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A GRAPH ASSOCIATED TO GROUPS BY AUOTOMORPHISMS OF THEM

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ABSTRACT. Let G be a group. By using the set of automorphisms of G, we associate a simple graph to G denoted by $\Gamma_{\mathrm{Aut}(G)}(G)$. In this paper we study some properties of this graph.

1. INTRODUCTION

Let G be a group and $Z(G)$ be its center. For an arbitrary element z in G, let I_z be the inner automorphism of G given by $I_z(t) = z^{-1}tz$ for all $t \in G$. Also, for $x \in G$, $\mathrm{Stab}(x) = \{f \in \mathrm{Aut}(G) : f(x) = x\}$. The *non-commuting graph* of G denoted by Γ_G (which was first introduced by Pual Erdös [5])is a simple graph which its vertex set is $G\backslash Z(G)$ and two distinct vertices x and y are adjacent if and only if $xy \neq yx$. In other words $xy \neq yx$ if and only if $I_x(y) \neq y$ (or $I_y(x) \neq x$). Clearly, $I_x \in \mathrm{Stab}(x)$ for all $x \in G$.

In this paper, we are inspired by this idea and by using the set of *automorphisms* of G, denoted by $\mathrm{Aut}(G)$, we associate a simple graph denoted by $\Gamma_{\mathrm{Aut}(G)}(G)$ as follows: Two distinct vertices $x, y \in G$ are adjacent if and only if there is $f \in \mathrm{Stab}(x)$ with $f(y) \neq y$ or $g \in \mathrm{Stab}(y)$

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with $g(x) \neq x$. Also, an element $x \in G$ is a vertex of $\Gamma_{\mathrm{Aut}(G)}(G)$ if there is an element y in G such that x and y are adjacent.

In this paper, we study some combinatorial properties of $\Gamma_{\mathrm{Aut}(G)}(G)$.

To begin with, we remind some notations in graph theory. Let X be a graph. We use the notations; diam(X), girth(X), α(X), γ(X) and X^c for the *diameter*, *girth*, *independence number*, *domination number* and *complement* of X, respectively. In this paper our terminologies on graphs and groups are derived from [3] and [4], respectively.

2. Main results

Note that, for every group G with at most two elements, $\mathrm{Aut}(G) = \{\mathrm{id}_G\}$, and so $\Gamma_{\mathrm{Aut}(G}(G)$ is the empty graph. So in this paper all groups G have at least three elements.

In the following proposition we determine the vertex set of $\Gamma_{\mathrm{Aut}(G}(G)$.

Proposition 2.1. *Let G be a group with at least three elements, then $V(\Gamma_{\mathrm{Aut}(G)}(G)) = G$.*

Example 2.2. Let p be a prime number with $p \neq 2$. Then $\Gamma_{\mathrm{Aut}(\mathbb{Z}_p)}(\mathbb{Z}_p)$ is a star graph.

In the following theorem, we study some basic properties of $\Gamma_{\mathrm{Aut}(G)}(G)$.

Theorem 2.3. *Let G be a group. Then the components of $(\Gamma_{\mathrm{Aut}(G)}(G))^c$ are complete graph, and consequently $\Gamma_{\mathrm{Aut}(G)}(G)$ is a complete r-partite graph. Moreover,* $\mathrm{diam}(\Gamma_{\mathrm{Aut}(G)}(G)) \leq 2$.

The next corollary immediately follows from Theorem 2.3.

Corollary 2.4. *Let G be a group. If the graph $\Gamma_{\mathrm{Aut}(G)}(G)$ has a cycle, then* $\mathrm{gr}(\Gamma_{\mathrm{Aut}(G)}(G)) \leq 4$.

Notation. Let G be a group. It is easy to see that the set of all elements of G which are fixed by every automorphism of G is a normal subgroup of G . For simplicity of notation, we denote it by $\mathcal{H}$.

In the following proposition we determine the structure of parts of $\Gamma_{\mathrm{Aut}(G)}(G)$.

Proposition 2.5. *Let G be a group. Then one of the parts of $\Gamma_{\mathrm{Aut}(G)}(G)$ is $\mathcal{H}$ and the other parts are a union of some $\mathcal{H}$-cosets.*

The following remark is useful in our proofs.

Remark 2.6. Suppose that G is a group and x, y are two adjacent vertices in the non-commuting graph of G. Then it is easy to see that x and y are adjacent in the graph $\Gamma_{\mathrm{Aut}(G)}(G)$. This means that if the vertices x and y are not adjacent in $\Gamma_{\mathrm{Aut}(G)}(G)$, then $xy = yx$. On the other hand, $\Gamma_{\mathrm{Aut}(G)}(G)$ is a complete r-partite graph. So, all vertices in a same part of $\Gamma_{\mathrm{Aut}(G)}(G)$ commute together.

