Measuring mutual aggregate uncertainty in evidence theory

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***Abstract*—Mutual information as a tool for measuring the amount of dependency is used in many applications in probability theory. But no similar measures have been introduced to calculate the mutual uncertainty between two variables in Dempster-Shafer theory. In this paper three mutual measures based on three uncertainty measures are proposed. These uncertainty measures are: 1) Aggregate Uncertainty (AU) proposed by Klir et al.; 2) Ambiguity Measure (AM) proposed by Jousselme *et al.;* and 3) Modified Ambiguity Measure (MAM) that is proposed in this paper. MAM is the modification of AM that resolves the non-subadditivity problem of AM.**

A threat assessment problem constructed by Dempster-Shafer network is used for testing these mutual measures. We use the proposed mutual measures to identify which input variables of the network are more influential on the threat value. Finally it is concluded that mutual uncertainty based on MAM is a justifiable measure to compute the relevancy in decision making applications.

***Keywords*—Mutual aggregate uncertainty measure, Dempster–Shafer networks, Threat assessment, Pignistic probability, Sub-additivity, Decision making**

I. Introduction

Computing the amount of uncertainty or information contained in an event is of crucial importance in many applications in decision-making systems. To calculate the amount of uncertainty, we need to define a measure. Shannon entropy, H(x), is an uncertainty measure in Probability Theory (PT) proposed by Shannon [1]. The different types of uncertainty proposed in various theories have been classified by Klir and Yuan in [2]. Dempster-Shafer Theory (DST) is an extension of the probability theory and the set theory, and as such, it is able to represent two types of uncertainty, i.e., nonspesifity and discord.

The Generalized Hartley (GH) measure, which was proposed by Hartley ($\mathrm{H(A)} = \log_2 |\mathrm{A}|$), quantifies the amount of nonspesifity contained in a basic probability assignment in DST [3,4]. GH satisfies five essential requirements of a nonspesifity measure and is thus an appropriate measure to calculate the amount of set uncertainty in DST.

Similar to the Shannon entropy in PT, a number of measures have been proposed to compute probabilistic information in DST including Confusion by Höhle in 1982 [5], Dissonance by Yager in 1983 [6], Discord by Klir and Ramer in 1990 [7], and Strife by Klir and Parviz in 1992 [8]. Not all of these probabilistic information measures are justifiable discord measures in evidence theory because they not only have some conceptual flaws, but also are non-subadditive, and differ from Shannon entropy equation [9]. Therefore, Klir proposed AU as an aggregated uncertainty measure that computes nonspecificity and discord simultaneously [9]. He posited that any aggregate uncertainty measure such as AU must satisfy five requirements including *Probability consistency*, *Set consistency*, *Range*, *Subadditivity* and *Additivity*. AU, however, had high computational complexity and was insensitive to changes in evidence; moreover, the two types of uncertainty (nonspecificity and discord) were not separable.

Jousselme *et al.* proposed another aggregated uncertainty measure called AM based on the classic pignistic probability [10]. They proved that AM satisfies the five requirements of an aggregate uncertainty measure, it has low computational complexity and is sensitive to changes in evidence. But, Klir and Lewis showed that the proof of AM subadditivity provided by Jousselme *et al.* was wrong [11]. They referred to the dependency of the pignistic probability on the marginalization process to support their argument. This means that the marginal probabilities derived from the joint pignistic probability are not the same as the pignistic distributions derived from the marginal belief functions. In this paper the classic pignistic probability is modified, and then the modified AM is proposed. Also, it is proved that the modified AM is subadditive.

Another issue of this paper is proposing mutual aggregate uncertainty measure in DST. Similar to the mutual information in PT, the proposed measure should be symmetric, always non-negative and be equal to zero if and only if the two variables are independent [11]. AU and AM are the available aggregate uncertainty measures that can be used for introducing the mutual uncertainty measure. But AU is not sensitive to changes in evidence and this defect is transmitted

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to the corresponding mutual measure in DST. AM also is not subadditive and so the mutual measure based on AM will not be necessarily non-negative. To this end, the modified AM is proposed that is sub-additive and the corresponding mutual measure will be a justifiable tool to calculate the amount of dependency or relevancy in DST. The defects of the proposed measures will be compared in two examples. In order to understand the effectiveness of the proposed measures, a Dempster-Shafer network for solving a threat assessment problem will be presented. These measures will be used to compute the dependency of the input variables of the network to the threat value.

The paper is organized as follows: In Section II some necessary theoretical concepts are described, the available aggregate uncertainty measures in Dempster-Shafer theory are reviewed, and the *invariance with respect to marginalization* property of the pignistic probability is described. In Section III the modified pignistic probability that is invariant under the marginalization process is proposed. Then, the modified AM based on the modified pignistic probability is defined and proved to be subadditive. In Section IV three mutual aggregate uncertainty measures are proposed. In Section V a Dempster-Shafer network is presented to solve a threat assessment problem and then the proposed mutual measures are used to compute the dependency of the input variables of the network to the threat value. Finally, in Section VI some concluding remarks are made.

II. THEORETICAL BACKGROUND

A. Dempster-Shafer theory

In the probability theory, a probability density function (PDF) $p: \Omega_X \to [0,1]$ assigns values to $\Omega_X = \{x_1, x_2, \dots, x_n\}$ (the state space of variable X), where, $p(x_i) \geq 0$, and $\sum_{i=1:n} p(x_i) = 1$. If Ω_X and Ω_Y are the state spaces of variables X and Y, then the joint state space is denoted by $\Omega_{XY} = \Omega_X \times \Omega_Y$ and $p_{XY}: \Omega_{XY} \to [0,1]$ is the corresponding joint PDF. Marginalization in the probability theory involves addition over the state space of the variables being eliminated. Suppose p_{XY} is a joint PDF for Ω_{XY}, then the marginal PDF for Ω_X is $p_X(x) = \sum_Y p_{XY}(x,y)$.

