This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

Using Mutual Aggregate Uncertainty Measures in a Threat Assessment Problem Constructed by Dempster–Shafer Network

Asghar Shahpari and Seyed A. Seyedin

***Abstract*—Mutual information as a tool for measuring the amount of dependency between two variables is used in many applications in probability theory. In this paper, three mutual measures based on three aggregate uncertainty (AU) measures in Dempster–Shafer theory (DST) are proposed. These uncertainty measures are: 1) AU; 2) ambiguity measure (AM); and 3) modified AM (MAM), which is proposed in this paper. MAM is the modification of AM which resolves the nonsubadditivity problem of AM. A threat assessment problem constructed by a Dempster–Shafer network is used for testing these mutual measures. We use the proposed mutual measures to identify which input variables of the network are more influential on the threat value. Finally, we conclude that mutual uncertainty based on MAM is a justifiable measure to compute the dependency between two variables in applications related to the DST.**

***Index Terms*—Ambiguity, Dempster–Shafer theory (DST), mutual aggregate uncertainty (AU) measure, pignistic probability, subadditivity, threat assessment.**

I. Introduction

Computing the amount of uncertainty or information contained in an event is of crucial importance in many applications in decision-making systems. To calculate the amount of uncertainty, we need to define a measure. Shannon entropy, *H(x)*, is an uncertainty measure in probability theory (PT) proposed by Shannon [1]. The different types of uncertainty proposed in various theories have been classified by Klir and Yuan [2]. Dempster–Shafer theory (DST) is an extension of the PT and the set theory, and as such, it is able to represent two types of uncertainty, i.e., nonspecifcity and discord.

The generalized Hartley (GH) measure, which was proposed by Hartley ($\mathrm{H(A)} = \log_2|\mathrm{A}|$), quantifies the amount of nonspecificity contained in a basic probability assignment (BPA) in DST [3] and [4]. GH satisfies five essential requirements of a nonspecificity measure and is thus an appropriate measure of calculating the amount of set uncertainty in DST.

Similar to the Shannon entropy in PT, a number of measures have been proposed to compute probabilistic information in DST including confusion by Höhle [5] in 1982, dissonance by Yager [6] in 1983, discord by Klir and Ramer [7] in 1990, and strife by Klir and Parviz [8] in 1992. Not all of these probabilistic information measures are justifiable discord measures in evidence theory because they not only have some conceptual flaws, but also are nonsubadditive, and differ from the Shannon entropy equation [9]. Therefore, Klir [9] proposed AU as an aggregated uncertainty measure that computes nonspecificity and discord simultaneously. He posited that any aggregate uncertainty (AU) measure must satisfy five requirements including probability consistency, set consistency, range, subadditivity, and additivity. AU, however, had high computational complexity and was insensitive to changes in evidence; moreover, the two types of uncertainty (nonspecificity and discord) were not separable. Also, Abellan and Masegosa [12] analyzed the requirements, presented in the literature, for total uncertainty measures in DST and their shortcomings.

Jousselme *et al.* [10] proposed another aggregated uncertainty measure called ambiguity measure (AM) based on the classic pignistic probability. They proved that AM satisfies the five requirements of an AU measure, it has low computational complexity and is sensitive to changes in evidence. However, Klir and Lewis [11] demonstrated that Jousselme *et al.*'s proof of AM subadditivity was wrong. They referred to the dependency of the pignistic probability on the marginalization process to support their argument. This means that the marginal probabilities derived from the joint pignistic probability are not the same as the pignistic distributions derived from the marginal belief functions. In this paper, the classic pignistic probability is modified, and based on this modification the modified AM (MAM) is proposed. Also, it is proved that the MAM is subadditive.

Another contribution of this paper is to propose the mutual AU measure in DST. Similar to the mutual information in PT, the proposed measure should be symmetric, always nonnegative and be equal to zero if and only if the two variables are independent [11]. AU and AM are the available AU measures that can be used for introducing the mutual uncertainty measure. However, there exist two problems: 1) AU is not sensitive to changes in evidence, resulting in corresponding mutual measure having this defect and 2) if AM is not subadditive, then the mutual measure based on AM will not necessarily be nonnegative. To solve the second problem, the MAM is proposed that is subadditive and the corresponding mutual measure will be nonnegative. The defects of the proposed measures will be compared in two examples. Moreover, in order

Manuscript received May 27, 2014; accepted October 21, 2014. This paper was recommended by Associate Editor J.-P. Barthès.

The authors are with the Department of Engineering, Ferdowsi University of Mashhad, Mashhad 9177948974, Iran (e-mail: shahpari@stu.um.ac.ir).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSMC.2014.2378213

to understand the effectiveness of the proposed measures, a threat assessment problem constructed by a Dempster–Shafer network will be used.

This paper is organized as follows. Section II presents some necessary theoretical concepts. In addition, the available AU measures in DST are reviewed in Section II and the invariance with respect to marginalization property of the pignistic probability is described. Section III proposes the modified pignistic probability which is invariant under the marginalization process. Then, based on the modified pignistic probability, the MAM is defined and proved to be subadditive. In Section IV, three mutual AU measures are proposed. In Section V, a Dempster–Shafer network is presented to solve a threat assessment problem and the proposed mutual measures are then used to compute the dependency of the network input variables to the threat value. Finally, in Section VI the conclusion is provided.

II. Theoretical Background

A. DST

In the PT, a probability density function (PDF) $p : \Omega_X \rightarrow [0, 1]$ assigns values to $\Omega_X = \{x_1, x_2, \ldots, x_n\}$ (the state space of variable X), where, $p(x_i) \geq 0$, and $\sum_{i=1:n} p(x_i) = 1$. If Ω_X and Ω_Y are the state spaces of variables X and Y, then the joint state space is denoted by $\Omega_{XY} = \Omega_X \times \Omega_Y$ and $p_{XY} : \Omega_{XY} \rightarrow [0, 1]$ is the corresponding joint PDF. Marginalization in the PT involves addition over the state space of the variables being eliminated. Suppose p_{XY} is a joint PDF for Ω_{XY}, then the marginal PDF for Ω_X is $p_X(x) = \sum_Y p_{XY}(x, y)$.

DST is an imprecise PT, in which a BPA assigns values to the subsets of the state space [13] and [14]. The function $m : 2^{\Omega_X} \rightarrow [0, 1]$ is a BPA on the power set of $\Omega_X = \{x_1, x_2, \ldots, x_n\}$, where $m(\varnothing) = 0$, $m(A) \geq 0$, and $\sum_{A \in 2^{\Omega_X}} m(A) = 1$. Any element in 2^{Ω_X} with a nonzero BPA is called a focal element. The other functions defined in DST are the belief function ($\mathrm{Bel}(A) = \sum_{\mathrm{B} \subseteq A} m(B)$) and the plausibility function ($Pl(A) = \sum_{A \cap B \neq \varnothing} m(\mathrm{B})$). Concepts which are significant for this paper such as joint state space, projection, combination, and marginalization are defined as follows.

Definition 1: If 2^{Ω_X} and 2^{Ω_Y} are the state spaces of variables X and Y with cardinalities $2^{|\Omega_X|}$ and $2^{|\Omega_Y|}$, then the joint state space is denoted by $2^{\Omega_{XY}}$ with the cardinality $2^{|\Omega_{XY}|}$, where, $\Omega_{XY} = \Omega_X \times \Omega_Y$.

