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Gegenbauer spectral method for time-fractional convection–diffusion equations with variable coefficients

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In this paper, we study the numerical solution to time-fractional partial differential equations with variable coefficients that involve temporal Caputo derivative. A spectral method based on Gegenbauer polynomials is taken for approximating the solution of the given time-fractional partial differential equation in time and a collocation method in space. The suggested method reduces this type of equation to the solution of a linear algebraic system. Finally, some numerical examples are presented to illustrate the efficiency and accuracy of the proposed method. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: spectral methods; Gegenbauer polynomials; Caputo derivative; time-fractional diffusion equations

1. Introduction

Recently, fractional differential operators are indisputably found to play a fundamental role in the modeling of a considerable number of phenomena. Because of the nonlocal property of fractional derivative, they can utilize for modeling of memory-dependent phenomena and complex media such as porous media and anomalous diffusion [1–4]. They have been used in modeling turbulent flow [5, 6], chaotic dynamics of classical conservation systems [7], and even finance [8, 9] (see [1] for more information).

Also, fractional calculus emerged as an important and efficient tool for the study of dynamical systems where classical methods reveal strong limitations. In [10], a fractional advection–dispersion equation is derived by extending Fick's first law from isotropic media to heterogeneous media and is particularly suitable for description of the highly skewed and heavy-tailed dispersion processes observed in rivers and other natural media.

In the last decade or so, extensive research has been carried out on the development of numerical methods for fractional partial differential equations, including finite difference method [11–13], finite element methods [14, 15], and spectral methods [16]. In [17–19], the authors used their proposed numerical schemes to solve Bagley–Torvik equation and other ordinary fractional differential equations. Gegenbauer polynomials have received much attention for their fundamental properties as well as for their use in applied mathematics. The described functions are a key ingredient to the implementation of spectral and pseudo-spectral methods to solve certain types of differential equations. Gegenbauer polynomials are a convenient basis for polynomial approximations because they are eigenfunctions of corresponding differential operators. For numerical methods, it is usually as convenient as efficient to convert between representations of a polynomial by expansion coefficients or by function values, respectively.

Spectral approximations, such as the Fourier approximation based upon trigonometric polynomials for periodic problems, and the Chebyshev, Legendre, or the general Gegenbauer approximation based upon polynomials for nonperiodic problems are exponentially accurate for analytic functions [20–23]. In [24], the authors provided collocation method for natural convection heat transfer equations embedded in porous medium by using the rational Gegenbauer polynomials. Gottlieb and Shu in [25] used the Gegenbuaer polynomials to construct an exponentially convergent approximation to overcome Gibbs phenomenon. Micheli and Viano in [26] presented a simple and fast algorithm for the computation of the Gegenbauer coefficients of the expansion of a function in Gegenbauer polynomials, which is known to be very useful in the development of spectral methods for the numerical solution of ordinary and partial differential equations. Kadem *et al*. in [27] applied the Chebyshev polynomials expansion method to find both an analytical solution of the fractional transport equation in the one-dimensional plane geometry and its numerical approximations. Parand *et al*. [28] have

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studied a collocation method using a weighted orthogonal system based on the rational Gegenbauer function for solving numerically a laminar boundary layer equation – a generalization of the well-known Blasius equation – on the halfline. Shamsi and Dehghan [29] proposed a Legendre pseudospectral method for solving approximately an inverse problem of determining an unknown control parameter, which is the coefficient of the solution in a diffusion equation in a three-dimensional region. In the present paper, we consider the following time-fractional convection–diffusion equation with variable coefficients:

$$D_t^{\alpha}u(x,t)+a(x)\frac{\partial u(x,t)}{\partial x}+b(x)\frac{\partial^2 u(x,t)}{\partial x^2}=f(x,t),\quad 0<x<1,\quad 0<t\le T, \tag{1.1}$$