Dempster-Shafer theory is an imprecise probability theory in which a basic probability assignment (BPA) assigns values to the subsets of the state space [12, 13]. The function $m: 2^{\Omega_X} \to [0,1]$ is a BPA on the power set of $\Omega_X = \{x_1, x_2, \dots, x_n\}$, where $m(\emptyset) = 0$, $m(A) \geq 0$, and $\sum_{A \in 2^{\Omega_X}} m(A) = 1$. Any element in 2^{Ω_X} with a non-zero BPA is called a focal element. Two other functions defined in DST are the belief function $(Bel(A) = \sum_{B \subseteq A} m(B))$ and the plausibility function $(Pl(A) = \sum_{A \cap B \neq \emptyset} m(B))$. Concepts such as joint state space, projection, combination, and marginalization, which are significant for this study, are defined as follows:

Definition 1. If 2^{Ω_X} and 2^{Ω_Y} are the state spaces of variables X and Y with cardinalities $2^{|\Omega_X|}$ and $2^{|\Omega_Y|}$, then the joint state space is denoted by $2^{\Omega_{XY}}$ with cardinality $2^{|\Omega_{XY}|}$, where, $\Omega_{XY} = \Omega_X \times \Omega_Y$.

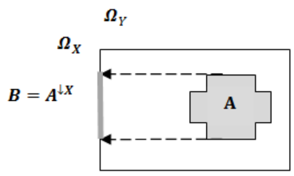

Fig. 1.Projection of subset $A \subseteq \Omega_{XY}$ on Ω_X.

Example 1. If $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2, y_3\}$ are the state spaces of X and Y, then the joint state space in DST will be $2^{\Omega_{XY}}$ ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1,y_1), (x_1,y_2), (x_1,y_3), (x_2,y_1), (x_2,y_2), (x_2,y_3)\}$), with $2^{|\Omega_{XY}|} = 2^6 = 64$ members. For simplicity, the following notations are introduced: $\Omega_{XY} = \Omega_X \times \Omega_Y = \{Z_{11}, Z_{12}, Z_{13}, Z_{21}, Z_{22}, Z_{23}\}$.

If Ω_X and Ω_Y are the state spaces of two variables and $\Omega_{XY} = \Omega_X \times \Omega_Y$ is the corresponding joint state space, then the projection of any subset $A \subseteq \Omega_{XY}$ on Ω_X is denoted by $A^{\downarrow \Omega_X}$. This projection is shown in Figure 1.

Definition 2. Let $m_X: 2^{\Omega_X} \to [0,1]$ and $m_Y: 2^{\Omega_Y} \to [0,1]$ be to equally reliable and independence BPAs, the combination is calculated by Dempster's rule of combination in the following manner:

$$m_{XY}(Z) = (m_X \oplus m_Y)(Z) = \frac{\sum_{X \cap Y = Z} m_X(X).m_Y(Y)}{1-K} \quad \forall\ X \subseteq \Omega_X\ \&\ Y \subseteq \Omega_Y \quad (1)$$

Where, $K = \sum_{X \cap Y = \emptyset} m_X(X).m_Y(Y)$ represents the basic probability mass associated with conflict.

Definition 3. If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is a joint BPA on Ω_{XY}, then the marginal of m_{XY} on Ω_X is denoted by $m_{XY}^{\downarrow \Omega_X}$, and given by:

$$m_X(B) = m_{XY}^{\downarrow \Omega_X}(B) = \sum_{A \subseteq \Omega_{XY},\, A^{\downarrow \Omega_X} = B} m_{XY}(A) \quad \forall B \subseteq \Omega_X \quad (2)$$

B. Available aggregate uncertainty measures in DST

AU and AM are two available aggregate uncertainty measures in DST, which are defined as follows:

Definition 4. [14-16] If $m: 2^{\Omega_X} \to [0,1]$ is a BPA on Ω_X and $P_{6.1}$ is the probability function obtained from the algorithm 6.1, then AU is calculated as follows:

$$AU(m) = -\sum_{x \in \Omega_X} P_{6.1}(x).\log_2\big(P_{6.1}(x)\big) \quad (3)$$

Definition 5. If $m: 2^{\Omega_X} \rightarrow [0,1]$ is a BPA on Ω_X and $BetP_m$ is the corresponding pignistic probability, then AM is defined as follows:

$$AM(m) = -\sum_{x \in \Omega_X} BetP_m(x).\log_2\big(BetP_m(x)\big) \qquad (4)$$

$BetP_m$, called pignistic probability, is a DST to PT transformation that transform a BPA in DST to the probability density function in PT and defined as follows:

Definition 6. [17] If $m: 2^{\Omega_X} \rightarrow [0,1]$ is a BPA on Ω_X, then $BetP_m$ denotes the corresponding probability function obtained using the pignistic probability, which is defined for each singleton $x \in \Omega_X$ as follows:

$$BetP_m(\{x\}) = \sum_{\substack{A \subseteq \Omega_X \\ x \in A}} {m(A)}/{|A|} \qquad (5)$$

C. Subadditivity

Klir and Wierman defined the following five requirements to be satisfied by an aggregate uncertainty measure including *Probability consistency*, *Set consistency*, *Range*, *Subadditivity*, and *Additivity* [18]. In this paper, only the subadditivity property is more significant and represented as follows:

Definition 7. If $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is an arbitrary joint BPA on Ω_{XY} and the associated marginal BPAs are m_X and m_Y, then;

$$AU(m_{XY}) \leq AU(m_X) + AU(m_Y) \qquad (6)$$

AM overcomes two deficiencies of AU, i.e., computational complexity and sensitivity to changes in evidence, but it is not subadditive. The dependency of the pignistic probability on marginalization process is the cause of this defect [11]. This concept is described in the following subsection.