Example 1: If $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2, y_3\}$ are the state spaces of X and Y, then the joint state space in DST will be $2^{\Omega_{XY}}$ ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$), with $2^{|\Omega_{XY}|} = 2^6 = 64$ members. For simplicity, the following notations are introduced: $\Omega_{XY} = \Omega_X \times \Omega_Y = \{\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{21}, \mathrm{Z}_{22}, \mathrm{Z}_{23}\}$.

If Ω_X and Ω_Y are the state spaces of two variables and $\Omega_{XY} = \Omega_X \times \Omega_Y$ is the corresponding joint state space, then the projection of any subset $A \subseteq \Omega_{XY}$ on Ω_X is denoted by $A^{\downarrow \Omega_X}$. This projection is shown in Fig. 1.

Definition 2: Let $m_X : 2^{\Omega_X} \rightarrow [0, 1]$ and $m_Y : 2^{\Omega_Y} \rightarrow [0, 1]$ be two equally reliable and independent BPAs, the combination is calculated by Dempster's rule of combination in the following manner:

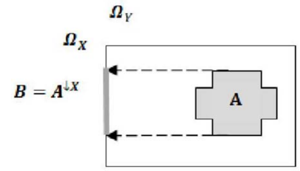

Fig. 1. Projection of subset $\mathrm{A} \subseteq \Omega_{XY}$ on Ω_X.

$$m_{XY}(Z) = (m_X \oplus m_Y)(Z) = \frac{\sum_{X \cap Y = Z} m_X(X) \cdot m_Y(Y)}{1 - K} \quad \forall X \subseteq \Omega_X \,\&\, Y \subseteq \Omega_Y \quad (1)$$

where, $K = \sum_{X \cap Y = \varnothing} m_X(X).m_Y(Y)$ represents the conflict.

Definition 3: If $m_{XY} : 2^{\Omega_{XY}} \rightarrow [0, 1]$ is a joint BPA on Ω_{XY}, then the marginal of m_{XY} on Ω_X is denoted by $m_{XY}^{\downarrow \Omega_X}$, and given by

$$m_X(B) = m_{XY}^{\downarrow \Omega_X}(B) = \sum_{A \subseteq \Omega_{XY}, A \downarrow \Omega_X = B} m_{XY}(A) \quad \forall B \subseteq \Omega_X. \quad (2)$$

It will show that the number of marginal singletons in the joint state space is a major parameter in this paper, as emphasized by the following definition.

Definition 4: If $m_{XY} : 2^{\Omega_{XY}} \rightarrow [0, 1]$ is a joint BPA on Ω_{XY}, then $x_i \in \Omega_X$ and $y_i \in \Omega_Y$ existing in A are the marginal singletons of subset $A \in 2^{\Omega_{XY}}$. The number of marginal singletons x_i is denoted by $\#(x_i \in A)$ and the number of marginal singletons y_i is denoted by $\#(y_i \in A)$.

To illustrate this point, the marginal singletons number has been calculated for the subset $A = \{\mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{23}\}$, which is a member of the joint state space in Example 1. The marginal singletons are x_1, x_2, y_2, and y_3. There are two x_1, one x_2, one y_2, and two y_3 in the subset A, which can be written as $\#(x_1 \in A) = 2$, $\#(x_2 \in A) = 1$, $\#(y_2 \in A) = 1$, and $\#(y_3 \in A) = 2$. Note that the number of marginal singletons of any member of a joint state space in PT is 1 (for instance, $(x_1, y_1) \in \Omega_{XY}$ has one x_1 and one y_1). However, the marginal singletons number of any subset of the joint state space in DST is not necessarily one.

B. Available AU Measures in DST

AU and AM are two available AU measures in DST and are defined as follows.

Definition 5 [15]–[17]: If $m : 2^{\Omega_X} \rightarrow [0, 1]$ is a BPA on Ω_X and Bel is a belief measure on Ω_X, then AU associated with Bel is computed by

$$\mathrm{AU}(\mathrm{Bel}) = \max_{\mathcal{P}_{\mathrm{Bel}}} \left\{ -\sum_{x \in \Omega_X} p_x \log_2 p_x \right\} \quad (3)$$

where $\mathcal{P}_{\mathrm{Bel}}$ consists of all probability distributions $< p_x | x \in \Omega_X >$ that satisfy the constraints

$$0 \leq p_x \leq 1 \; \forall x \in \Omega_X$$

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

and

$$\sum_{x \in \Omega_X} p_x = 1$$

and

$$\mathrm{Bel}\,(A) \leq \sum_{x \in A} p_x \ \forall A \subseteq \Omega_X.$$

Definition 6 [10]: If $m : 2^{\Omega_X} \to [0, 1]$ is a BPA on Ω_X and $BetP_m$ is the corresponding pignistic probability, then AM is defined as follows:

$$\mathrm{AM}\,(m) = -\sum_{x \in \Omega_X} BetP_m\,(x) \cdot \log_2\ (BetP_m\,(x)). \tag{4}$$

$BetP_m$, called the pignistic probability, is a DST to PT transformation that transforms a BPA in DST into the PDF in PT, as defined in the following.

Definition 7 [18]: If $m : 2^{\Omega_X} \to [0, 1]$ is a BPA on Ω_X, then $BetP_m$ denotes the pignistic probability, which is defined for each singleton $x \in \Omega_X$ as follows:

$$BetP_m\,(\{x\}) = \sum_{\substack{A \subseteq \Omega_X \\ x \in \mathrm{A}}} m\,(A)/|A|. \tag{5}$$

C. Five Requirements For AU

Klir and Wierman [19] defined the following five requirements which an AU measure must satisfy.

1) Probability Consistency: If all focal elements are singletons ($P\,(x_i) = m\,(x_i)\ \ \forall x_i \in \Omega_X$), then the uncertainty measure equals the Shannon entropy

$$\mathrm{AU}\,(m) = -\sum_{x \in \Omega_X} m\,(x) \cdot \log_2\ (m\,(x)). \tag{6}$$

2) Set Consistency: If there is one focal element, such as $m\,(A) = 1$ and $m\,(B) = 0\ \forall B \in 2^{\Omega_X}$ and $\mathrm{B} \neq \mathrm{A}$, then the uncertainty measure equals the Hartley measure

$$\mathrm{AU}\,(m) = \log_2 |A|. \tag{7}$$

3) Range: The range of AM is $[0,\ \log_2 |\Omega_X|]$, meaning that

$$0\ \leq \mathrm{AU} \leq log_2 |\Omega_X|. \tag{8}$$

4) Additivity: If $m_{XY} : 2^{\Omega_{XY}} \to [0, 1]$ is a noninteractive joint BPA on Ω_{XY} ($m_{XY} = m_X \times m_Y$) and the associated marginal BPAs are m_X and m_Y, then

$$\mathrm{AU}\,(m_{XY}) = \mathrm{AU}\,(m_X) + \mathrm{AU}\,(m_Y). \tag{9}$$

5) Subadditivity: If $m_{XY} : 2^{\Omega_{XY}} \to [0, 1]$ is an arbitrary joint BPA on Ω_{XY} and the associated marginal BPAs are m_X and m_Y, then

$$\mathrm{AU}\,(m_{XY}) \leq \mathrm{AU}\,(m_X) + \mathrm{AU}\,(m_Y). \tag{10}$$

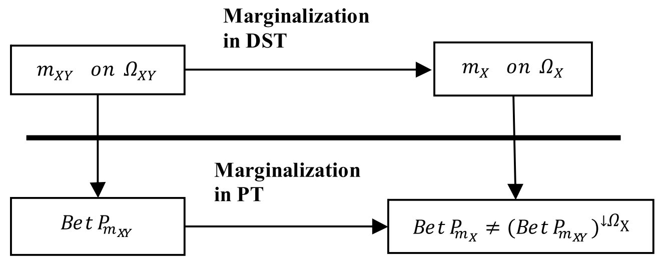

Fig. 2. Pignistic probability is dependent on the marginalization process.