with initial condition

$$u(x,0)=g(x),\quad 0<x<1, \tag{1.2}$$

and boundary conditions

$$u(0,t)=h_1(t),\quad u(1,t)=h_2(t),\quad 0<t\le T, \tag{1.3}$$

where $a(x), b(x) \neq 0$ are continues functions and $0 < \alpha \le 1$. Here, the time-fractional derivative is defined as the Caputo fractional derivative. Saadatmandi *et al.* [30] used the Sinc–Legendre collocation method for the solution of Equation 1.1 with homogeneous boundary conditions, and Uddin and Haq [31] applied radial basis functions for solving this problem with constant coefficients. In [32] and [33], numerical methods based on finite difference and finite element, respectively, are utilized for the cases $a(x) = 0$ and $b(x) = -1$. The author of [34] developed implicit unconditionally stable numerical methods to solve (1.1) on the condition that $a(x) = 0, b(x) = -1$ and $f(x,t) = 0$. Sakar and Erdogan [35] introduced homotopy analysis method and Adomians decomposition method (ADM) for solving time-fractional Fornberg–Whitham equation. Also in [36], the approximate analytical solutions to the nonlinear Fornberg–Whitham equation with fractional time derivative has been obtained by using a reliable algorithm like the variational iteration method (VIM). Authors of [37] investigated the homotopy analysis method to solve nonlinear fractional partial differential equations such as fractional KdV, K(2, 2), Burgers, BBM-Burgers, cubic Boussinesq, coupled KdV, and Boussinesq-likeB(m,n)equations. Authors of [38] investigated the high-order and unconditionally stable difference scheme for the solution of modified anomalous fractional sub-diffusion equation by the inclusion of a secondary fractional time derivative acting on a diffusion operator. In [39], a numerical method developed for solving the fractional Fisher's equation by the quadratic spline functions. A truncated Legendre series together with the Legendre operational matrix of fractional derivatives are used for numerical integration of fractional differential equations is introduced in [40]. A numerical scheme for solving the fractional convection–diffusion equation presented in [41] that is applied biorthogonal multiwavelet basis to construct operational matrix of fractional derivative.

The main advantage of spectral methods lies in their accuracy for given number of unknowns. Spectral methods are a nice and powerful approach for numerical solution of fractional partial differential equations, due to the being global of fractional operator and the being global of basis functions of the method. When solving a fractional partial differential equation to high accuracy and if the data defining the problem are smooth, then spectral methods are usually the best tool. In the present paper, we extend the application of spectral methods with Gegenbauer polynomials for solution of fractional partial differential equations, which we refer to it as Gegenbauer spectral method (GSM). Issues regarding the convergence of the GSM will be addressed in Section 2. The remaining part of this paper is organized as follows. In Section 2, we introduce the Caputo type of fractional derivative and the Gegenbauer polynomials and discuss their properties, in particular exponential rate of convergence of the Gegenbauer approximation of a function. We continue by introducing collocation spectral method and use it to solve the problem that is developed in Section 3. In Section 4, numerical results will show the efficiency of the GSM approach.

2. Preliminaries

2.1. The fractional derivative in the Caputo sense

Among different approaches to the generalization of the notation of differentiation in fractional sense (Grünwald-Letnikov, Riemann-Liouville, etc.), we pay attention to the approach suggested by Caputo [42], because of its possible usefulness for the formulation and solution of applied problems and their transparency. Indeed, the Caputo's approach allows the formulation of initial conditions for initial-value problems for fractional-order differential equations in a form involving only the limit values of integer-order derivatives at the initial time $t = 0$. We describe some useful definitions and mathematical preliminaries of the fractional calculus theory in the Caputo sense, which is required for our development [43].

Definition 1
A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exist a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2
Caputo type of the fractional order derivative of $f \in C_\mu^n, \mu \ge -1$ is defined as

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(\zeta)}{(x-\zeta)^{\alpha+1-n}}d\zeta,\quad n-1<\alpha<n,\quad n\in\mathbb{N},$$

where $\alpha > 0$ is the order of the derivative, $\Gamma(\cdot)$ is the Euler's Gamma function, $n = [\alpha] + 1$, with $[\alpha]$ denoting the integer part of α.

Diethelm in [44] discussed the relationship between the Caputo fractional derivative and Riemann–Liouville approach. Similar to integer-order differentiation, Caputo fractional differentiation is a linear operator,

$$D^{\alpha}(c_1 f_1(x) + c_2 f_2(x)) = c_1 D^{\alpha} f_1(x) + c_2 D^{\alpha} f_2(x),$$

where c_1 and c_2 are constants. $D_t^{\alpha} u(x,t)$ in the Equation 1.1 is the Caputo fractional derivative of order $0 < \alpha \leq 1$ in time and is defined as

$$D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\zeta)}{\partial \zeta} \frac{d\zeta}{(t-\zeta)^{\alpha}}.$$

An important property of fractional derivatives is that when the order goes to an integer, the fractional derivative approaches to the integer-order derivative. For $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order. For the Caputo derivative, we have [43]

$$D^{\alpha} C = 0, \quad (C \text{ is a constant})$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \alpha \rfloor \end{cases}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α. Also $\mathbb{N}$ and $\mathbb{N}_0$ stand for $\{1, 2, \cdots\}$ and $\{0, 1, 2, \cdots\}$, respectively.

