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Ministry of Science,
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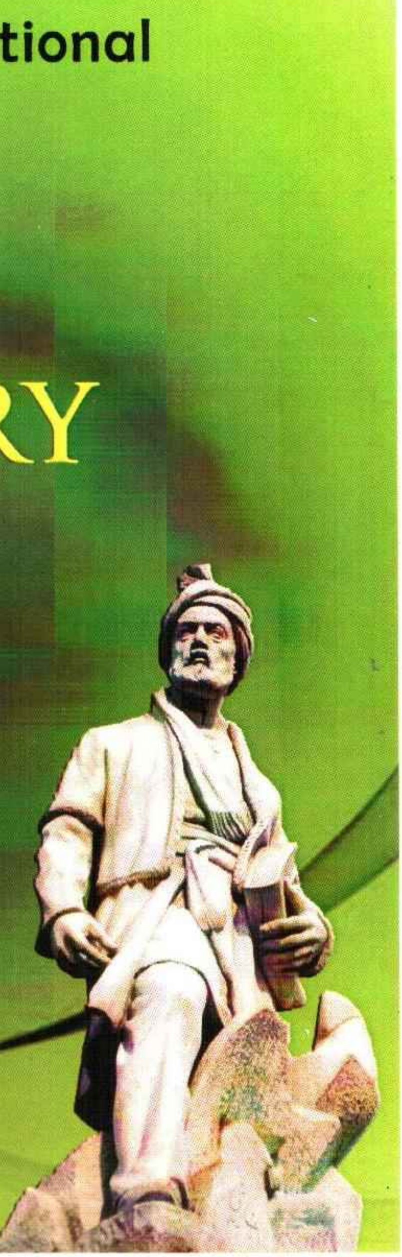
Ferdowsi University
of Mashhad

Extended Abstracts

of The 3rd Biennial International

GROUP THEORY

Conference



28-31 January 2015
Faculty of Mathematical Sciences
Ferdowsi University of Mashhad
Mashhad, Iran

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3[rd] Biannual International Group Theory Conference (3BIGTC)
January. 28-31(2015), pp. 143-146
Ferdowsi University of Mashhad, I. R. Iran

Some results on the generalized non-commuting graph of a finite group

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Abstract. In this talk, we define the generalized non-commuting graph denoted by $\Gamma_{(H,K)}$, where H and K are two subgroups of a non-ablian group G. Take $(H \cup K) \setminus (C_H(K) \cup C_K(H))$ as the vertex set of the graph and two distinct vertices x and y join by an edge, whenever x or y in H and $[x, y] \neq 1$. We obtain diameter and girth of this graph and discuss about dominating set and planarity of $\Gamma_{(H,K)}$. Moreover, we try to find a connection between $\Gamma_{(H,K)}$ and the relative commutativity degree of two subgroups denoted by $d(H, K)$. Furthermore, we prove that if $\Gamma_{(H_1,G)} \cong \Gamma_{(H_2,G)}$ then $\Gamma_{H_1} \cong \Gamma_{H_2}$.

1 Introduction

A simple graph Γ_G is associated to a group G, whose vertex set is $G \setminus Z(G)$ and the edge set is all pairs (x, y), where x and y are distinct non-central elements such that $[x, y] = x^{-1}y^{-1}xy \neq 1$. The non-commuting graph of G was introduced by Erdös. In the next section, we introduce the generalized non-commuting graph $\Gamma_{(H,K)}$. We state some of the basic graph theoretical properties of $\Gamma_{(H,K)}$ which are mostly new or a generalization of some results in [3]. For instance determining diameter, dominating set, domination number and planarity of the graph. The third section is managed to state a connection between the generalized non-commuting graph and the commutativity degree. We will

2010 Mathematical Subject Classification. 05C25
Keywords. Commutativity degree, relative commutativity degree, non-commuting graph, relative non-commuting graph
* Speaker

present a formula for the number of edges of $\Gamma_{(H,K)}$ in terms of $d(H)$ and $d(H,K)$. Moreover, we observe that the generalized non-commuting star graph exists, although in [3] we see there is no relative non-commuting star graph. We also present some conditions under which we have generalized non-commuting complete bipartite and bipartite graph. In the last section, we explain some properties of $\Gamma_{(H,K)}$, where $K = G$.

2 The generalized non-commuting graphs

In this section, we define the generalized non-commuting graph for any non-abelian group G and subgroups H, K.

Definition 1. Let H and K be subgroups of non-abelian group G. We associate a graph $\Gamma_{(H,K)}$ to the subgroups H and K as follows. Take $(H \cup K) \setminus (C_H(K) \cup C_K(H))$ as the vertices of the graph and two distinct vertices x and y adjacent, whenever x or y in H and $[x, y] \neq 1$. We call it as the generalized non-commuting graph of subgroups H and K of G.

Proposition 1. Suppose $\Gamma_{(H,K)}$ is the generalized non-commuting graph of the non-abelian group G and its subgroups H and K.

(i) If $x \in H \setminus K$, then $\deg(x) = |H \cup K| - |C_H(x) \cup C_K(x) \cup C_H(K)|$.

(ii) $\deg(x) = |H \cup K| - |C_H(x) \cup C_K(x)|$ for $x \in H \cap K$.

(iii) If $x \in K \setminus H$, then $\deg(x) = |H| - |C_H(x) \cup C_K(H)|$.

Theorem 1. For non-abelian group G and its subgroups H, K with trivial center, $\text{diam}(\Gamma_{(H,K)}) \leq 3$. Moreover, $\text{girth}(\Gamma_{(H,K)}) \leq 4$.