Although AM satisfies the requirements $R1$–$R4$, it is not subadditive. The dependency of the pignistic probability on the marginalization process is the cause of this defect [11]. The following section describes this concept.

D. Invariance With Respect to the Marginalization Process

The invariance with respect to the marginalization property of DST to PT transformations was illustrated in [20]–[22]. This property is represented by the following definition.

Definition 8: Let $m_{XY} : 2^{\Omega_{XY}} \to [0, 1]$ be an arbitrary joint BPA on Ω_{XY}, and m_X be the marginal BPA on Ω_X. If $P_{m_{XY}}$ and P_{m_X} are the DST to PT transformations of m_{XY} and m_X, respectively, then P_m is invariant under marginalization if and only if

$$P_{m_X} = \left(P_{m_{XY}}\right)^{\downarrow \Omega_X}. \tag{11}$$

However, using an example, Klir and Lewis [11] showed that the pignistic probability is dependent on the marginalization process. The dependency of the pignistic probability on the marginalization process is depicted in Fig. 2.

III. MAM

In this section, the reasons for the dependency of the pignistic probability on the marginalization process are discussed. To this purpose, the focus should be on the projection method in DST. It can be shown that some probabilistic information is lost in the classic DST projection process. Consider $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2, y_3\}$ as marginal state spaces and $m_{XY} : 2^{\Omega_{XY}} \to [0, 1]$ as the joint BPA with one focal element, $m_{XY}(A = \{\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{23}\}) = 1$. In the classic computation of the pignistic probability, first, the projection of subset A over Ω_X is $B = \{x_1, x_2\} = \{\mathrm{Z}_{11}, \mathrm{Z}_{12}, \mathrm{Z}_{13}, \mathrm{Z}_{23}\}^{\downarrow \Omega_X}$, then $m_X(\mathrm{B}) = 1$, and finally $BetP_{m_X}(x_1) = BetP_{m_X}(x_2) = m_X(\mathrm{B})/|B| = 1/2$. In other words, the mass of the joint subset A is distributed equally among the members of $A^{\downarrow \Omega_X}$ (x_1 and x_2). However, in this case, there are three x_1 and one x_2 on the subset A. It can be said that the number of marginal singletons ($\#(x_1 \in A) = 3$ and $\#(x_2 \in A) = 1$) are probabilistic information that are not taken into account in the marginal probability distribution.

The question is how to retain this crucial information. To this end, it is sufficient to distribute the mass of the joint subset A among the members of $A^{\downarrow \Omega_X}$ in proportion to their number of marginal singletons. Therefore, the pignistic probability is calculated as follows:

$$BetP_{m_X}(x_1) = m_X\left(A^{\downarrow\Omega_X}\right)\cdot\frac{\#(x_1\in A)}{(\#(x_1\in A)+\#(x_2\in A))} = \frac{3}{4}$$

$$BetP_{m_X}(x_2) = m_X\left(A^{\downarrow\Omega_X}\right)\cdot\frac{\#(x_2\in A)}{(\#(x_1\in A)+\#(x_2\in A))} = 1/4.$$

Consider the more complex joint BPA with three focal elements: $m_{XY}(A_1 = \{Z_{12}, Z_{13}, Z_{23}\}) = 1/3$, $m_{XY}(A_2 = \{Z_{21}, Z_{22}, Z_{23}\}) = 1/3$, and $m_{XY}(A_3 = \{Z_{11}, Z_{12}, Z_{13}, Z_{23}\}) = 1/3$. According to the proposed method, we have $m_X(A_1{}^{\downarrow\Omega_X} = \{x_1, x_2\}) = m_{XY}(A_1) = 1/3$, $\#(x_1 \in A_1) = 2$, and $\#(x_2 \in A_1) = 1$. $m_X(A_2{}^{\downarrow\Omega_X} = \{x_2\}) = m_{XY}(A_2) = 1/3$,$\#(x_1 \in A_2) = 0$, and $\#(x_2 \in A_2) = 3$. $m_X(A_3{}^{\downarrow\Omega_X} = \{x_1, x_2\}) = m_{XY}(A_3) = 1/3$, $\#(x_1 \in A_3) = 3$, and $\#(x_2 \in A_3) = 1$.

Then

$$\begin{aligned} Bet\ P_{m_X}(x_1) &= m_X\left(A_1{}^{\downarrow\Omega_X}\right)\cdot\frac{\#(x_1\in A_1)}{(\#(x_1\in A_1)+\#(x_2\in A_1))} \\ &\quad + m_X\left(A_2{}^{\downarrow\Omega_X}\right)\cdot\frac{\#(x_1\in A_2)}{(\#(x_1\in A_2)+\#(x_2\in A_2))} \\ &\quad + m_X\left(A_3{}^{\downarrow\Omega_X}\right)\cdot\frac{\#(x_1\in A_3)}{(\#(x_1\in A_3)+\#(x_2\in A_3))} \\ &= \frac{1}{3}.\frac{2}{3}+\frac{1}{3}\cdot\frac{0}{3}+\frac{1}{3}\cdot\frac{3}{4} = \frac{17}{36} \end{aligned}$$

and

$$BetP_{m_X}(x_2) = \frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{3}{3}+\frac{1}{3}\cdot\frac{1}{4} = \frac{19}{36}.$$

It is observed that, in the marginalization process, the mass of the subset A is transmitted to $A^{\downarrow\Omega_X}$ and $m_X(A^{\downarrow\Omega_X})$ can be replaced by $m_{XY}(A)$. Also, there is $\#(x_1 \in A) + \#(x_2 \in A) = |A|$. This modification leads us to propose the modified $BetP_m$ as follows.

Definition 9: If $m_{XY} : 2^{\Omega_{XY}} \to [0, 1]$ is a joint BPA defined over Ω_{XY}, then the modified pignistic probability is defined for each singleton $Z_i \in \Omega_{XY}$ and $x_i \in \Omega_X$ as follows:

$$MBet_{m_X}(x_i) \sum_{B\subseteq\Omega_X,\, x_i\in B}\ \sum_{\substack{A\subseteq\Omega_{XY}\\ B=A^{\downarrow\Omega_X}}} \frac{m_{XY}(A)\cdot\#(x_i\in A)}{|A|}\ \forall x_i\in\Omega_X \tag{12}$$

$$MBet_{m_{XY}}\left((x_i, y_j)\right) = \sum_{\substack{A\subseteq\Omega_{XY}\\ (x_i,y_j)\in A}} \frac{m_{XY}(A)}{|A|}\ \forall\ (x_i, y_j)\in\Omega_{XY} \tag{13}$$

where $\#(x_i \in A)$ is the number of x_i in the subset A, and $|A|$ denotes the cardinality of A.

Corollary 1: In cases where the projection process is not used, the modified pignistic probability is computed employing 13 and is reduced to the pignistic probability, i.e., $MBet_{m_X} = BetP_{m_X}$.

Here, the invariance with respect to the marginalization of the modified pignistic probability is expressed by the following proposition.