2.2. *Gegenbauer polynomials*

In this section, we recall some useful results about the Gegenbauer polynomials $C_n^{\lambda}(z)$ of degree n and associated with the real parameter λ that are a family of orthogonal polynomials and possess many applications [45].

Definition 3
The Gegenbauer polynomial $C_n^{\lambda}(z)$ is defined for $\lambda > -\frac{1}{2}, \lambda \neq 0$ by the Rodrigues' formula

$$C_n^{\lambda}(z) = \left(-\frac{1}{2}\right)^n \frac{\Gamma\left(\lambda + \frac{1}{2}\right) \Gamma(n+2\lambda)}{n! \Gamma(2\lambda) \Gamma\left(n + \lambda + \frac{1}{2}\right)} \left(1 - z^2\right)^{\frac{1}{2} - \lambda} \frac{d^n}{dz^n} \left(\left(1 - z^2\right)^{n + \lambda - \frac{1}{2}}\right). \tag{2.1}$$

It turns out that the Gegenbauer polynomials, $C_n^{\lambda}(z)$, appear as the eigensolutions to the following singular Sturm–Liouville problem restricted to the finite domain $[-1, 1]$ [45,46],

$$\frac{d}{dz}\left(\left(1 - z^2\right)^{\lambda + \frac{1}{2}} \frac{dC_n^{\lambda}(z)}{dz}\right) + n(n+2\lambda)\left(1 - z^2\right)^{\lambda - \frac{1}{2}} C_n^{\lambda}(z) = 0,$$

that is, they are essentially the symmetric Jacobi polynomials, $P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(z)$, although normalized differently because

$$C_n^{\lambda}(z) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right) \Gamma(n+2\lambda)}{\Gamma(2\lambda) \Gamma\left(n + \lambda + \frac{1}{2}\right)} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(z).$$

There exist useful relations between Legendre polynomials $L_n(z)$ and Chebyshev polynomials of the first kind and second kind, $T_n(z), U_n(z)$, respectively, and the Gegenbauer polynomials $C_n^{\lambda}(z)$ as [45],

$$L_n(z) = C_n^{1/2}(z), \quad U_n(z) = C_n^1(z),$$

and

$$T_n(z) = \frac{n}{2} \lim_{\lambda \to 0} \lambda^{-1} C_n^{\lambda}(z), \quad n \geq 1.$$

The Gegenbauer polynomials satisfy the following orthogonality relation [47]

$$\int_{-1}^{1} C_m^{(\lambda)}(z) C_n^{(\lambda)}(z) w^{\lambda}(z)\, dz = \gamma_n^{\lambda} \delta_{mn},$$

where the even function $w^{\lambda}(z) = \left(1 - z^2\right)^{\lambda - \frac{1}{2}}$ is the weight function for the Gegenbauer polynomials and

$$\gamma_n^{\lambda} = \frac{\pi 2^{1-2\lambda} \Gamma(n+2\lambda)}{\Gamma(n+1)(n+\lambda)[\Gamma(\lambda)]^2},$$

is the normalization factor and δ_{mn} is the Kronecker delta function. The Gegenbauer polynomials can be generated through the following recursion formula:

$$(n+1)C_{n+1}^{\lambda}(z) = 2z(\lambda+n)C_n^{\lambda}(z) - (2\lambda+n-1)C_{n-1}^{\lambda}(z), \quad n \geq 1,$$

where the starting terms are

$$C_0^{\lambda}(z) = 1, \quad C_1^{\lambda}(z) = 2\lambda z.$$

An explicit formula for the Gegenbauer polynomials is given by the following finite series:

$$C_n^{\lambda}(z) = \sum_{m=0}^{n} \frac{(-1)^{n-m}(2\lambda)_{n+m}}{m!(n-m)!\left(\lambda+\frac{1}{2}\right)_m} \left(\frac{z+1}{2}\right)^m,$$

where $(\theta)_n$ is the Pochhammer symbol, means $\theta(\theta+1)(\theta+2)\cdots(\theta+n-1)$ for $n \geq 1$ and $(\theta)_0 = 1$.