D. Invariance with respect to marginalization process

Invariance with respect to marginalization property of a DST to PT transformations was illustrated in [19, 20, 21]. This property is illustrated for the pignistic probability by the following definition:

Definition 8. Let $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ be an arbitrary joint BPA on Ω_{XY}, and m_X be the marginal BPA on Ω_X. If $BetP_{m_{XY}}$ and $BetP_{m_X}$ are the pignistic probabilities of m_{XY} and m_X respectively, then $BetP_m$ is invariant under marginalization if and only if:

$$BetP_{m_X} = (BetP_{m_{XY}})^{\downarrow \Omega_X}. \qquad (7)$$

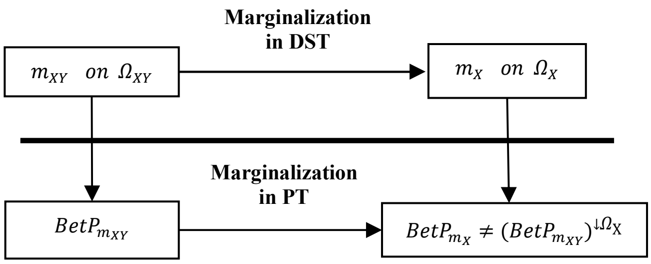

Fig. 2. The pignistic probability is dependent to the marginalization process.

But, Klir and Lewis showed that the pignistic probability is dependent to the marginalization process by an example [11]. This issue has been shown in Fig. 2.

III. MODIFIED AM

The main reason of the dependency of the pignistic probability on the marginalization process is in the projection process. As we know, in PT, the state space of the joint PDF is the Cartesian product of the two marginal state spaces ($\Omega_{XY} = \Omega_X \times \Omega_Y$). Also the PDF is invariant under the marginalization process because, $p_X(x) = \sum_Y p_{XY}(x,y)$. But in DST, the joint state space of the variables X over Ω_X and Y over Ω_Y is $2^{\Omega_{XY}}$ and it is not equal to the Cartesian product of the two marginal state spaces: $2^{\Omega_X} \times 2^{\Omega_Y}$. Now the question arises that if a BPA on DST has focal elements of $\{2^{\Omega_X} \times 2^{\Omega_Y}\}$, then similar to the PDF in PT, it will be invariant under marginalization process. To answer this question let us consider that $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is a joint BPA on $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2\}$. In this case, the joint state space is $2^{\Omega_{XY}} = \{\emptyset, \{Z_{11}\}, \{Z_{12}\}, \{Z_{21}\}, \{Z_{22}\}, \{Z_{11}, Z_{12}\}, \{Z_{11}, Z_{21}\}, \{Z_{11}, Z_{22}\}, \{Z_{12}, Z_{21}\}, \{Z_{12}, Z_{22}\}, \{Z_{21}, Z_{22}\}, \{Z_{11}, Z_{12}, Z_{21}\}, \{Z_{11}, Z_{12}, Z_{22}\}, \{Z_{11}, Z_{21}, Z_{22}\}, \{Z_{12}, Z_{21}, Z_{22}\}, \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}\}$. But the Cartesian product of 2^{Ω_X} and 2^{Ω_Y} is $\{2^{\Omega_X} \times 2^{\Omega_Y}\} = \{\emptyset, \{Z_{11}\}, \{Z_{12}\}, \{Z_{21}\}, \{Z_{22}\}, \{Z_{11}, Z_{12}\}, \{Z_{11}, Z_{21}\}, \{Z_{12}, Z_{22}\}, \{Z_{21}, Z_{22}\}, \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}\}$. Suppose m_{XY} has one focal element that is not a member of $\{2^{\Omega_X} \times 2^{\Omega_Y}\}$ i.e. $m_{XY}(\{Z_{12}, Z_{21}, Z_{22}\}) = 1$, then the joint pignistic probabilities are: $BetP_{m_{XY}}(Z_{12}) = BetP_{m_{XY}}(Z_{21}) = BetP_{m_{XY}}(Z_{22}) = 1/3$ and $BetP_{m_{XY}}(Z_{11}) = 0$. The marginal of m_{XY} on Ω_X is $m_X\big(B = \{x_1, x_2\} = \{Z_{12}, Z_{21}, Z_{22}\}^{\downarrow \Omega_X}\big) = 1$ and then $BetP_{m_X}(x_1) = BetP_{m_X}(x_2) = 1/2$. Similarly, the marginal of m_{XY} on Ω_Y is $m_Y\big(B = \{y_1, y_2\} = \{Z_{12}, Z_{21}, Z_{22}\}^{\downarrow \Omega_Y}\big) = 1$ and then $BetP_{m_y}(y_1) = BetP_{m_y}(y_2) = 1/2$. It can see that $BetP_{m_X}(x_1) \neq BetP_{m_{XY}}(Z_{11}) + BetP_{m_{XY}}(Z_{12})$.