Proposition 1: Suppose m_{XY}: $2^{\Omega_{XY}} \to [0, 1]$ is a joint BPA over Ω_{XY}, $MBet_{m_{XY}}$ is the joint modified pignistic probability and $MBet_{m_X}$ is the modified pignistic probability of marginal m_X, then

$$MBet_{m_X} = (MBet_{m_{XY}})^{\downarrow\Omega_X}. \tag{14}$$

Proof: It will be proved that: $\sum_{j=1}^{|\Omega_Y|} MBet_{m_{XY}}((x_1, y_j)) = MBet_{m_X}(x_1)$. Suppose that $\Omega_X = \{x_1, x_2, \ldots, x_N\}$ and $\Omega_Y = \{y_1, y_2, \ldots, y_M\}$ are the state spaces of X and Y with the cardinalities N and M, and the joint state space is $\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1, y_1), \ldots, (x_1, y_M), (x_2, y_1), \ldots, (x_2, y_M), \ldots, (x_N, y_M)\} = \{Z_{11}, \ldots, Z_{1M}, Z_{21}, \ldots, Z_{2M}, \ldots, Z_{NM}\}$. Now, we have

$$\begin{aligned} &\sum_{j=1}^{|\Omega_Y|} MBet_{m_{XY}}\left((x_1, y_j)\right) \\ &= \sum_{j=1}^{M} MBet_{m_{XY}}\left(Z_{1j}\right) \\ &= MBet_{m_{XY}}(Z_{11}) + \cdots + MBet_{m_{XY}}(Z_{1M}) \\ &= \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A}} \frac{m_{XY}(A)}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A}} \frac{m_{XY}(A)}{|A|}. \end{aligned}$$

We can separate the subset $\{A|A \subseteq \Omega_{XY}, Z_{11} \in A\}$ into the M new subsets: $\{A|A \subseteq \Omega_{XY}, Z_{11} \in A,\ \#(x_1 \in A) = 1\}$, $\{A|A \subseteq \Omega_{XY}, Z_{11} \in A,\ \#(x_1 \in A) = 2\}$,..., and $\{A|A \subseteq \Omega_{XY}, Z_{11} \in A,\ \#(x_1 \in A) = M\}$. For example, if $N = 2$ and $M = 2$, we have

$$\begin{aligned} &\{A\,|A\subseteq\Omega_{XY},\ Z_{11}\in A\} \\ &= \left\{\begin{array}{l}\{Z_{11}\}, \{Z_{11}, Z_{12}\}, \{Z_{11}, Z_{21}\}, \{Z_{11}, Z_{22}\}, \{Z_{11}, Z_{12}, Z_{21}\}, \\ \{Z_{11}, Z_{12}, Z_{22}\}, \{Z_{11}, Z_{21}, Z_{22}\}, \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}\end{array}\right\} \\ &= \{\{Z_{11}\}, \{Z_{11}, Z_{21}\}, \{Z_{11}, Z_{22}\}, \{Z_{11}, Z_{21}, Z_{22}\}\} \\ &\quad \cup \{\{Z_{11}, Z_{12}\}, \{Z_{11}, Z_{12}, Z_{21}\}, \{Z_{11}, Z_{12}, Z_{22}\}, \\ &\qquad \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}\} \\ &= \{A|A\subseteq\Omega_{XY},\ Z_{11}\in A,\ \#(x_1\in A) = 1\} \\ &\quad \cup \{A|A\subseteq\Omega_{XY}, Z_{11}\in A,\ \#(x_1\in A) = 2\}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A}} \frac{m_{XY}(A)}{|A|} &= \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A)}{|A|} \\ &+ \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A)}{|A|} + \cdots \\ &+ \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A)}{|A|} \end{aligned}$$

and

$$\sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A}} \frac{m_{XY}(A\ )}{|A|} = \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|} + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|}.$$

Now

$$\begin{aligned}\sum_{j=1}^{M} MBet_{m_{XY}}\left(Z_{1j}\right) &= \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|} + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|} + \cdots\\
&\quad + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|} + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|}\\
&= \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|}\\
&\quad \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} + \cdots\\
&\quad + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|}.\end{aligned}$$

We have

$$\sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|} = \sum_{\substack{A\subseteq\Omega_{XY}\\ \#(x_1\in A)=1}} \frac{m_{XY}(A\ )}{|A|}$$

and

$$\sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|} = 2\sum_{\substack{A\subseteq\Omega_{XY}\\ \#(x_1\in A)=2}} \frac{m_{XY}(A\ )}{|A|}.$$

Because $\{A|A\subseteq\Omega_{XY},\ Z_{11}\in A,\ \#(x_1\in A)=2\}\cup\ \cdots\ \cup\{A|A\subseteq\Omega_{XY}, Z_{1M}\in A,\ \#(x_1\in A)=2\}=\{A|A\subseteq\Omega_{XY}, \#(x_1\in A)=2\}$, but each member is repeated twice in this subset. So, the summation is multiplied by 2. Also $\{A|A\subseteq\Omega_{XY},\ Z_{11}\in A,\ \#(x_1\in A)=M\}\cup\cdots\cup\{A|A\subseteq\Omega_{XY}, Z_{1M}\in A, \#(x_1\in A)=M\}=\{A|A\subseteq\Omega_{XY}, \#(x_1\in A)=M\}$ and each member is repeated M times in this subset. So the summation is multiplied by M as follows:

$$\sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{11}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|} + \cdots + \sum_{\substack{A\subseteq\Omega_{XY}\\ Z_{1M}\in A\ \&\#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|} = M\sum_{\substack{A\subseteq\Omega_{XY}\\ \#(x_1\in A)=M}} \frac{m_{XY}(A\ )}{|A|}.$$

Then, we have

$$\sum_{j=1}^{M} MBet_{m_{XY}}\left(\mathrm{Z_{1j}}\right) = \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = 1}} \frac{m_{XY}(A\ )}{|A|} + 2 \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = 2}} \frac{m_{XY}(A\ )}{|A|} + \cdots + M \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = M}} \frac{m_{XY}(A\ )}{|A|}.$$

Now, we can write

$$\begin{aligned}\sum_{j=1}^{M} MBet_{m_{XY}}\left(\mathrm{Z_{1j}}\right) &= \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = 1}} \frac{m_{XY}(A\ ) \cdot \#(x_1 \in A)}{|A|} \\ &+ \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = 2}} \frac{m_{XY}(A\ ) \cdot \#(x_1 \in A)}{|A|} + \ldots \\ &+ \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = M}} \frac{m_{XY}(A\ ) \cdot \#(x_1 \in A)}{|A|} \\ &= \sum_{\substack{A \subseteq \Omega_{XY} \\ \#(x_1 \in A) = 1\ OR\ \#(x_1 \in A) = 2 OR \ldots OR\ \#(x_1 \in A) = M}} \frac{m_{XY}(A\ ) \cdot\ \#(x_1 \in A)}{|A|}.\end{aligned}$$

Also, we have

$$\{A | A \subseteq \Omega_{XY},\ \#(x_1 \in A) = 1\} \cup \cdots \cup \{A | A \subseteq \Omega_{XY},\ \#(x_1 \in A) = M\} = \{A | B = A^{\downarrow \Omega_X}, B \subseteq \Omega_X, x_1 \in B, A \subseteq \Omega_{XY}\}.$$

Hence

$$\sum_{j=1}^{M} MBet_{m_{XY}}\left(\mathrm{Z_{1j}}\right) = \sum_{B \subseteq \Omega_X,\, x_1 \in B} \sum_{\substack{A \subseteq \Omega_{XY} \\ B = A^{\downarrow \Omega_X}}} \frac{m_{XY}(A\ ) \cdot \#(x_1 \in A)}{|A|} = MBet_{m_X}(x_1).$$

■

The modified pignistic probability is utilized in a new AU measure called modified ambiguity, which is denoted by MAM and defined as follows.