The first four Gegenbauer polynomials of degree n and associated with the parameter λ can be read as follows:

$$\begin{aligned} C_0^{\lambda}(z) &= 1, \\ C_1^{\lambda}(z) &= 2\lambda z, \\ C_2^{\lambda}(z) &= \frac{(\lambda)_2}{2!}(2z)^2 - \lambda, \\ C_3^{\lambda}(z) &= \frac{(\lambda)_3}{3!}(2z)^3 - 2(\lambda)_2 z. \end{aligned}$$

The formal Gegenbauer expansion of a function $f(x)$, defined in the interval $[-1,1]$, reads [46]

$$f(x) = \sum_{n=0}^{\infty} f_n^{\lambda} C_n^{(\lambda)}(x), \tag{2.2}$$

where the Gegenbauer coefficients are given by

$$f_n^{\lambda} = \frac{1}{\gamma_n^{\lambda}} \int_{-1}^{1} f(x) C_n^{(\lambda)}(x) w^{\lambda}(x)\, dx.$$

Spectral convergence of Gegenbauer expansion (2.2) is obtained for sufficiently smooth functions $f(x)$. In fact, let $f(x)$ be analytic on the closed segment $[-1,1]$. Then the expansion (2.2) of $f(x)$ is convergent (in the complex plane $\mathbb{C}$) within the greatest ellipse with foci at ± 1, in which $f(x)$ is regular (stated for the more general family of Jacobi polynomials in ([45], Theorem 9.1.1). In particular, there exists some constant $0 < \rho_0 < 1$ such that the function $f(x)$ has a unique analytic extension onto the elliptical region:

$$D_{\rho} := \left\{ z \in \mathbb{C} : 2z = \rho e^{i\theta} + \left(\rho e^{i\theta}\right)^{-1}, 0 \leq \theta \leq 2\pi, 0 < \rho \leq \rho_0 \right\}. \tag{2.3}$$

Consider the $(N+1)$-term truncated Gegenbauer expansion of $f(x)$

$$f_{N+1}^{(\lambda)}(x) := \sum_{n=0}^{N} f_n^{\lambda} C_n^{(\lambda)}(x),$$

then, the series (2.2) converges at exponential rate [48]

$$\max_{x \in [-1,1]} |f(x) - f_{N+1}^{(\lambda)}(x)| \leq C \left(\frac{(N+\lambda)(N+2\lambda)^{N+2\lambda}}{N^{N-1}(2\lambda)^{2\lambda}} \right)^{\frac{1}{2}} \rho^N,$$

ρ being defined in (2.3) and C is a generic constant. This very fast rate of convergence motivates the use of expansions as (2.2) in numerical computations, in particular in spectral methods for fractional partial differential equations. For practical use of Gegenbauer polynomials on the time interval of interest $t \in [0,T]$, it is necessary to shift the defining domain by the following variable substitution:

$$z = \frac{2}{T}t - 1, \quad 0 \leq t \leq T.$$

Let the shifted Gegenbauer polynomials $C_n^{\lambda}\left(\frac{2}{T}t - 1\right)$ be denoted by $G_n^{\lambda}(t)$, then $G_n^{\lambda}(t)$ can be obtained by

$$G_n^{\lambda}(t) = \sum_{m=0}^{n} \frac{1}{T^m} \frac{(-1)^{n-m}(2\lambda)_{n+m}}{m!(n-m)!\left(\lambda+\frac{1}{2}\right)_m} t^m, \quad 0 \leq t \leq T. \tag{2.4}$$

Note that the values $G_n^{\lambda}(0) = (-1)^n \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}$ and $G_n^{\lambda}(T) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}$ are fulfilled at the endpoints. We determine the Caputo fractional derivative of $G_n^{\lambda}(t)$ in the following theorem.

Theorem 1

Let $G_n^\lambda(t), 0 \le t \le T$ denote the shifted Gegenbauer polynomials of degree n and associated with the parameter λ, and also, suppose $\alpha > 0$ then the derivative of order α in the Caputo sense for $G_n^\lambda(t)$ is

$$D^\alpha\left(G_n^\lambda(t)\right) = \sum_{m=\lceil\alpha\rceil}^{n} b_{n,m}^{(\alpha,\lambda)} t^{m-\alpha},$$

where

$$b_{n,m}^{(\alpha,\lambda)} = \frac{(-1)^{n-m}(2\lambda)_{n+m}}{T^m(n-m)!\left(\lambda+\frac{1}{2}\right)_m \Gamma(m+1-\alpha)}.$$

Proof

Taking the fractional derivative of order α on the definition of $G_n^\lambda(t)$ in finite series (2.4), we have

$$\begin{aligned} D^\alpha\left(G_n^\lambda(t)\right) &= \sum_{m=0}^{n} \frac{1}{T^m} \frac{(-1)^{n-m}(2\lambda)_{n+m}}{m!(n-m)!\left(\lambda+\frac{1}{2}\right)_m} D^\alpha(t^m) \\ &= \sum_{m=\lceil\alpha\rceil}^{n} \frac{1}{T^m} \frac{(-1)^{n-m}(2\lambda)_{n+m}\Gamma(m+1)}{m!(n-m)!\left(\lambda+\frac{1}{2}\right)_m \Gamma(m+1-\alpha)} t^{m-\alpha} \\ &= \sum_{m=\lceil\alpha\rceil}^{n} \frac{1}{T^m} \frac{(-1)^{n-m}(2\lambda)_{n+m}}{(n-m)!\left(\lambda+\frac{1}{2}\right)_m \Gamma(m+1-\alpha)} t^{m-\alpha} \end{aligned}$$

