_1u_2-bu_3\\ \frac{\mathrm{d}u_4}{\mathrm{d}t}&=u_1u_3+du_4\end{aligned}\tag{5.1}$$

where u_i are state variables, $i=1,2,3,4$, while a, b, c, and d are real constants. It was demonstrated in [23] that when $a=36$, $b=3$, $c=20$, $-1.03\leqslant d\leqslant 0.46$, the system (see [19]) has periodic orbit. If $a=36$, $b=3$, $c=20$, $-0.46\leqslant d\leqslant -0.35$, the system has chaotic attractor. If $a=36$, $b=3$, $c=20$, $-0.35\leqslant d\leqslant 1.3$ the system (see [19]) has hyperchaotic attractor.

For the Lu system (5.2), the parameters used in the SHAM and PSHAM algorithms described in the previous section are $\alpha_{11}=a$, $\alpha_{12}=-a$, $\alpha_{14}=-1$, $\alpha_{22}=-c$, $\alpha_{33}=b$, $\alpha_{44}=-d$ with all other $\alpha_{pq}=0$ (for $p,q=1,2,3,4$)

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and

$$R^i_{r,m-1} = \mathscr{L}_r[u^i_{r,m-1}] + Q^i_{r,m-1}, \quad Q^i_{r,m-1} = \begin{bmatrix} 0 \\ \sum_{j=0}^{m-1} u^i_{1,j} u^i_{3,m-1-j} \\ -\sum_{j=0}^{m-1} u^i_{1,j} u^i_{2,m-1-j} \\ -\sum_{j=0}^{m-1} u^i_{1,j} u^i_{3,m-1-j} \end{bmatrix}. \quad (5.2)$$

With these definitions, the PSHAM algorithm gives

$$\mathbf{AW}^i_{r,m} = (\chi_m + \hbar_r)\mathbf{AW}^i_{r,m-1} + \hbar_r \mathbf{Q}^i_{r,m-1}. \quad (5.3)$$

Because the right hand side of equation (5.3) is known, the solution can easily be obtained by using methods for solving linear system of equations.

In the remainder of this section, we present the results of the numerical simulations of system (5.2) which were conducted using the PSHAM algorithm. Unless otherwise specified, all the PSHAM results presented in this section were obtained using $N = 10$ collocation points and ten iterations (that is $M = 10$) in each $[t^{i-1}, t^i]$ interval. The width of each interval $\Delta t = t^i - t^{i-1}$ was taken to be $\Delta t = 0.1$. We remark that like in the homotopy analysis method case [17], the convergence of the PSHAM can be adjusted by altering the value of the convergence controlling parameter $\hbar$. However, for illustration purposes, a fixed value of $\hbar = -1$ was used in this study. We also fix the values of the parameters $a = 36$, $b = 3$, $c = 20$ with $d = -0.91$ for the periodic case, $d = -0.35$ for the chaotic case and $d = 1$ for the hyper-chaotic case. The initial conditions are $u_1(0) = 4, u_2(0) = 8, u_3(0) = -1$ and $u_4(0) = -3$. The accuracy of the proposed method was validated against in-built MATLAB based Runge–Kutta routines. In addition the graphical results obtained from this study were qualitatively compared with previous results from literature where the same problem was solved using other methods of solution.

The results of the PSHAM simulation of the Lu system (5.2) for the periodic, chaotic, and hyper-chaotic case are shown in Figs. 1, 2, and 3, respectively. It can be observed that results are all qualitatively the same as those reported in [19] for all the three cases considered in this study. This validates the applicability of the PSHAM method as a possible tool for solving other complex initial value problems.

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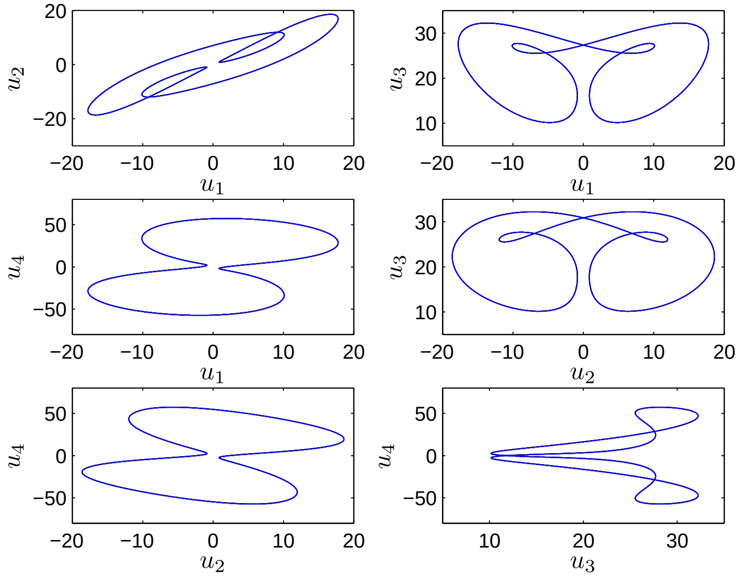

Figure 1. Phase portraits for the periodic case.