Definition 10: If $m : 2^{\Omega_X} \to [0, 1]$ is a BPA on Ω_X and $MBet_m$ is the modified pignistic probability, then

$$\mathrm{MAM} = -\sum_{x \in \Omega_X} MBet_m(x) \cdot \log_2 (MBet_m(x)). \tag{15}$$

The MAM based on the modified pignistic probability, satisfies the five requirements of AU. This contribution is presented in the following proposition.

Proposition 2: The MAM satisfies the five requirements of AU.

Proof of the probability consistency: Similar to the proof of [10, Th. 1], if all focal elements are singletons, there is $MBet_{m_X}(x_i) = m(x_i)\ \ \forall x_i \in \Omega_X$ and then $\mathrm{MAM}(m) = -\sum_{x \in \Omega_X} m(x) \cdot \log_2\ (m(x))$. Thus, MAM is equal to the Shannon entropy measure. ■

Proof of the set consistency: Similar to the proof of [10, Th. 2], if there is one focal element such as $m(A) = 1$ and $m(B) = 0\ \forall B \in 2^{\Omega_X}$ and $\mathrm{B} \neq \mathrm{A}$, then

$$MBet_{m_X}(x_i) = \frac{1}{|\mathrm{A}|} m(A) = \frac{1}{|\mathrm{A}|}\ \forall x_i \in \mathrm{A}.$$

Hence, $\mathrm{MAM}(m) = -\sum_{x_i \in \Omega_X} MBet_{m_X}(x_i) \cdot \log_2 (MBet_{m_X}(x_i)) = \log_2 |A|$, which is also set consistent. ■

Proof of the range: Similar to the proof of [10, Th. 3], we have $\sum_{x_i \in \Omega_X} MBet_{m_X}(x_i) = 1$. If there is one singleton focal element, $A, (\ m(A) = 1$ and $|A| = 1$ and $m(B) = 0 \forall B \subseteq \Omega_X$ and $B \neq A)$, then MAM will be minimized and equal to zero. However, if there is one focal element with the maximum cardinality $|\Omega_X|$ $(\ m(A) = 1$ and $|A| = |\Omega_X|$ and $m(B) = 0\ \forall B \subseteq \Omega_X$ and $B \neq A)$, then MAM will be maximized and equal to $\log_2 |\Omega_X|$. Thus, the range of MAM is $[0,\ \log_2 |\Omega_X|]$. ■

Proof of the subadditivity and additivity: The proof of the subadditivity property is the same as that presented by Jousselme *et al.* [10]

$$\mathrm{MAM}(m_{XY}) = -\sum_{(x,y) \in \Omega_{XY}} MBet_{m_{XY}}(x, y) \cdot \log_2 \left(MBet_{m_{XY}}(x, y)\right).$$

According to the Gibb's inequality, $-\sum_{i=1}^{n} p_i \log_2 p_i \leq -\sum_{i=1}^{n} p_i \log_2 q_i$

$$\begin{aligned}\mathrm{MAM}(m_{XY}) &\leq -\sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} MBet_{m_{XY}}(x, y) \cdot \log_2 (MBet_{m_X}(x) \\ &\quad \times MBet_{m_Y}(y)) \\ &= -\sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} MBet_{m_{XY}}(x, y) \cdot \log_2 (MBet_{m_X}(x)) \\ &\quad -\sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} MBet_{m_{XY}}(x, y) \cdot \log_2 (MBet_{m_Y}(y)) \\ &= -\sum_{x \in \Omega_X} \log_2 \left(MBet_{m_X}(x)\right) \sum_{y \in \Omega_Y} MBet_{m_{XY}}(x, y) \\ &\quad -\sum_{y \in \Omega_Y} \log_2 \left(MBet_{m_Y}(y)\right) \sum_{x \in \Omega_X} MBet_{m_{XY}}(x, y).\end{aligned}$$

The modified pignistic probability is invariant under marginalization, $MBet_{m_X}(X) = \sum_Y MBet_{m_{XY}}(X, Y)$ and $MBet_{m_Y}(Y) = \sum_X MBet_{m_{XY}}(X, Y)$. Then, we have

$$\begin{aligned}\mathrm{MAM}(m_{XY}) &\leq -\sum_{x \in \Omega_X} \log_2 \left(MBet_{m_X}(x)\right) \cdot MBet_{m_X}(x) \\ &\quad -\sum_{y \in \Omega_Y} \log_2 \left(MBet_{m_Y}(y)\right) \cdot MBet_{m_Y}(y) \\ &= \mathrm{MAM}(m_X) + \mathrm{MAM}(m_Y).\end{aligned}$$

Thus, MAM is subadditive.

If BPA is noninteractive ($MBet_{m_{XY}}(x,y) = MBet_{m_X}(x) \cdot MBet_{m_Y}(y)$), inequality is transformed into equality in line 2 of the proof, then MAM is additive. ∎

IV. Mutual AU Measure in DST

In the PT, entropy is a measure of uncertainty in a random variable and mutual information measures how much information one random variable contains about another one. It describes how much information two random variables share with each other. On the other hand, mutual information, $I(X;Y)$, is often used as a tool to measure similarity and dependency. Mutual information is defined as follows.

Definition 11 [23]: If $p_{XY} : \Omega_{XY} \rightarrow [0,1]$ is an arbitrary joint PDF on Ω_{XY} with joint entropy $H(X,Y)$, the associated marginal PDFs are p_X and p_Y with entropies $H(X)$ and $H(Y)$, then the mutual information between X and Y, is given by

$$I(X;Y) = H(X) + H(Y) - H(X,Y). \quad (16)$$

Based on [23], mutual information has three requirements: 1) it is symmetric, because entropy is a symmetric measure; 2) it is always nonnegative, because entropy is a subadditive measure ($H(X,Y) \le H(X) + H(Y)$); and 3) it is equal to zero if and only if X and Y are independent, because entropy is an additive measure ($H(X,Y) = H(X) + H(Y)$ iff $p_{XY}(x,y) = p_X(x)p_Y(y)$).

Similar to the $I(X;Y)$, we propose mutual AU measure in DST as follows.

Definition 12: If $m_{XY} : 2^{\Omega_{XY}} \rightarrow [0,1]$ is an arbitrary joint BPA on Ω_{XY}, the associated marginal BPAs are m_X and m_Y, then three mutual AU measures, based on AU, AM, and MAM, are defined by

$$\mathrm{AU}(X;Y) = \mathrm{AU}(X) + \mathrm{AU}(Y) - \mathrm{AU}(X,Y) \quad (17)$$

$$\mathrm{AM}(X;Y) = \mathrm{AM}(X) + \mathrm{AM}(Y) - \mathrm{AM}(X,Y) \quad (18)$$

$$\mathrm{MAM}(X;Y) = \mathrm{MAM}(X) + \mathrm{MAM}(Y) - \mathrm{MAM}(X,Y). \quad (19)$$

Similar to the mutual information in PT, the proposed mutual measure should be symmetric, always nonnegative and equal to zero if and only if the two variables are independent. In this regard and due to the specifications of AU, AM, and MAM, the following can be said.

1) $\mathrm{AU}(X;Y)$, $\mathrm{AM}(X;Y)$, and $\mathrm{MAM}(X;Y)$ are symmetric, because AU, AM and MAM are symmetric.
2) $\mathrm{AU}(X;Y)$ and $\mathrm{MAM}(X;Y)$ are nonnegative, because AU and MAM are subadditive. However, according to the nonsubadditivity of AM, $\mathrm{AM}(X;Y)$ is not necessarily nonnegative.
3) $\mathrm{AU}(X;Y)$, $\mathrm{AM}(X;Y)$, and $\mathrm{MAM}(X;Y)$ are equal to zero if and only if the variables are noninteractive, because AU, AM, and MAM are additive.