We could easily conclude the proof by noting that $D^\alpha(t^m) = 0$ for $m = 0, 1, \dots, \lceil\alpha\rceil - 1$ and $\alpha > 0$. □

We use a stable formula for computing coefficients $b_{n,m}^{(\alpha,\lambda)}$ when $0 < \alpha \le 1$:

$$b_{n,m+1}^{(\alpha,\lambda)} = \frac{-(2\lambda+n+m)(n-m)}{T(\lambda+\frac{1}{2}+m)(m+1-\alpha)} b_{n,m}^{(\alpha,\lambda)}, \qquad m = 1, 2, \dots,$$

where we start with the following equation:

$$b_{n,1}^{(\alpha,\lambda)} = \frac{(-1)^{n-1}(2\lambda)_{n+1}}{T(n-1)!\left(\lambda+\frac{1}{2}\right)\Gamma(2-\alpha)}.$$

3. Solving the problem with spectral method

Let $x_0, x_1, \dots, x_M$ be $M+1$ distinct nodes in $[0,1]$, and $\varphi_m(x), m = 0, 1, \dots, M$ be the m-th Lagrange interpolation polynomials based on the aforementioned nodes, which are expressed as

$$\varphi_m(x) = \prod_{j=0, j\neq m}^{M} \frac{x-x_j}{x_m-x_j}, \quad m = 0, 1, \dots, M.$$

By noting that $\varphi_m''(x)$ is a polynomial of degree $m-2$, we have

$$\varphi_m''(x) = \sum_{k=0}^{M} \varphi_m''(x_k)\varphi_k(x),$$

or in the matrix form

$$\Phi_{xx} = H^{(2)}\Phi,$$

while the vector Φ is given by $\Phi = [\varphi_0(x) \quad \varphi_1(x) \quad \cdots \varphi_M(x)]^T$, and Φ_{xx} refers to the second-order derivative of the Φ, and $H^{(2)} = \left[h_{mk}^{(2)}\right] = \left[\varphi_m''(x_k)\right]$ is the second-order derivative matrix. For calculating the entries of $H^{(2)}$, we need to compute the entries of differentiation matrix $H = [h_{mk}] = \left[\varphi_m'(x_k)\right]$. According to [49]

$$h_{mk} = \begin{cases} \frac{d_k}{d_m}(x_k - x_m)^{-1}, & m \neq k, \\ -\sum\limits_{l=0, l\neq k}^{M} h_{lk}, & m = k, \end{cases}$$

where $d_j = \prod_{l=0,l\neq j}^{M}(x_j - x_l)$. The computation of ration $\frac{d_m}{d_k}$ in h_{mk} may cause round-off error; therefore, to avoid this problem as mentioned in [49], we can compute them as follows:

$$q_j = \sum_{l=0,l\neq j}^{M} \ln(|x_j - x_l|), \qquad \frac{d_k}{d_m} = (-1)^{k+m} e^{q_k - q_m}.$$

Now, the entries of matrix $H^{(2)}$ can be computed recursively by the entries of H as follows:

$$h_{mk}^{(2)} = \begin{cases} 2(h_{kk}h_{mk} - (x_k - x_m)^{-1}h_{mk}), & m \neq k, \\ -\sum\limits_{l=0,l\neq k}^{M} h_{lk}^{(2)}, & m = k. \end{cases}$$

To solve the problem (1.1)-(1.3), we approximate $u(x,t)$ by $(N+1)$ shifted Gegenbauer polynomials truncated as $(N+1)$-term truncated Gegenbauer expansion of (2.2) and $(M+1)$ Lagrange polynomials in the following form:

$$u_{M,N}^{(\lambda)}(x,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(x)G_n^{\lambda}(t). \tag{3.1}$$

We derive in the following lemma an appropriate form for the fractional and classical partial derivative of the approximate solution (3.1).