Proposition 2.7. *Let G be a non-abelian group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is Hamiltonian.*

In the following theorem, we determine when $\Gamma_{\mathrm{Aut}(G)}(G)$ is a complete graph.

Theorem 2.8. *Let G be a group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is a complete graph if and only if $G \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell-times}$, for some $\ell \geq 2$.*

Corollary 2.9. *Let G be a non-abelian group. Then $\mathrm{gr}(\Gamma_{\mathrm{Aut}(G)}(G)) = 3$ and $\mathrm{diam}(\Gamma_{\mathrm{Aut}(G)}(G)) = 2$.*

Proposition 2.10. *Let G be a group. Then the graph $\Gamma_{\mathrm{Aut}(G)}(G)$ is a tree if and only if it is a star graph.*

In the following proposition, we determine when $\Gamma_{\mathrm{Aut}(G)}(G)$ is a star graph.

Proposition 2.11. *Let G be a group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is a star graph if and only if $\mathrm{Fix}(f) = \{e\}$ for every non-identity automorphism f of G, where e is the identity element in G.*

Corollary 2.12. *$\Gamma_{\mathrm{Aut}(\mathbb{Q})}(\mathbb{Q})$ and $\Gamma_{\mathrm{Aut}(\mathbb{Z})}(\mathbb{Z})$ are star graph, where $\mathbb{Q}$ and $\mathbb{Z}$ are the set of rational numbers and integer numbers, respectively.*

In the following proposition, we determine when $\Gamma_{\mathrm{Aut}(G)}(G)$ is a cycle.

Proposition 2.13. *Let G be a group. Then the graph $\Gamma_{\mathrm{Aut}(G)}(G)$ is a cycle if and only if $G \cong \mathbb{Z}_4$.*

Proposition 2.14. *Let G be a group and p be the smallest prime number, which divides $|G|$. Then $\alpha(\Gamma_{\mathrm{Aut}(G)}(G)) \geq p$.*

Proposition 2.15. *Let G be a group. Then*

$$\gamma(\Gamma_{\mathrm{Aut}(G)}(G)) = \begin{cases} 1 & \text{if } \mathcal{H} = \{e\}, \\ 2 & \text{otherwise.} \end{cases}$$

Proposition 2.16. *Let G be a cyclic group. Then the elements in G of a same order, lie in the same part of $\Gamma_{\mathrm{Aut}(G)}(G)$.*

Note that the converse of Proposition 2.16 is not true in general.

The planarity is one of the important properties in the study of a simple graph. We are going to investigate the planarity of $\Gamma_{\mathrm{Aut}(G)}(G)$, for some groups G.

Proposition 2.17. *Let G be a non-abelian group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is non-planar.*

We need the following lemma in the sequal.

Lemma 2.18. *The following statements hold.*

(a) $\Gamma_{\mathrm{Aut}(\mathbb{Z}_2\times\mathbb{Z}_4)}(\mathbb{Z}_2\times\mathbb{Z}_4)$ *is non-planar.*
(b) $\Gamma_{\mathrm{Aut}(\mathbb{Z}_2\times\mathbb{Z})}(\mathbb{Z}_2\times\mathbb{Z})$ *is non-planar.*
(c) *If $n\geq 3$, then $\Gamma_{\mathrm{Aut}(\mathbb{Z}_{2^n})}(\mathbb{Z}_{2^n})$ is non-planar.*
(d) *Let p be a prime number with $p\geq 3$ and $n\geq 2$. Then $\Gamma_{\mathrm{Aut}(\mathbb{Z}_{p^n})}(\mathbb{Z}_{p^n})$ is non-planar.*
(e) *Let p be a prime number. Then $\Gamma_{\mathrm{Aut}(\mathbb{Z}_{p^\infty})}(\mathbb{Z}_{p^\infty})$ is non-planar.*

In the following propositions we determine some abelian groups G with planar $\Gamma_{\mathrm{Aut}(G)}(G)$.

Proposition 2.19. *Let G be a free abelian group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is planar if and only if $G\cong\mathbb{Z}$.*

Proposition 2.20. *Let G be a finitely generated abelian group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is planar if and only if G is isomorphic to one of the following groups; $\mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_p$ and $\mathbb{Z}_2\times\mathbb{Z}_p$, where p is a prime number.*

Corollary 2.21. *The following statements hold.*

(a) *Let G be a finite group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is a star graph if and only if $G\cong\mathbb{Z}_p$ for some p with $p\neq 2$.*
(b) *Let G be an infinite finitely generated group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is a star graph if and only if $G\cong\mathbb{Z}$.*

Proposition 2.22. *Let G be a divisible abelian group. Then $\Gamma_{\mathrm{Aut}(G)}(G)$ is planar if and only if G is isomorphic to $\mathbb{Q}$.*

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