Proposition 2. Let H, K be subgroups of non-abelian group G and $S \subseteq V(\Gamma_{(H,K)})$. Then S is a dominating set for $\Gamma_{(H,K)}$ if and only if $C_K(S) \cup C_H(S) \subseteq C_K(H) \cup C_H(K) \cup S$.

In graph theory an independent set is a set of vertices in a graph, no two of which are adjacent. It is clear that $V(\Gamma_{(H,K)}) \cap K$ is an independent set for $\Gamma_{(H,K)}$.

Now, we deal with the planarity of $\Gamma_{(H,K)}$. As we have seen in [2], the non-commuting graph Γ_G is planar whenever G is isomorphic to S_3 or D_8 or Q_8. Since $\Gamma_{(H,K)}$ is the subgraph of Γ_G, then it is obvious that $\Gamma_{(H,K)}$ is planar if $G \cong S_3$ or D_8 or Q_8. Furthermore, one can easily check that if $H \neq S_3$ or D_8 or Q_8, then $\Gamma_{H,K}$ is not planar. Since Γ_H is a subgraph of $\Gamma_{(H,K)}$. Indeed, we can see that $\Gamma_{(S_3,S_4)}$ is not a planar graph because we can obtain complete graph K_5 with vertex set $\{(1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (2\ 3\ 4)\}$ which is a subgraph of $\Gamma_{(S_3,S_4)}$. In general, $\Gamma_{(S_3,S_n)}$ is not planar for every $n \geq 4$

3 The generalized non-commuting graphs and $d(H, K)$

For any finite group G, the commutativity degree of G, denoted by $d(G)$ is the probability that two randomly chosen elements of G commute with each other [5]. It can be defined as the following ratio:

$$d(G) = \frac{1}{|G|^2}|\{(x, y) \in G \times G : [x, y] = 1\}|.$$

Similarly, if H and K are two subgroups of G, then the generalized commutativity degree of H, K in G is defined as follows

$$d(H,K) = \frac{1}{|H||K|}|\{(h,k) \in H \times K : [h,k] = 1\}|.$$

It is clear that if G is abelian or one of H or K is a central subgroup, then $d(H,K) = 1$ (see [2]). In this section, we present a formula for the number of edges of the generalized non-commuting graph $\Gamma_{(H,K)}$. Consequently we will give an upper bound for $|E(\Gamma_{(H,K)})|$.

Proposition 3. Let H, K be subgroups of non-abelian group G. Then the number of edges for the generalized non-commuting graph is obtained by,

$$|E(\Gamma_{(H,K)})| = |H||K|(1 - d(H,K)) + \frac{|H|^2}{2}(1 - d(H)) - \frac{|H \cap K|^2}{2}(1 - d(H \cap K)). \quad (1)$$

Example 1. In this example we compute the number of edges for some certain groups.

(i) Suppose $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1}\rangle$ is the dihedral group of order 8, $H = \langle ab\rangle$ and $K = \langle b\rangle$ are two subgroups of D_8. Obviously $V(\Gamma_{(H,K)}) = \{ab, b\}$, $d(H) = 1$, $d(H,K) = 3/4$, $|E(\Gamma_{(H,K)})| = 1$ and $\Gamma_{(H,K)}$ is K_2.

(ii) Let $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ be the symmetric group of order 6, $H = \{e, (1\ 2)\}$ and $K = \{e, (1\ 3)\}$ be subgroups of S_3. It is clear that again $\Gamma_{(H,K)} = K_2$.

Corollary 1. Let $\Gamma_{(H,K)}$ be a generalized non-commuting graph. Then

$$|E(\Gamma_{(H,K)})| \leq |H|(|K| + \frac{3}{16}|H| - 1) - |C_H(K)|(|K| - 1).$$

Now, we recall that the star graph as a tree on n vertexes in which one vertices of degree $n-1$ and the others of degree 1.

Example 2. Let $D_{2n} = \langle a, b; a^n = b^2 = 1, a^b = a^{-1}\rangle$ be the dihedral group of order $2n$, $H = \langle a\rangle$ and $K = \langle b\rangle$. Then $\Gamma_{(H,K)}$ is a star graph.
If n is an even number, then $V(\Gamma_{(H,K)}) = n - 1$, $\deg(a^i) = 1, i \neq \frac{n}{2}, 1 \leq i \leq n-1$ and $\deg(b) = n-2$. Therefore $\Gamma_{(H,K)}$ is a star graph. Moreover, $d(H,K) = (n+2)/2n$ and by Proposition 3 or by the fact $\Gamma_{(H,K)}$ is a star graph follows $|E(\Gamma_{(H,K)})| = n-2$. If n is an odd number, then $V(\Gamma_{(H,K)}) = n$. Furthermore, $\deg(a^i) = 1$, $1 \leq i \leq n-1$ and $\deg(b) = n-1$. Hence $\Gamma_{(H,K)}$ is a star graph. We deduce $d(H,K) = (n+1)/2n$ and so $|E(\Gamma_{(H,K)})| = n-1$.

As a consequence of the above corollary, one can see that $K = G$ then $\Gamma_{(H,G)}$ is empty graph if and only if H is abelian subgroup of G, with this properties we can prove the following theorem.

Theorem 2. Let H_1 and H_2 be subgroups of non-abelian group G such that $\Gamma_{(H_1,G)} \cong \Gamma_{(H_2,G)}$. Then $\Gamma_{H_1} \cong \Gamma_{H_2}$.

Theorem 3. Let H be an on-abelian subgroup of G such that $\Gamma_{(H,G)} \cong \Gamma_S$, for some non-abelian finite simple group S. Then $H = G \cong S$.

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