6. Conclusion

In this paper we presented a new application of the spectral homotopy analysis method in solving a class of nonlinear differential equations whose solutions show periodic, chaotic and hyperchaotic behaviour. The proposed method, (referred to as the Piecewise Spectral Homotopy Analysis Method, or PSHAM) of solution extends the application of the Spectral Homotopy Analysis Method (SHAM) to complex nonlinear initial value problems. The PSHAM approach was tested on a four dimensional system of nonlinear initial value problem that is well known to display periodic, chaotic and hyper-chaotic behaviour under carefully selected values of its governing parameters. The present numerical results were validated against results from literature and Runge–Kutta based schemes. From this preliminary investigation of the possible application of extended versions of the SHAM we conclude that the PSHAM promises to be a useful tool for solving highly nonlinear initial value problems including those with behaviour that is difficult to resolve mathematically such as the chaotic and hyper-chaotic nature of the system considered in this study.

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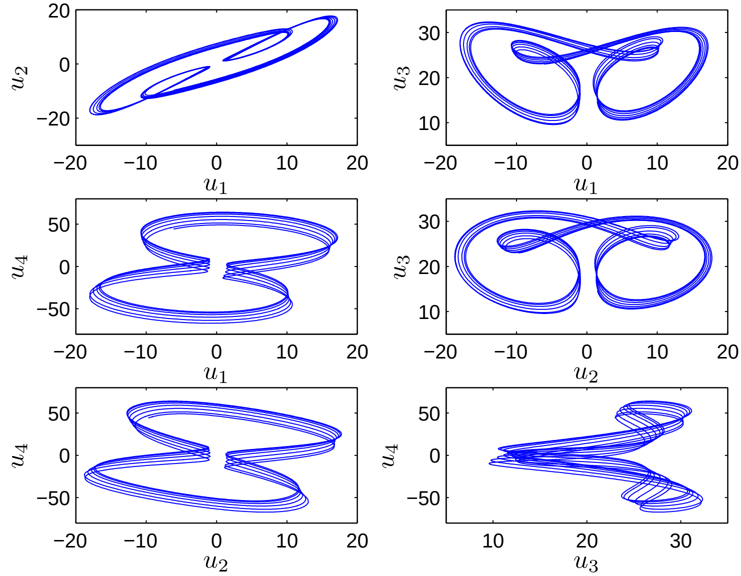

Figure 2. Phase portraits for the chaotic case.

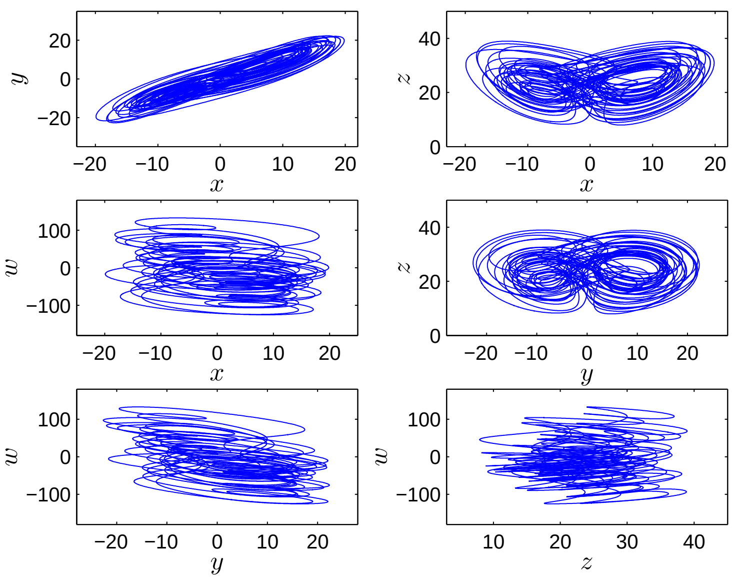

Figure 3. Phase portraits for the hyper chaotic case.

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