In the other case, Suppose m_{XY} has one focal element of $\{2^{\Omega_X} \times 2^{\Omega_Y}\}$ i.e. $m_{XY}(\{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}) = 1$, the joint pignistic probabilities are $BetP_{m_{XY}}(Z_{11}) = BetP_{m_{XY}}(Z_{12}) = BetP_{m_{XY}}(Z_{21}) = BetP_{m_{XY}}(Z_{22}) = 1/4$. The corresponding marginal pignistic probabilities are computed similar to the previous case. So, in this case, the pignistic probability is invariant under the marginalization process, $BetP_{m_X}(x_1) = BetP_{m_{XY}}(Z_{11}) + BetP_{m_{XY}}(Z_{12})$. To solve this problem that is enough to instead of distribution $m_{XY}(C = \{Z_{11}, Z_{12}, Z_{21}\})$ equally between Z_{11}, Z_{12}, and Z_{21}, it is distributed equally among Z_{11}, Z_{12}, Z_{21}, and Z_{22} which are the members of $\{C^{\downarrow\Omega_X} \times C^{\downarrow\Omega_Y}\} = \{(x_1, x_2) \times (y_1, y_2)\} = \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}$. Now, due to the above discussion the pignistic probability can be modified as follows:

Definition 9. If $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ is a joint BPA defined over Ω_{XY}, then the modified pignistic probability is defined for each singleton $Z_{ij} \in \Omega_{XY}$ given by,

$$MBet_{m_{XY}}(Z_{ij}) = \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{ij} \in Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} \quad \forall\, Z_{ij} \in \Omega_{XY}. \quad (8)$$

Where, $Cart(C) = \{C^{\downarrow\Omega_X} \times C^{\downarrow\Omega_Y}\}$, and $|.|$ denotes the cardinality operator.

Corollary 1. In one-dimensional state space, $Cart(C) = C$ and $m_{XY} = m_X$. Then the modified pignistic probability will be equal to the pignistic probability, $MBet_{m_X} = BetP_{m_X}$.

Here, the *invariance with respect to marginalization* of the modified pignistic probability is expressed in the following proposition.

Proposition 1. *Suppose* $m_{XY}: 2^{\Omega_{XY}} \to [0,1]$ *is a joint BPA over* Ω_{XY}, $MBet_{m_{XY}}$ *is the joint modified pignistic probability and* $MBet_{m_X}$ *is the modified pignistic probability of marginal* m_X, *then we have:*

$$MBet_{m_X} = (MBet_{m_{XY}})^{\downarrow\Omega_X}\ . \quad (9)$$

Proof. We must prove that, $\sum_{j=1}^{|\Omega_Y|} MBet_{m_{XY}}((x_1, y_j)) = MBet_{m_X}(x_1)$. To simplify the problem, we suppose that $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2\}$ be the state spaces of X and Y, and ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\} = \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}$). Now we have,

$$\sum_{j=1}^{|\Omega_Y|} MBet_{m_{XY}}\left((x_1, y_j)\right) = \sum_{j=1}^{2} MBet_{m_{XY}}(Z_{1j}) = MBet_{m_{XY}}(Z_{11}) + MBet_{m_{XY}}(Z_{12})$$

$$= \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{11} \in Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} + \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{12} \in Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|}$$

We can separate the subset $\{C | C \subseteq \Omega_{XY}, Z_{11} \in C\}$ to the two new subsets: $\{C | C \subseteq \Omega_{XY}, Z_{11} \in C, Z_{12} \notin C\}$ and $\{C | C \subseteq \Omega_{XY}, \{Z_{11}, Z_{12}\} \in C\}$. Then, we have,

$$\sum_{j=1}^{2} MBet_{m_{XY}}(Z_{1j}) = \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{11} \in Cart(C) \text{ and } Z_{12} \notin Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} + \sum_{\substack{C \subseteq \Omega_{XY} \\ \{Z_{11}, Z_{12}\} \in Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} + \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{12} \in Cart(C) \text{ and } Z_{11} \notin Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} + \sum_{\substack{C \subseteq \Omega_{XY} \\ \{Z_{11}, Z_{12}\} \in Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|}$$

$$= \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{11} \in Cart(C) \text{ and } Z_{12} \notin Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} + \sum_{\substack{C \subseteq \Omega_{XY} \\ Z_{12} \in Cart(C) \text{ and } Z_{11} \notin Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|} + 2 \sum_{\substack{C \subseteq \Omega_{XY} \\ \{Z_{11}, Z_{12}\} \in Cart(C)}} \frac{m_{XY}(C)}{|Cart(C)|}$$

The new subsets can be rewritten as follows:
$\alpha = \{C | C \subseteq \Omega_{XY}, Z_{11} \in Cart(C), Z_{12} \notin Cart(C)\} = \{C | C \subseteq \Omega_{XY}, x_1 \in C^{\downarrow\Omega_X}, y_2 \notin C^{\downarrow\Omega_Y}\}$, and
$\beta = \{C | C \subseteq \Omega_{XY}, Z_{12} \in Cart(C), Z_{11} \notin Cart(C)\} = \{C | C \subseteq \Omega_{XY}, x_1 \in C^{\downarrow\Omega_X}, y_1 \notin C^{\downarrow\Omega_Y}\}$. Then we have,
$\alpha \cup \beta = \{C | C \subseteq \Omega_{XY}, x_1 \in C^{\downarrow\Omega_X}, |C^{\downarrow\Omega_Y}| = 1\}$.
Also,
$\{C | C \subseteq \Omega_{XY}, \{Z_{11}, Z_{12}\} \in Cart(C)\} = \{C | C \subseteq \Omega_{XY}, x_1 \in C^{\downarrow\Omega_X}, |C^{\downarrow\Omega_Y}| = 2\}$, and also $|Cart(C)| = |\{C^{\downarrow\Omega_X} \times C^{\downarrow\Omega_Y}\}| = |C^{\downarrow\Omega_X}|.|C^{\downarrow\Omega_Y}|$. So we have,

$$\sum_{j=1}^{2} MBet_{m_{XY}}(Z_{1j}) = \sum_{\substack{C \subseteq \Omega_{XY} \\ x_1 \in C^{\downarrow\Omega_X}, |C^{\downarrow\Omega_Y}| = 1}} \frac{m_{XY}(C)}{|C^{\downarrow\Omega_X}|.|C^{\downarrow\Omega_Y}|} + 2 \sum_{\substack{C \subseteq \Omega_{XY} \\ x_1 \in C^{\downarrow\Omega_X}, |C^{\downarrow\Omega_Y}| = 2}} \frac{m_{XY}(C)}{|C^{\downarrow\Omega_X}|.|C^{\downarrow\Omega_Y}|}$$