Lemma 3.1
Let $\alpha > 0$ and x_k be the spatial collocation points. Then the following relations hold:

$$D_t^{\alpha}\left(u_{M,N}^{(\lambda)}(x_k,t)\right) = \sum_{n=\lceil\alpha\rceil}^{N}\sum_{r=\lceil\alpha\rceil}^{n} c_{kn} b_{n,r}^{(\lambda,\alpha)} t^{r-\alpha},$$

$$\frac{\partial u_{M,N}^{(\lambda)}(x_k,t)}{\partial x} = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} h_{mk} G_n^{\lambda}(t),$$

$$\frac{\partial^2 u_{M,N}^{(\lambda)}(x_k,t)}{\partial x^2} = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} h_{mk}^{(2)} G_n^{\lambda}(t).$$

Proof

$$\begin{aligned} D_t^{\alpha}\left(u_{M,N}^{(\lambda)}(x_k,t)\right) &= \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(x_k) D^{\alpha} G_n^{\lambda}(t) \\ &= \sum_{m=0}^{M}\sum_{n=\lceil\alpha\rceil}^{N}\sum_{r=\lceil\alpha\rceil}^{n} c_{mn} b_{n,r}^{(\lambda,\alpha)} \varphi_m(x_k) t^{r-\alpha} \\ &= \sum_{m=0}^{M}\sum_{n=\lceil\alpha\rceil}^{N}\sum_{r=\lceil\alpha\rceil}^{n} c_{mn} b_{n,r}^{(\lambda,\alpha)} \delta_{m,k} t^{r-\alpha} \\ &= \sum_{n=\lceil\alpha\rceil}^{N}\sum_{r=\lceil\alpha\rceil}^{n} c_{kn} b_{n,r}^{(\lambda,\alpha)} t^{r-\alpha}, \end{aligned}$$

$$\begin{aligned} \frac{\partial u_{M,N}^{(\lambda)}(x_k,t)}{\partial x} &= \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m'(x_k) G_n^{\lambda}(t) \\ &= \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} h_{mk} G_n^{\lambda}(t), \\ \frac{\partial^2 u_{M,N}^{(\lambda)}(x_k,t)}{\partial x^2} &= \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m''(x_k) G_n^{\lambda}(t) \\ &= \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} h_{mk}^{(2)} G_n^{\lambda}(t). \square \end{aligned}$$

Now, we are ready to solve (1.1)-(1.3). By substituting Equation (3.1) in Equation (1.1), we obtain

$$D_t^{\alpha} u_{M,N}^{(\lambda)}(x,t) + a(x)\frac{\partial u_{M,N}^{(\lambda)}(x,t)}{\partial x} + b(x)\frac{\partial^2 u_{M,N}^{(\lambda)}(x,t)}{\partial x^2} = f(x,t). \tag{3.2}$$

We collocate Equation (3.2) in certain nodes right now. A good choice for these nodes is the Chebyshev–Gauss nodes associated with interval [0, 1] for spatial collocation, that is,

$$x_k = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{(2k+1)\pi}{2(M+1)}\right), \quad k = 0,1,2,\dots,M.$$

For suitable collocation points in time, we use the shifted Gegenbauer roots $\tau_l, l = 0,1,2,\dots,N$ of $G_{N+1}^{\lambda}(t)$ associated with interval $[0,T]$. So using Lemma 3.1, we have

$$\begin{aligned}\sum_{n=\lceil\alpha\rceil}^{N}\sum_{r=\lceil\alpha\rceil}^{n} c_{kn} b_{n,r}^{(\lambda,\alpha)} \tau_l^{r-\alpha} + a(x_k)\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} h_{mk} G_n^{\lambda}(\tau_l) + b(x_k)\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} h_{mk}^{(2)} G_n^{\lambda}(\tau_l) \\ = f(x_k,\tau_l), \quad k = 1,\dots,M-1, \quad l = 0,1,\dots,N-1.\end{aligned} \tag{3.3}$$

Also by applying Equation (3.1) in the initial and boundary conditions (1.2) and (1.3), respectively, one can write

$$\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(x) G_n^{\lambda}(0) = g(x), \tag{3.4}$$

$$\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(0) G_n^{\lambda}(t) = h_1(t), \tag{3.5}$$

$$\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(1) G_n^{\lambda}(t) = h_2(t). \tag{3.6}$$

Collocating Equation (3.4) in $M-1$ points x_k and Equations (3.5) and (3.6) in N points τ_l, we have

$$\sum_{n=0}^{N} c_{kn}(-1)^n \frac{(2\lambda)_n}{n!} = g(x_k), \quad k = 0,\dots,M, \tag{3.7}$$

$$\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(0) G_n^{\lambda}(\tau_l) = h_1(\tau_l), \quad l = 0,1,\dots,N-1, \tag{3.8}$$

$$\sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}\varphi_m(1) G_n^{\lambda}(\tau_l) = h_2(\tau_l), \quad l = 0,1,\dots,N-1. \tag{3.9}$$

The number of the unknown coefficients c_{mn} is equal to $(N+1)(M+1)$ and can be obtained from Equations (3.3), (3.7)–(3.9). Consequently, $u_{M,N}^{(\lambda)}(x,t)$ given in Equation (3.1) can be calculated.

