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Nilpotent groups with respect to a certain automorphism

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Abstract. The aim of this article is to give a new extension of nilpotent groups in terms of an automorphism of the group which is an ordinary nilpotent group, whenever we put an identity automorphism. In this talk, we state some results on the ordinary nilpotent groups which are still valid for the extended nilpotent groups. Furthermore, some conditions under with the ordinary and extended nilpotent groups are coincidence also are stated.

1 Introduction

Nilpotent groups and the relation between them and commutators have been investigated in a lot of researches. Also the generalization of commutators has been studied by different authors. Hegarty extended the definition of commutators and stated important results related to this generalization (see [1]). The relation between commutators and nilpotent groups encourages us to introduce a generalization of nilpotent groups according to the extension of commutators. In this article, we define a commutator with respect to a certain automorphism and our aim is to introduce the notion of α-nilpotent for the given group G which is a generalization of nilponent property and based on the fixed automorphism $\alpha \in \mathrm{Aut}(G)$.

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2 Main results

First, we start with some definitions.

Definition 1. Let G be a group and $\alpha \in \mathrm{Aut(G)}$, element

$$x^{-1}y^{-1}xy^{\alpha}$$

is called α-commutator of x, y and is denoted by $[x, y]_{\alpha}$.

Also, we can define an α-commutator of weight n for all $n \geq 3$ inductively, as follows

$$[x_1, x_2, \cdots, x_n]_{\alpha} = [x_1, [x_2, \cdots, x_n]_{\alpha}]_{\alpha}.$$

Definition 2. Let G be a group and X_1, X_2 be non-empty subsets of G. We define α-commutator subgroup of X_1 and X_2 as the following

$$[X_1, X_2]_{\alpha} = \langle [x_1, x_2]_{\alpha} : x_1 \in X_1, x_2 \in X_2 \rangle.$$

The subgroup

$$Z^{\alpha}(G) = \{y \in G : [x, y]_{\alpha} = 1 \text{ for all } x \in G\},$$

is called the α-center of a group G. It is clear that if α is an identity automorphism then $Z^{\alpha}(G) = Z(G)$. It is not difficult to see that $Z^{\alpha}(G) \trianglelefteq G$ and is invariant under α. There is an important relation between center and α-center of a group in studing α-nilpotent groups, which is $Z^{\alpha}(G) = Z(G) \cap \mathrm{Fix}(\alpha)$, where $\mathrm{Fix}(\alpha) = \{\mathrm{g} \in \mathrm{G} : \mathrm{g}^{\alpha} = \mathrm{g}\}$.

Definition 3. Let N be a normal subgroup of G and $N^{\alpha} = N$. By $\overline{\alpha}$ we mean that an automorphism of G/N by rule $(xN)^{\overline{\alpha}} = x^{\alpha}N$.

To introduce an α-nilpotent group we need to define a central α-series.

Definition 4. A central α-series of G is a normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

such that $G_i^{\alpha} = G_i$ and $G_{i+1}/G_i \leq Z^{\overline{\alpha}}(G/G_i)$, $0 \leq i \leq n-1$.

Definition 5. An α-nilpotent group is a group which has at least a central α-series.

It is clear that every α-nilpotent group is nilpotent but the converse is not true. For example the cyclic group $\mathbb{Z}_p$ is not α-nilpotent unless α is an identity automorphism, where p is a prime number.
We are interested to provide some necessary and sufficient conditions for a group to be an α-nilpotent. So we define the following normal series. Put $Z_0^{\alpha}(G) = 1$, $Z_1^{\alpha}(G) = Z^{\alpha}(G)$ and define $Z^{\bar{\alpha}}(\frac{G}{Z_{i-1}^{\alpha}(G)}) = \frac{Z_i^{\alpha}(G)}{Z_{i-1}^{\alpha}(G)}$ for all $i \in \mathbb{N}$. Then the normal series

$$1 = Z_0^{\alpha}(G) \trianglelefteq Z_1^{\alpha}(G) \trianglelefteq Z_2^{\alpha}(G) \trianglelefteq \ldots$$

is said to be an upper central α-series. It is easy to see that $(Z_i^{\alpha}(G))^{\alpha} = Z_i^{\alpha}(G)$.
Now, some of results on the α-nilpotent groups are the following.

Theorem 1. A group G is α-nilpotent if and only if $Z_s^{\alpha}(G) = G$, for some positive integer s.

Theorem 2. If α is a central automorphism and $Z^{\alpha}(G) = Z(G)$, then G is α-nilpotent if and only if it is nilpotent.

Theorem 3. If G is an α-nilpotent group, $N \trianglelefteq G$ and $N^{\alpha} = N$, then $N \cap Z^{\alpha}(G) \neq 1$, in particular $Z^{\alpha}(G) \neq 1$.

Corollary 1. If α is a fixed point free automorphism of group G then G is not α-nilpotent.

Theorem 4. Let $\alpha \in \mathrm{Aut(G)}$, $1 \neq H \leq G$, $H^{\alpha} = H$ and $H = [G, H]_{\alpha}$, then G is not α-nilpotent.

Theorem 5. Assume that M and N are two normal subgroups of group G which are both invariant under α and α-nilpotent. Then MN is an α-nilpotent group.

References

[1] P. Hegarty, *The absolute centre of a group*, J. of Algebra. 169 (1994), no. 3, 929--935.