In the denominator of the second term, $|C^{\downarrow\Omega_Y}| = 2$, and $\{C | C \subseteq \Omega_{XY}, x_1 \in C^{\downarrow\Omega_X}, |C^{\downarrow\Omega_Y}| = 1\} \cup \{C | C \subseteq \Omega_{XY}, x_1 \in C^{\downarrow\Omega_X}, |C^{\downarrow\Omega_Y}| = 2\} = \{C | B = C^{\downarrow\Omega_X}, B \subseteq \Omega_X, x_1 \in B, C \subseteq \Omega_{XY}\}$. Also $m_X(C^{\downarrow\Omega_X}) = m_{XY}(C)$, because in the marginalization process the mass of the joint subset C is transmited to the $C^{\downarrow\Omega_X}$. Hence,

$$\sum_{j=1}^{2} MBet_{m_{XY}}(Z_{1j}) = \sum_{\substack{B \subseteq \Omega_X \\ x_1 \in B, B = C^{\downarrow\Omega_X}}} \frac{m_X(B)}{|B|} = MBet_{m_X}(x_1). \quad \square$$

The modified pignistic probability is used in a new aggregate uncertainty measure called modified ambiguity, which is denoted by MAM and is defined as follows:

Definition 10. If $m: 2^{\Omega_X} \rightarrow [0,1]$ is a BPA on Ω_X and $MBet_m$ is the modified pignistic probability, then we have:

$$MAM = -\sum_{x\in\Omega_X} MBet_m(x).\log_2\big(MBet_m(x)\big) \quad (10)$$

The modified AM based on the modified pignistic probability satisfies the five requirements of AU. This issue is presented in the following proposition.

Proposition 2. *The modified AM satisfies the five requirements of AU.*

Proof. In marginal state space, the pignistic probability and the modified pignistic probability are equal (Corollary 1). Thus, this measure satisfies the *Probability consistency*, *Set consistency* and *Range* requirements as proved in [10].

The proof of the subadditivity property is the same as the one proposed by Jousselme *et al.* in [10].

$$MAM(m_{XY}) = -\sum_{(x,y)\in\Omega_{XY}} MBet_{m_{XY}}(x,y).\log_2\left(MBet_{m_{XY}}(x,y)\right).$$

According to the Gibb's inequality, $-\sum_{i=1}^{n} p_i \log_2 p_i \leq -\sum_{i=1}^{n} p_i \log_2 q_i$:

$$\begin{aligned}
&MAM(m_{XY}) \\
&\leq -\sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} MBet_{m_{XY}}(x,y).\log_2(MBet_{m_X}(x).MBet_{m_Y}(y)) \\
&= -\sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} MBet_{m_{XY}}(x,y).\log_2(MBet_{m_X}(x)) \\
&\qquad -\sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} MBet_{m_{XY}}(x,y).\log_2(MBet_{m_Y}(y)) \\
&= -\sum_{x\in\Omega_X}\log_2\left(MBet_{m_X}(x)\right)\sum_{y\in\Omega_Y} MBet_{m_{XY}}(x,y) \\
&\qquad -\sum_{y\in\Omega_Y}\log_2\left(MBet_{m_Y}(y)\right)\sum_{x\in\Omega_X} MBet_{m_{XY}}(x,y).
\end{aligned}$$

The modified pignistic probability is invariant under marginalization, $MBet_{m_X}(X) = \sum_Y MBet_{m_{XY}}(X,Y)$ and $MBet_{m_Y}(Y) = \sum_X MBet_{m_{XY}}(X,Y)$. Then we have,

$$\begin{aligned}
MAM(m_{XY}) &\leq -\sum_{x\in\Omega_X}\log_2\left(MBet_{m_X}(x)\right).MBet_{m_X}(x) \\
&\qquad -\sum_{y\in\Omega_Y}\log_2\left(MBet_{m_Y}(y)\right).MBet_{m_Y}(y) \\
&= MAM(m_X) + MAM(m_Y).
\end{aligned}$$

Thus, MAM is subadditive.

If BPA is non-interactive ($MBet_{m_{XY}}(x,y) = MBet_{m_X}(x).MBet_{m_Y}(y)$), inequality is transformed into equality in line 2 of the proof, then MAM is additive. □

IV. Mutual Uncertainty Measure in DST

In probability theory mutual information, $I(X;Y)$, is often used as a measure of similarity and relevancy [22]. Mutual information is defined as follows:

Definition 11. If $p_{XY}: \Omega_{XY} \rightarrow [0,1]$ is an arbitrary joint PDF on Ω_{XY} with joint entropy $H(X,Y)$, the associated marginal PDFs are p_X and p_Y with entropies $H(X)$ and $H(Y)$, then the mutual information between $X$and Y, is given by:

$$I(X;Y) = H(X) + H(Y) - H(X,Y) \quad (11)$$

Mutual information has three requirements: 1) it is symmetric, because entropy is a symmetric measure; 2) it is always non-negative, because entropy is a subadditive measure ($H(X,Y) \leq H(X) + H(Y)$); and 3) it is equal to zero if and only if X and Y are independent, because entropy is an additive measure ($H(X,Y) = H(X) + H(Y)\ iff\ p_{XY}(x,y) = p_X(x)p_Y(y)$). Similar to the $I(X;Y)$, we propose mutual aggregate uncertainty measure in DST as follows:

Definition 12. If $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ is an arbitrary joint BPA on Ω_{XY}, the associated marginal BPAs are m_X and m_Y, then three mutual aggregate uncertainty measures based on, AU, AM, and MAM are given by:

$$AU(X;Y) = AU(X) + AU(Y) - AU(X,Y) \quad (12)$$

$$AM(X;Y) = AM(X) + AM(Y) - AM(X,Y) \quad (13)$$

$$MAM(X;Y) = MAM(X) + MAM(Y) - MAM(X,Y)\,. \quad (14)$$

Three requirements of the mutual information should be satisfied by the proposed mutual measures. In this regard and due to the specifications of AU, AM, and MAM, it can be said that:

- $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$ are symmetric, because AU, AM and MAM are symmetric.
- $AU(X;Y)$ and $MAM(X;Y)$ are non-negative, because AU and MAM are subadditive.
- $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$ are equal to zero if the variables are non-interactive, because AU, AM, and MAM are additive.
- But according to the non-subadditivity of AM, $AM(X;Y)$ is not necessarily non-negative.
- The insensitivity property of AU to changes in evidence is transmited to $AU(X;Y)$.

The latter two concepts are described in the following examples:

Example 2. [11] Let $m_{XY}: 2^{\Omega_{XY}} \rightarrow [0,1]$ be the joint BPA for $\Omega_X = \{x_1, x_2\}$ and $\Omega_Y = \{y_1, y_2\}$ ($\Omega_{XY} = \Omega_X \times \Omega_Y = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2)\} = \{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}$). We want to compute $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$.

$$\begin{cases} m_{XY}(\{Z_{11}, Z_{12}, Z_{21}\}) = a \\ m_{XY}(\{Z_{22}\}) = 1 - a \ ,0 \le a \le 1 \end{cases}.$$

The marginal of m_{XY} on Ω_X and Ω_Y are, $m_X(\{x_1\}) = m_Y(\{y_1\}) = 0$, $m_X(\{x_2\}) = m_Y(\{y_2\}) = 1 - \mathrm{a}$, and $m_X(\{x_1, x_2\}) = m_Y(\{y_1, y_2\}) = \mathrm{a}$. Now, the DST to PT transformations are computed and are listed in Table I.

Now, the changes of $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$ according to the changing a, are shown in Fig. 2. It is observed that AM will be negative and so it couldn't be an acceptable mutual uncertainty measure.

Example 3. Consider the BPA's of the example 2 with a little change as follows:

$$\begin{cases} m_{XY}(\{Z_{11}, Z_{12}, Z_{21}, Z_{22}\}) = a \\ m_{XY}(\{Z_{22}\}) = 1 - a \ ,0 \le a \le 1 \end{cases}.$$

The DST to PT transformations are computed and are listed in Table II. And variations of $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$ according to the changing a are shown in Fig. 3. According to the Fig. 3, it is concluded that AU is not sensitive in some cases and so it is not an appropriate measure for computing the amount of mutual uncertainty. The two above examples shows that the mutual modified ambiguity measure is better than the others.

I. THREAT ASSESSMENT USING DEMPSTER-SHAFER NETWORK

A. A Threat model

In 1989 Shenoy introduced the concept of a valuation-based system (VBS) as alternative to Bayesian networks which provides a general framework for managing uncertainty in expert systems [23, 24]. VBS can be modeled with each of the three major theories of uncertainty, namely probability theory, possibility theory and the theory of evidence. In this paper the VBS based on Dempster-Shafer theory called evidential networks or Dempster-Shafer networks are used. A. Benavoli *et al.* have been used an evidential network to model a threat assessment problem [25]. Also, in [19, 20] a threat assessment problem with Bayesian networks and evidential networks are compared.

In a VBS, knowledge is represented by a network of variables (nodes) corresponding to the entities of the domain (and their states), and of links (edges) representing the relationships between these entities [23]. For solving a particular problem, we first need to build a network model in terms of these nodes and links. Inference within a VBS is performed via two operators called combination and marginalization. The reasoning within a VBS would be to compute the joint valuation for the entire network and then to marginalize it to the subset of variables of interest for decision making [23, 24].

In this paper we use a VBS based on DST to solve a typical threat assessment problem. The model is shown in the form of a Dempster-Shafer network in Fig. 4, where the variables are represented by circular nodes and the valuations (BPAs) by diamond shapes.

TABLE I. $P_{6.1}$, BETP$_M$, AND MBETP$_M$ OF EXAMPLE 1

	$P_{6.1}$	$BetP_m$	$MBet_m$
m_{XY}	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}$	$\begin{bmatrix} a/4 & a/4 \\ a/4 & 1-3\frac{a}{4} \end{bmatrix}$
m_X	$\begin{cases} \begin{bmatrix} a \\ 1-a \end{bmatrix}; & if a < 1/2 \\ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}; & if a \ge 1/2 \end{cases}$	$\begin{bmatrix} a/2 \\ 1-a/2 \end{bmatrix}$	$\begin{bmatrix} a/2 \\ 1-a/2 \end{bmatrix}$
m_Y	$\begin{cases} [a \quad 1-a]; & if a < 1/2 \\ [1/2 \quad 1/2]; & if a \ge 1/2 \end{cases}$	$[a/2 \quad 1-a/2]$	$[a/2 \quad 1-a/2]$

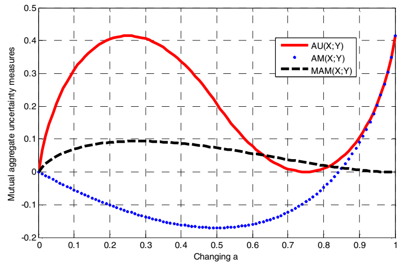

Fig. 2. Changes in $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$ according to the changing a.