4. Numerical illustrations

Example 4.1
Consider the following time-fractional diffusion equation

$$D_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = 2\left(\frac{1}{\Gamma(3-\alpha)} t^{2-\alpha} - 1\right), \quad 0 < x < 1, \quad 0 < t \le 1, \quad 0 < \alpha \le 1,$$

where

$$u(0,t) = t^2 \quad u(1,t) = 1 + t^2 \quad u(x,0) = x^2$$

It is easy to see that the exact solution to this problem is

$$u(x,t) = x^2 + t^2.$$

Set $N = 2$ and $M = 2$ with $\lambda = \alpha$ in applying the GSM. We obtain the following result for $\alpha = 0.5$:

$$u_{M,N}^{(\lambda)}(x,t) = 7.40572 \times 10^{-11}x + 4.58429 \times 10^{-10}t + x^2 + 1.31176 \times 10^{-10} - 1.76000 \times 10^{-8}x^2t + 1.56081 \times 10^{-8}xt + 1.56000 \times 10^{-8}x^2t^2 - 1.27421 \times 10^{-8}xt^2 + t^2.$$

We obtain the following result for $\alpha = \lambda = 1, M = 2, N = 2$:

$$u_{M,N}^{(\lambda)}(x,t) = -5.45000 \times 10^{-10}x + 2.05794 \times 0^{-10}t + x^2 + t^2 - 3.88000 \times 10^{-9}x^2t - 3.92959 \times 10^{-9}xt + 1.09609 \times 10^{-8}xt^2 + 1.15000 \times 10^{-10} - 4.30000 \times 10^{-10}x^2t^2$$

In the aforementioned results, we get approximate solution of the problem that is very close to the exact solution. To make a comparison, in Table I, we bring results of applying biorthogonal flatlet multiwavelets scheme [41] for numerical solution of the problem with $\alpha = 0.5$ by taking different values of m and J in time $t = 0.25$.

Example 4.2
Let us consider the one-dimensional fractional heat-like equation

$$D_t^{\alpha}u(x,t) = \frac{1}{2}x^2\frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < 1, \quad 0 < t \le 1, \quad 0 < \alpha \le 1,$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = e^t,$$

and the initial condition

$$u(x,0) = x^2.$$

When $\alpha = 1$, the exact solution of the aforementioned problem is $u(x,t) = x^2e^t$. Taking $\alpha = 0.75$ in Table II, we compare our method with $\lambda = \alpha, M = 3$ and different values of M together with the result obtained by using four-term of the VIM, ADM, and also Sinc–Legendre method with $n = 6$ (time discretization) and $m = 15$ (spatial discretization) given in [50,51], and [30], respectively. Also in Table III, the absolute error function $|u(x,t) - u_{M,N}^{(\lambda)}(x,t)|$ obtained by the present method with $\lambda = 0.9, M = 2$ and $N = 12$ has been compared with Sinc–Legendre method [30] and VIM [50]. The results show the accuracy of the GSM with small discretization of time

Table I. Absolute error for $\alpha = 0.5$ and different values of m for Example 4.1.

	BFM [41] method		
x	$J = 1, m = 2$	$J = 1, m = 3$	$J = 2, m = 2$
0.2	3.3×10^{-2}	4.4×10^{-3}	8.8×10^{-2}
0.4	1.9×10^{-2}	5.1×10^{-2}	9.8×10^{-2}
0.6	1.6×10^{-2}	7.1×10^{-2}	3.4×10^{-1}
0.8	1.2×10^{-1}	2.8×10^{-2}	4.3×10^{-1}

Table II. Comparison of $u(x,t)$ for $\alpha = 0.75$ for Example 4.2.