In the following example, the nonnegativity property of the mutual measures is illustrated.

Example 2 [11]: Let $m_{XY} : 2^{\Omega_{XY}} \rightarrow [0,1]$ be the joint BPA for $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2\}$ ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2)\} = \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}$).

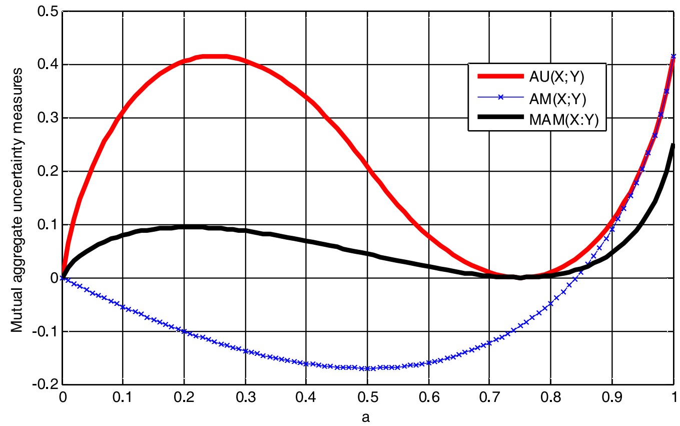

Fig. 3. Variations of AU $(X;Y)$, AM $(X;Y)$, and MAM $(X;Y)$ according to the changes in the parameter a.

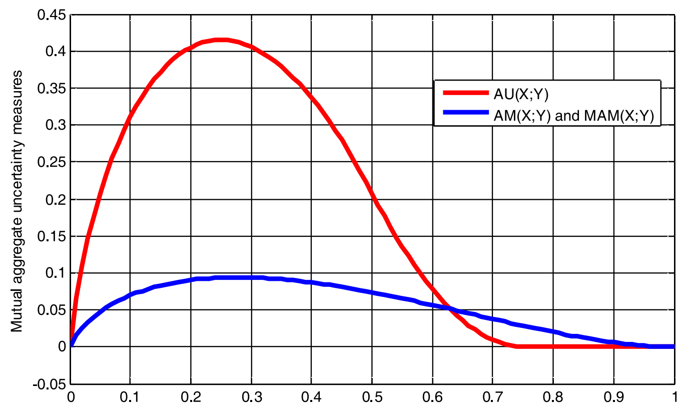

Fig. 4. Insensitivity of AU $(X;Y)$ according to the changes in the parameter a for $a > 0.7$.

We want to compute AU $(X;Y)$, AM $(X;Y)$, and MAM $(X;Y)$

$$\begin{cases} m_{XY}(\{Z_{11}, Z_{12}, Z_{21}\}) = a \\ m_{XY}(\{Z_{22}\}) = 1-a, 0 \le a \le 1. \end{cases}$$

The marginal of m_{XY} on Ω_X and Ω_Y are $m_X(\{x_1\}) = m_Y(\{y_1\}) = 0$, $m_X(\{x_2\}) = m_Y(\{y_2\}) = 1-a$, and $m_X(\{x_1,x_2\}) = m_Y(\{y_1,y_2\}) = a$. Now, the changes of AU $(X;Y)$, AM $(X;Y)$, and MAM $(X;Y)$, according to the changes in parameter a, are shown in Fig. 3. It can be observed that AM will be negative and cannot be an acceptable mutual uncertainty measure.

As mentioned in Section I, AU is not sensitive to changes in evidence. Thus, AU $(X;Y)$ poses this shortcoming, which is illustrated by the following example.

Example 3: Consider the BPAs of Example 2 with the following minor change:

$$\begin{cases} m_{XY}(\{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}) = a \\ m_{XY}(\{Z_{22}\}) = 1-a, 0 \le a \le 1. \end{cases}$$

According to the changing a, variations of AU $(X;Y)$, AM $(X;Y)$, and MAM $(X;Y)$ are depicted in Fig. 4. In Fig. 4, for large values of a, AU $(X;Y)$ is almost fixed and not sensitive to evidence changes. Therefore, it is not an appropriate measure for computing the amount of mutual uncertainty. The two examples above indicate that the mutual MAM is superior to the others.

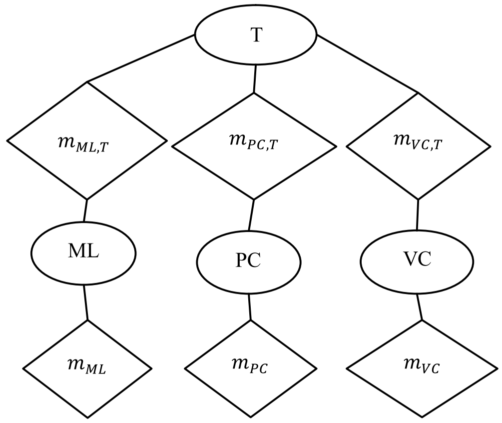

Fig. 5. Threat model using Dempster–Shafer network.

V. Using the Proposed Mutual Measures in Threat Assessment Problem

A. Threat Model Constructed by Dempster–Shafer Network

In 1989, Shenoy [24], [25] introduced the concept of a valuation-based system (VBS) as an alternative to Bayesian networks which provides a general framework for managing uncertainty in expert systems. This paper employs a VBS which is based on DST called evidential networks or Dempster–Shafer networks. Benavoli *et al*. [26] have utilized a Dempster–Shafer network to model a threat assessment problem. Also, in [20] and [21], Bayesian networks and evidential networks were compared in a threat assessment problem.

In a VBS, knowledge is represented by a network of variables (nodes) corresponding to the entities of the domain (and their states), and their links (edges) represent the relationships between these entities [24]. For solving a particular problem, network model must first be built in terms of these nodes and links. Inference within a VBS is performed via two operators called combination and marginalization. The reasoning within a VBS is to compute the joint valuation for the entire network and then to marginalize it to the subset of variables of interest for decision making [24], [25].

This paper utilizes a VBS based on DST in order to solve a typical threat assessment problem. The model is shown in the form of a Dempster–Shafer network in Fig. 5, where the variables are represented by circular nodes and the valuations (BPAs) by diamond shapes. The network includes four variables: missile launch (ML) over the state space $\Omega_{ML} = \{ml = 0, ml = 1\}$, voice communication (VC) over the state space $\Omega_{VC} = \{vc = 0, vc = 1\}$, political climate (PC) over the state space $\Omega_{PC} = \{pc = 0, pc = 1\}$, and threat (T) over the state space $\Omega_T = \{t = 0, t = 1\}$.

The valuations or the joint BPAs are obtained by the implication rules addressed in [26]–[28]. To obtain a joint BPA, the expert knowledge is first expressed in the form of an uncertain implication rule, such as "if A then B," and with a certain degree of confidence. For instance, suppose there are two disjoint variables: X over the state space Ω_X and Y over the state space Ω_Y. An implication rule is an expression of the form $A \subseteq \Omega_X \Rightarrow B \subseteq \Omega_Y$ with a probability (confidence) p such that p $\in [\alpha, \beta]$, with $0 \leq \alpha \leq \beta \leq 1$. Next, the implication rule can be expressed by a BPA consisting of three focal sets on the joint domain $\Omega_X \cup \Omega_Y$ as follows:

$$m_{XY}(\mathrm{C}) = \begin{cases} \alpha, & \textit{if } C = (A \times B) \cup (A^C \times \Omega_Y) \\ 1-\beta, & \textit{if } C = \left(A \times B^C\right) \cup (A^C \times \Omega_Y) \\ \beta - \alpha, & \textit{if } C = \Omega_X \times \Omega_Y \end{cases} \quad (20)$$

where A^C is the complement of A in Ω_X and B^C is the complement of Bin Ω_Y.