TABLE II. $P_{6.1}$, BETP$_M$, AND MBETP$_M$ OF EXAMPLE 2

	$P_{6.1}$	$BetP_m$	$MBet_m$
m_{XY}	$\begin{cases} \begin{bmatrix} a/3 & a/3 \\ a/3 & 1-a \end{bmatrix}; & if a < 3/4 \\ \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}; & if a \ge 3/4 \end{cases}$	$\begin{bmatrix} a/4 & a/4 \\ a/4 & 1-3\frac{a}{4} \end{bmatrix}$	$\begin{bmatrix} a/4 & a/4 \\ a/4 & 1-3\frac{a}{4} \end{bmatrix}$
m_X	$\begin{cases} \begin{bmatrix} a \\ 1-a \end{bmatrix}; & if a < 1/2 \\ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}; & if a \ge 1/2 \end{cases}$	$\begin{bmatrix} a/2 \\ 1-a/2 \end{bmatrix}$	$\begin{bmatrix} a/2 \\ 1-a/2 \end{bmatrix}$
m_Y	$\begin{cases} [a \quad 1-a]; & if a < 1/2 \\ [1/2 \quad 1/2]; & if a \ge 1/2 \end{cases}$	$[a/2 \quad 1-a/2]$	$[a/2 \quad 1-a/2]$

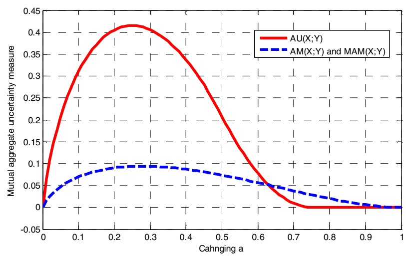

Fig. 3. Changes in $AU(X;Y)$, $AM(X;Y)$, and $MAM(X;Y)$ according to the changing a.

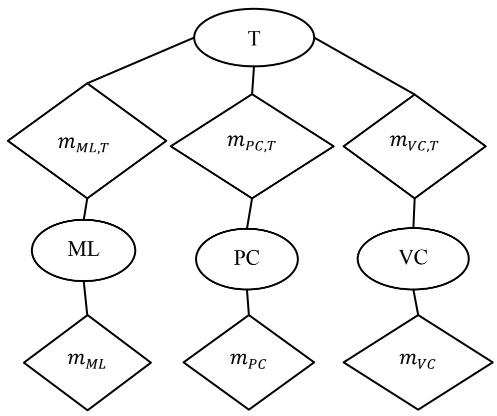

Fig. 4. Threat model using Dempster-Shafer network

The network includes four variables: missile launch (ML) that $\Omega_{ML} = \{ml = 0, ml = 1\}$, voice communication (VC) that $\Omega_{VC} = \{vc = 0, vc = 1\}$, political climate (PC) that $\Omega_{PC} = \{pc = 0, pc = 1\}$, and threat (T) that $\Omega_T = \{t = 0, t = 1\}$. We can summarize "expert" knowledge about the three input variables of the Dempster-Shafer network by the following set of independent rules, and then according to the implication rule in [25-27], each of the rules can be represented by a BPA. Then the BPAs are combined by Dempster's rule, and the joint BPAs are constructed as follows:

For ML and T it is supposed that: If the missile has been fired by the target, then its threat level is between 0.9 to 0.98, and if the missile has not been fired, then with certainty between 0.7 to 0.9 it is not a threat. These rules are rewritten as: "(ML=1)→(T=1) with confidence between 0.9 to 0.98" and "(ML=0)→(T=0) with confidence between 0.7 to 0.9". Then the joint state space will be the power set of $\Omega_{ML,T} = \Omega_{ML} \times \Omega_T = \{(0,0), (0,1), (1,0), (1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and we have,

$$\begin{cases} m_{ML,T}(\{Z_{00}, Z_{10}\}) = 0.04\ , m_{ML,T}(\{Z_{01}, Z_{10}\}) = 0.002 \\ m_{ML,T}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.004, m_{ML,T}(\{Z_{00}, Z_{11}\}) = 0.63 \\ m_{ML,T}(\{Z_{01}, Z_{11}\}) = 0.09, m_{ML,T}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.18 \\ m_{ML,T}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.056, m_{ML,T}(\{Z_{01}, Z_{10}, Z_{11}\}) = 0.008 \\ m_{ML,T}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.016 \end{cases}$$

For VC and T the rules are: If voice communication request has been answered by the target, then with certainty between 0.9 to 0.1 it is not a threat, and when it has not been received any response to the voice communication request, then its threat level is between 0.7 to 0.9. These rules are rewritten as: "(VC=1)→(T=0) with confidence between 0.9 to 1" and "(VC=0)→(T=1) with confidence between 0.7 to 0.9", then the joint state space will be the power set of $\Omega_{VC,T} = \Omega_{VC} \times \Omega_T = \{(0,0), (0,1), (1,0), (1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and we have,

$$\begin{cases} m_{VC,T}(\{Z_{00}, Z_{10}\}) = 0.09, m_{VC,T}(\{Z_{01}, Z_{10}\}) = 0.63 \\ m_{VC,T}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.18, m_{VC,T}(\{Z_{00}, Z_{11}\}) = 0.01 \\ m_{VC,T}(\{Z_{01}, Z_{11}\}) = 0.07, m_{VC,T}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.02 \end{cases}$$

