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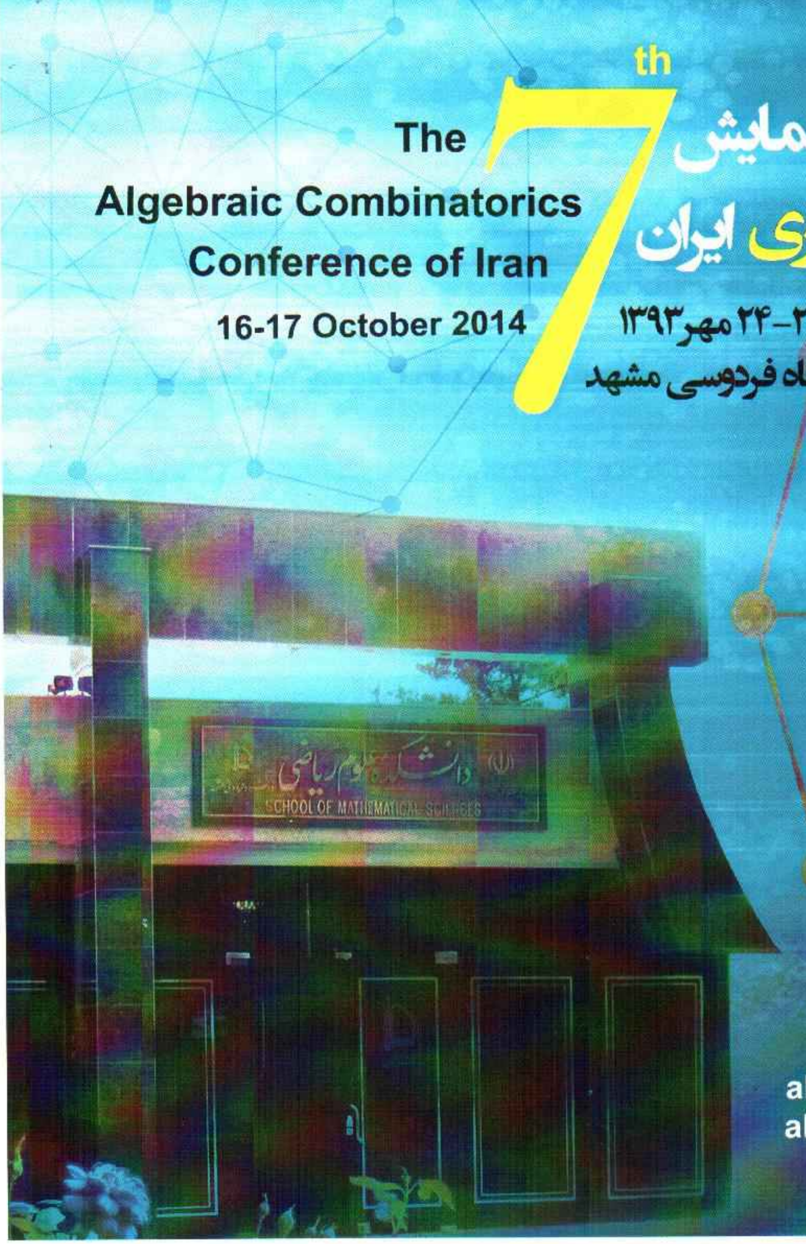

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PLANAR, TOROIDAL AND PROJECTIVE GENERALIZED PETERSEN GRAPHS

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ABSTRACT. The aim of this talk is a determination of all generalized Petersen graphs $P(n, k)$ which can be embedded on the plane, torus or projective plane. We give some necessary and sufficient conditions on the values n and k in which $P(n, k)$ to be planar, toroidal or projective graph.

1. INTRODUCTION

In 1969, a class of generalized Petersen graphs was defined by Watkins [5] as follows: for positive integers $n \geq 3$ and $1 \leq k < n/2$, the generalized Petersen graph $P(n, k)$ is defined on the set of $2n$ vertices $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ whose edges are $\{u_i, u_{i+1}\}$, $\{u_i, v_i\}$ and $\{v_i, v_{i+k}\}$, where i runs over $\{1, 2, \ldots, n\}$ and that all indices are taken in modulo n.

Recall that a graph is *planar* if it can be drawn in the plane such that its edges intersect only at their end points. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem [2] states that a graph is planar if and only if it contains no subdivisions

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of K_5 and $K_{3,3}$, where K_n is the *complete graph* with n vertices and $K_{m,n}$ is the *complete bipartite graph* with parts of sizes m and n.

It is well-known that a compact surface is homeomorphic to a sphere, a connected sum of g tori, or a connected sum of k projective planes (see [3, Theorem 5.1]). We denote S_0 for the sphere and S_g $(g \geq 1)$ for the surface formed by a connected sum of g tori, and N_k for the one formed by a connected sum of k projective planes. The number g is called the *genus* of the surface S_g and k is called the *crosscap* of N_k. When considering the orientability, the surfaces S_g and sphere are among the orientable class of surfaces and the surfaces N_k are among the non-orientable one.

A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. Similarly, if a simple graph can be embedded in N_k but not in N_{k-1}, then we call it a graph of crosscap k. The notations $\gamma(\Gamma)$ and $\overline{\gamma}(\Gamma)$ stand for the genus and crosscap of a graph Γ, respectively. It is easy to see that $\gamma(\Gamma_0) \leq \gamma(\Gamma)$ and $\overline{\gamma}(\Gamma_0) \leq \overline{\gamma}(\Gamma)$, for all subgraphs Γ_0 of Γ. Clearly, a graph Γ is planar if $\gamma(\Gamma) = 0$. A graph Γ such that $\gamma(\Gamma) = 1$ is called a *toroidal* graph. Also, a graph Γ such that $\overline{\gamma}(\Gamma) = 1$ is called a *projective* graph.

In this article, we determine all generalized Petersen graphs which can be embedded on the plane, torus or projective plane.

2. Main results

Theorem 2.1 ([4]). *For positive integers m and n, we have*

(1) $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$ *if* $n \geq 3$,

(2) $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$ *if* $m, n \geq 2$.

A block in a graph is a maximal subgraph with no cut point. The following theorem gives a formula for computing the genus of a graph using its blocks genus.

Theorem 2.2 ([1]). *If Γ is a graph with blocks $B_1, \ldots, B_n$, then*

$$\gamma(\Gamma) = \gamma(B_1) + \cdots + \gamma(B_n).$$

The following theorems state a characterization of Planar, toroidal and projective generalized Petersen graphs in terms of values n and k.

Theorem 2.3. *The generalized Petersen graph $P(n,k)$ is planar if and only if $k = 1$ or 2*

Theorem 2.4. *The generalized Petersen graph $P(n,k)$ is toroidal if and only if $n = 2k+1$ and $k \geq 3$*

Theorem 2.5. *The generalized Petersen graph $P(n,k)$ is projective if and only if $n = 2k+1$ and $k \geq 3$*

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