Finally, if there is more than one expression, each of the rules can be represented by a BPA. Then, the BPAs are combined by Dempster's rule and the joint BPAs are computed.

The BPAs of these network variables can now be computed as follows.

For ML and T, it is supposed that: if a missile has been fired by the target, then its threat level is between 0.9 and 0.98 and, if the missile has not been fired, then, with a certainty between 0.7 and 0.9, it is not a threat. These rules are rewritten as: "$(ML = 1) \rightarrow (T = 1)$ with a confidence between 0.9 and 0.98." and "$(ML = 0) \rightarrow (T = 0)$ with a confidence between 0.7 and 0.9." The joint state space will then be the power set of $\Omega_{ML,T} = \Omega_{ML} \times \Omega_T = \{(0,0), (0,1), (1,0), (1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and, we have

$$\begin{cases} m_{ML,T}(\{Z_{00}, Z_{10}\}) = 0.04, \ m_{ML,T}(\{Z_{01}, Z_{10}\}) = 0.002 \\ m_{ML,T}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.004, m_{ML,T}(\{Z_{00}, Z_{11}\}) = 0.63 \\ m_{ML,T}(\{Z_{01}, Z_{11}\}) = 0.09, m_{ML,T}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.18 \\ m_{ML,T}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.056 \\ m_{ML,T}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.008 \\ m_{ML,T}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.016. \end{cases}$$

For VC and T, the rules are: if a VC request has been answered by the target, then, with a certainty between 0.9 and 1, it is not a threat, and when there has not been any response to the VC request, then its threat level is between 0.7 and 0.9. These rules are rewritten as: "$(\mathrm{VC} = 1) \rightarrow (T = 0)$ with a confidence between 0.9 and 1" and "$(\mathrm{VC} = 0) \rightarrow (T = 1)$ with a confidence between 0.7 and 0.9," then the joint state space will be the power set of $\Omega_{VC,T} = \Omega_{VC} \times \Omega_T = \{(0,0), (0,1), (1,0), (1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and we have

$$\begin{cases} m_{VC,T}(\{Z_{00}, Z_{10}\}) = 0.09, \ m_{VC,T}(\{Z_{01}, Z_{10}\}) = 0.63 \\ m_{VC,T}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.18, m_{VC,T}(\{Z_{00}, Z_{11}\}) = 0.01 \\ m_{VC,T}(\{Z_{01}, Z_{11}\}) = 0.07, m_{VC,T}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.02. \end{cases}$$

For PC and T, "$(PC = 1) \rightarrow (T = 1)$ with a confidence between 0.6 and 0.8" and "$(PC = 0) \rightarrow (T = 0)$ with a confidence between 0.8 and 1." Then, the joint state space is the power set of $\Omega_{PC,T} = \Omega_{PC} \times \Omega_T = \{(0,0), (0,1), (1,0), (1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and the joint BPA will be

$$\begin{cases} m_{PC,T}(\{Z_{00}, Z_{10}\}) = 0.1333, m_{PC,T}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.0333 \\ m_{PC,T}(\{Z_{00}, Z_{11}\}) = 0.4, m_{PC,T}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.1 \\ m_{PC,T}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.2667 \\ m_{PC,T}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.0667. \end{cases}$$

Now, to make an inference within the network, the joint BPA for the entire variables is computed using the Dempster's

TABLE I
SENSITIVITY OF THE THREAT VALUE TO THE INPUT VARIABLES

	Threat	**pignistic probability**	**Euclidean distance**
BPA	$m_T(\{0\}) m_T(\{1\}) m_T(\{0,$	T=0 T=1	
$\boldsymbol{m_{ML}(\{0,1\}) = 1}$ $\boldsymbol{m_{VC}(\{0,1\}) = 1}$ $\boldsymbol{m_{PC}(\{0,1\}) = 1}$	0.199 0.129 0.671	0.535 0.465	
$\boldsymbol{m_{ML}(\{1\}) = 1}$	T1 : 0.043 0.885 0.072	0.079 0.921	d(T1,T2) =1.053
$\boldsymbol{m_{ML}(\{0\}) = 1}$	T2: 0.745 0.098 0.157	0.823 0.177	
$\boldsymbol{m_{VC}(\{0\}) = 1}$	T1 : 0.133 0.692 0.175	0.22 0.78	d(T1,T2) =0.97
$\boldsymbol{m_{VC}(\{1\}) = 1}$	T2: 0.905 0.094 0	0.905 0.095	
$\boldsymbol{m_{PC}(\{1\}) = 1}$	T1: 0.187 0.540 0.272	0.323 0.677	d(T1,T2) =0.78
$\boldsymbol{m_{PC}(\{0\}) = 1}$	T2 : 0.795 0.033 0.172	0.881 0.119	

TABLE II
MUTUAL AU OF THE INPUT VARIABLES TO THE THREAT VALUE

	ML,T	*VC,T*	*PC,T*
$\boldsymbol{MAM(X;Y)}$	0.4157	0.3731	0.2191
$\boldsymbol{AU(X;Y)}$	0.1078	0.1078	0.0101

rule of combination as follows:

$$m_{ML,PC,VC,T}(\cdot) = (m_{ML} \oplus m_{ML,T}) \oplus (m_{PC} \oplus m_{PC,T}) \oplus (m_{VC} \oplus m_{VC,T}).$$

It is then marginalized to the threat (T), $m_T(\cdot) = m_{ML,PC,VC,T}^{\downarrow \Omega_T}(\cdot)$. Finally, the pignistic probability is used to make a decision.

B. Sensitivity Analysis

Sensitivity analysis identifies which input variables of the network are more influential on decision making and how these input variables affect the decision process. In this process, the BPA of the input variable x is changed, then the BPA of the threat is computed, and then the effect of the change on the BPA of the T is evaluated. Similar to [26], this paper investigates how the changes of the input BPAs on the three input variables (ML, VC, and PC) affect the BPA of the decision variable T. Table I presents the results of the sensitivity analysis for this case. Input BPAs on ML, VC, and PC take two contrasting values: either all mass is assigned to state "0" or to state "1". By comparing the pignistic probability of the threat variable for the considered cases, it can be seen that T has most dependency on ML. A 2-D Euclidean distance, $d\,(T1, T2) = \sqrt{(T1_1 - T2_1)^2 + (T1_2 - T2_2)^2}$ is used for measuring the difference between the pignistic probabilities of threat values which are changed according to the maximum changes of the input variables, for instance, changing the BPA of ML from $m_{ML}\,(\{1\}) = 1$ (maximum threat) to $m_{ML}\,(\{0\}) = 1$ (minimum threat).

C. Using the Mutual Uncertainty Measures

In this section, AU(X; Y) and MAM(X; Y) are employed to compute the dependency of the paired variables (ML, T), (VC, T), and (PC, T). From Table II, it can be observed that ML has the most influence on the threat and PC has a minimum effect. These results are similar to the results obtained from the sensitivity analysis procedure. As shown in Table II, the amounts of dependency of the paired variables, (ML, T) and (VC, T), which were computed by AU(X; Y), are equal. Therefore, it is important to again note that AU(X; Y) is insensitive to changes in evidence and it is not a proper measure in decision-making applications.