For PC and T we have: "(PC=1)→(T=1) with confidence between 0.6 to 0.8" and "(PC=0)→(T=0) with confidence between 0.8 to 1". Then the joint state space is the power set of $\Omega_{PC,T} = \Omega_{PC} \times \Omega_T = \{(0,0), (0,1), (1,0), (1,1)\} = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and the joint BPA will be,

$$\begin{cases} m_{PC,T}(\{Z_{00}, Z_{10}\}) = 0.1333, m_{PC,T}(\{Z_{00}, Z_{01}, Z_{10}\}) = 0.0333 \\ m_{PC,T}(\{Z_{00}, Z_{11}\}) = 0.4, m_{PC,T}(\{Z_{00}, Z_{01}, Z_{11}\}) = 0.1 \\ m_{PC,T}(\{Z_{00}, Z_{10}, Z_{11}\}) = 0.2667 \\ m_{PC,T}(\{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}) = 0.0667 \end{cases}$$

Now to make an inference within the network the joint BPA for the entire variables is computed using the Dempster's rule of combination as follows:

$$m_{ML,PC,VC,T}(.) = (m_{ML} \oplus m_{ML,T}) \oplus (m_{PC} \oplus m_{PC,T}) \oplus (m_{VC} \oplus m_{VC,T})$$

Then it is marginalized to the threat (T), $m_T(.) = m_{ML,PC,VC,T}^{\downarrow \Omega_T}(.)$. Finally the pignistic probability is used to make a decision.

B. *Sensitivity analysis*

Sensitivity analysis studies the effect of the changes in the input valuations on the valuation of the output (decision) variable. In this way, sensitivity analysis helps us to identify which inputs are more influential on decision making and how they affect the decision process. In this process the valuation of the input variable x is changed, then the valuation of the threat is computed, and then the effect of the change on the valuation of the T is evaluated. Similar to the work in [25], we investigate how the changes of the input BPAs on three input variables (ML, VC and PC) affect the BPA of the decision variable T. Table III presents the results of the sensitivity analysis for this case. Input BPAs on ML, VC and PC take two contrasting values: either all mass is assigned to state "0" or to state "1". Comparing the pignistic probability of the threat variable for the considered cases, it can be seen that ML is most dependence to T and PC has minimum dependency. Two dimensional Euclidean distance, $d(T1, T2) = \sqrt{(T1_1 - T2_1)^2 + (T1_2 - T2_2)^2}$ is used for measuring the difference between the pignistic probabilities of threat values are changed according to the maximum changing of the input variables, for instant changing the BPA of ML from $m_{ML}(\{1\}) = 1$ (maximum threat) to $m_{ML}(\{0\}) = 1$ (minimum threat).

C. *Using the mutual uncertainty measures*

Now, in the other hand, we use $AU(X;Y)$ and $MAM(X;Y)$ to compute the dependency of the paired variables (ML,T), (VC,T), and (PC,T). From Table IV it is shown that the ML

TABLE III. SENSITIVITY OF THE THREAT VALUE TO THE INPUT VARIABLES.

	Threat	pignistic probability	Euclidean distance
BPA	$m_T(\{0\})$ $m_T(\{1\})$ $m_T(\{0,1\})$	T=0 T=1	
$m_{ML}(\{0,1\}) = 1$ $m_{VC}(\{0,1\}) = 1$ $m_{PC}(\{0,1\}) = 1$	0.199 0.129 0.671	0.535 0.465	
$m_{ML}(\{1\}) = 1$	T1 : 0.043 0.885 0.072	0.079 0.921	d(T1,T2) =1.053
$m_{ML}(\{0\}) = 1$	T2: 0.745 0.098 0.157	0.823 0.177	
$m_{VC}(\{0\}) = 1$	T1 : 0.133 0.692 0.175	0.22 0.78	d(T1,T2) =0.97
$m_{VC}(\{1\}) = 1$	T2: 0.905 0.094 0	0.905 0.095	
$m_{PC}(\{1\}) = 1$	T1: 0.187 0.540 0.272	0.323 0.677	d(T1,T2) =0.78
$m_{PC}(\{0\}) = 1$	T2 : 0.795 0.033 0.172	0.881 0.119	

TABLE IV. MUTUAL AGGREGATE UNCERTAINTY OF THE INPUT VARIABLES TO THE THREAT VALUE.

	ML,T	***VC,T***	***PC,T***
$\boldsymbol{MAM(X;Y)}$	0.4645	0.4093	0.2964
$\boldsymbol{AU(X;Y)}$	0.1078	0.1078	0.0101

has most influence to the threat and PC has minimum affect. These results are similar to the results obtained from the sensitivity analysis procedure. Another important point that was previously referred, is that the amounts of dependency of the paired variables (ML,T) and (VC,T) that computed by $AU(X;Y)$ are equal. So it is important to note that again, $AU(X;Y)$ is insensitive to changes in evidence and it is not a good measure in decision making applications. We hope that the considerable results help us to use these measures in diverse applications in DST.

II. CONCLUSION

In this paper the modified ambiguity measure, called MAM, was introduced. MAM against AM was subaditivite. This advantage was achieved by modification of the classic pignistic probability.

Based on the available aggregate uncertainty measures, those are AU and AM, and the new proposed measure i.e. MAM, three mutual measures were proposed. The proposed measures were examined in a threat assessment problem modeled by a Dempster-Shafer network. Due to the specifications of the measures and the testing results, we can conclude that:

- $AM(X;Y)$ is not an acceptable mutual measure, because it does not satisfy the non-negativity property of a mutual measure.
- $AU(X;Y)$ is not a good mutual measure, because it is insensitive to changes in evidence. And it should be used cautiously in application.
- $MAM(X;Y)$ is a justifiable mutual aggregate uncertainty measures in DST. Similar to the mutual information in PT, it satisfies the requirements of a mutual measure and can be used in decision making applications for computing the amounts of dependency or relevancy.

According to the many applications of the mutual information in PT, these mutual measures can be used in the future in various applications.

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