VI. CONCLUSION

This paper introduced MAM. MAM compared to AM was subadditive, and this advantage was achieved by the modification of the classic pignistic probability.

Based on the available AU measures, AU and AM, and the new proposed measure, MAM, three mutual measures were presented. The proposed measures were examined in a threat assessment problem modeled by a Dempster–Shafer network. Due to the specifications of the measures and the testing results, it can be concluded as below.

1) AM(X; Y) is not an acceptable mutual measure, because it does not satisfy the nonnegativity property of a mutual measure.
2) AU(X; Y) is not a proper mutual measure, because it is insensitive to changes in evidence. Furthermore, it should be used cautiously in applications.
3) MAM(X; Y) is a justifiable mutual AU measure in DST. Similar to the mutual information in PT, MAM satisfies the requirements of a mutual measure and can be used in decision-making applications for computing the amounts of dependency between two variables.

According to the many mutual information applications in PT, these mutual measures can be used in the future by researchers in various applications.

REFERENCES

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, pp. 379–423, Jul. 1948.

[2] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Upper Saddle River, NJ, USA: Prentice-Hall, 1995.

[3] D. Dubois and H. Prade, "A note on measures of specificity for fuzzy sets," *Int. J. Gen. Syst.*, vol. 10, no. 4, pp. 279–283, 1985.

[4] A. Ramer, "Uniqueness of information measure in the theory of evidence," *Fuzzy Sets Syst.*, vol. 24, no. 2, pp. 183–196, 1987.

[5] U. Höhle, "Entropy with respect to plausibility measures," in *Proc. 12th IEEE Int. Symp. Mult.-Valued Log.*, Paris, France, 1982, pp. 167–169.

[6] R. R. Yager, "Entropy and specificity in a mathematical theory of evidence," *Int. J. Gen. Syst.*, vol. 9, no. 4, pp. 249–260, 1983.

[7] G. J. Klir and A. Ramer, "Uncertainty in the Dempster–Shafer theory: A critical re-examination," *Int. J. Gen. Syst.*, vol. 18, no. 2, pp. 155–166, 1990.

[8] G. J. Klir and B. Parviz, "Probability possibility transformations: A comparison," *Int. J. Gen. Syst.*, vol. 21, no. 3, pp. 291–310, 1992.

[9] G. J. Klir, *Uncertainty and Information: Foundations of Generalized Information Theory*. Hoboken, NJ, USA: Wiley, 2006.

[10] A. L. Jousselme, C. Liu, D. Grenier, and E. Bossé, "Measuring ambiguity in the evidence theory," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 36, no. 5, pp. 890–903, Sep. 2006.

[11] G. J. Klir and H. W. Lewis, "Remarks on 'measuring ambiguity in the evidence theory,"' *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 38, no. 4, pp. 995–999, Jul. 2008.

[12] J Abellán and A. R. Masegosa, "Requirements for total uncertainty measures in Dempster–Shafer theory of evidence," *Int. J. Gen. Syst.*, vol. 37, no. 6, pp. 733–747, 2008.

[13] G. Shafer, *A Mathematical Theory of Evidence*. Princeton, NJ, USA: Princeton Univ. Press, 1976.

[14] A. Dempster, "Upper and lower probabilities induced by multivalued mapping," *Ann. Math. Stat.*, vol. 38, no. 2, pp. 325–339, 1967.

[15] Y. Maeda, H. T. Nguyen, and H. Ichihashi, "Maximum entropy algorithms for uncertainty measures," *Int. J. Uncertain. Fuzz. Knowl.-Based Syst.*, vol. 1, no. 1, pp. 69–93. 1993.

[16] D. Harmanec and G. J. Klir, "Measuring total uncertainty in Dempster–Shafer theory," *Int. J. Gen. Syst.*, vol. 22, no. 4, pp. 405–419, 1994.

[17] J. Abellán and S. Moral, "Completing a total uncertainty measure in the Dempster–Shafer theory," *Int. J. Gen. Syst.*, vol. 28, nos. 4–5, pp. 299–314, 1999.

[18] P. Smets, "Constructing the pignistic probability function in a context of uncertainty," in *Uncertainty in Artificial Intelligence*, vol. 5, R. Henrion, R. D. Shachter, L. N. Kanal, and J. F. Lemmer, Eds. Amsterdam, The Netherlands: North Holland, 1990, pp. 29–40.

[19] G. J. Klir and M. J. Wierman, *Uncertainty-Based Information* (Studies in Fuzziness and Soft Computing), 2nd ed. New York, NY, USA: Physica-Verlag, 1999.

[20] B. R. Cobb and P. P. Shenoy, "On the plausibility transformation method for translating belief function models to probability models," *Int. J. Approx. Reasoning*, vol. 41, no. 3, pp. 314–330, 2006.

[21] B. R. Cobb and P. P. Shenoy, "A comparison of Bayesian and belief function reasoning," *Inf. Syst. Front.*, vol. 5, no. 4, pp. 345–358, 2003.

[22] B. R. Cobb and P. P. Shenoy, "A comparison of methods for transforming belief function models to probability models," in *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. Lecture Notes in Artificial Intelligence, vol. 2711, T. D. Nielsen and N. L. Zhang, Eds. Berlin, Germany: Springer, 2003, pp. 255–266.

[23] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York, NY, USA: Wiley, 1990.

[24] P. P. Shenoy, "A valuation-based language for expert systems," *Int. J. Approx. Reasoning*, vol. 3, no. 5, pp. 383–411, 1989.

[25] P. P. Shenoy, "Valuation based systems: A framework for managing uncertainty in expert systems," in *Fuzzy Logic and the Management of Uncertainty*, L. A. Zadeh and J. Kacprzyk, Eds. New York, NY, USA: Wiley, 1992, pp. 83–104.

[26] A. Benavoli, B. Ristic, A. Farina, M. X. Dsto, and L. Chisci, "An application of evidential networks to threat assessment," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 45, no. 2, pp. 620–639, Apr. 2009.

[27] B. Ristic and P. Smets, "Target identification using belief functions and implication rules," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 41, no. 3, pp. 1097–1103, Jul. 2005.

[28] P. Smets, "Belief functions: The disjunctive rule of combination and the generalized Bayesian theorem," *Int. J. Approx. Reas.*, vol. 9, no. 1, pp. 1–35, 1993.

Asghar Shahpari received the B.S. degree in electrical engineering from the Shahroud University of Technology, Shahrud, Iran, in 1999, and the M.S. degree in communication engineering from the Ferdowsi University of Mashhad, Mashhad, Iran, in 2001, where he is currently pursuing the Ph.D. degree in communication engineering.

His current research interests include the field of data fusion, evidence theory, uncertainty, advanced signal processing techniques, and pattern recognition.

Seyed A. Seyedin received the B.S. degree in electronics engineering from the Isfahan University of Technology, Isfahan, Iran, the M.S. degree in control and guidance engineering from Roorkee University, Roorkee, India, and the Ph.D. degree from the University of New South Wales, Sydney, NSW, Australia, in 1986, 1992, and 1996, respectively.

In 1996, he joined the Faculty of Engineering, Ferdowsi University of Mashhad, Mashhad, Iran, as an Assistant Professor, where he has been an Associate Professor since 2007. He was the Head of the Department of the Electrical Engineering from 1998 to 2000. His current research interests include machine vision, DSP-based signal processing algorithms, and digital image processing techniques.

Dr. Seyedin is a member of the Technical Committee of the Iranian Conference on Machine Vision and Image Processing.