				Sinc–Legendre method ($n = 6$)	GSM ($M = 3$)	
t	x	VIM	ADM	$m = 15$	$N = 10$	$N = 14$
0.25	0.3	1.293 e-01	1.346 e-01	1.312 e-01	0.1312579	0.1312847
	0.6	5.175 e-01	5.385 e-01	4.957 e-01	0.4967283	0.4966314
	0.9	1.164 e00	1.211 e00	1.055 e-01	1.0591932	1.0591091
0.5	0.3	1.695 e-01	1.795 e-01	1.685 e-01	0.1689450	0.1689744
	0.6	6.780 e-01	7.183 e-01	6.303 e-01	0.6289597	0.6288702
	0.9	1.525 e00	1.616 e00	1.352 e00	1.3532178	1.3531380
0.75	0.3	2.154 e-01	2.313 e-01	2.118 e-01	0.2132472	0.2132235
	0.6	8.618 e-01	9.255 e-01	7.962 e-01	0.7968170	0.7961966
	0.9	1.939 e00	2.082 e00	1.733 e00	1.7310568	1.7306016
1.00	0.3	2.687 e-01	2.909 e-01	2.645 e-01	0.2669056	0.2668259
	0.6	1.075 e00	1.163 e00	9.745 e-01	0.9777382	0.9885615
	0.9	2.419 e00	2.618 e00	2.014 e00	2.1916358	2.1997874

Table III. Comparison of absolute error for $\alpha = 1$ for Example 4.2.

			Sinc–Legendre method ($n = 6$)			GSM ($\lambda = 0.9$)
t	x	VIM	$m = 10$	$m = 15$	$m = 25$	$M = 2$ $N = 12$
0.25	0.3	1.54e-05	5.65e-06	1.09e-06	9.92e-08	1.04e-09
	0.6	6.16e-05	1.28e-04	2.96e-05	2.70e-06	1.17e-10
	0.9	1.38e-04	4.29e-04	9.94e-05	1.02e-05	2.57e-09
0.5	0.3	2.60e-04	2.66e-05	6.45e-06	5.56e-07	2.58e-10
	0.6	1.03e-03	2.23e-04	5.24e-05	4.87e-06	2.67e-10
	0.9	2.34e-03	6.37e-04	1.47e-04	1.30e-05	1.60e-10
0.75	0.3	1.39e-03	5.12e-05	1.40e-05	1.14e-06	1.66e-10
	0.6	5.56e-03	3.16e-04	7.67e-05	6.90e-06	1.01e-09
	0.9	1..25e-02	8.80e-04	2.01e-04	1.59e-05	3.60e-09
1.00	0.3	4.64e-03	2.34e-04	1.83e-05	9.83e-07	1.31e-08
	0.6	1.85e-02	8.23e-04	9.40e-05	6.40e-06	2.20e-08
	0.9	4.18e-02	2.16e-04	4.26e-04	2.58e-05	1.58e-07

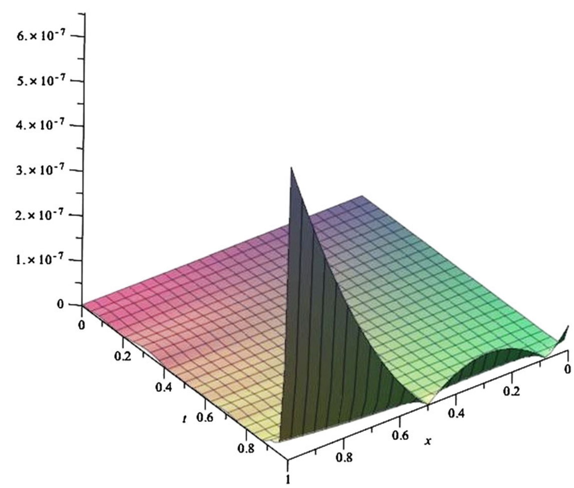

Figure 1. Plot of the absolute error obtained by GSM with $N = 2, M = 12, \lambda = 0.9$ for Example 4.2 when $\alpha = 1$.

and space compared with the mentioned methods. Figure 1 shows the absolute error of our method that indicates low values of the absolute error.

Example 4.3
Consider the initial boundary values problem of fractional partial differential equation of order α, $\quad 0 < \alpha < 1$

$$D_t^{\alpha} u(x,t) + x\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 2t^{\alpha} + 2x^2 + 2, \quad 0 < x < 1, \quad 0 < t < 1,$$

subject to the boundary conditions

$$u(0,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}, \quad u(1,t) = 1 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha},$$

and the initial condition $u(x,0) = x^2$.

The exact solution of this problem is [52, 53]

$$u(x,t) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}.$$

Figure 2 shows the approximate solution of this problem for $\alpha = 0.4$ obtained by the GSM with $M = 2, \quad N = 4, \lambda = 0.8$ together with the exact solution of this problem. To make a comparison, in Table IV, we bring results of absolute error of the Sinc–Legendre method [30] with $n = 7$ and $m = 15, 25$ together with the result obtained by using wavelet method given in [52], for $\alpha = 0.5$. We take great pleasure in telling that GSM yields the exact solution of the problem for very small discretization, that is, $M = 2, N = 1$.

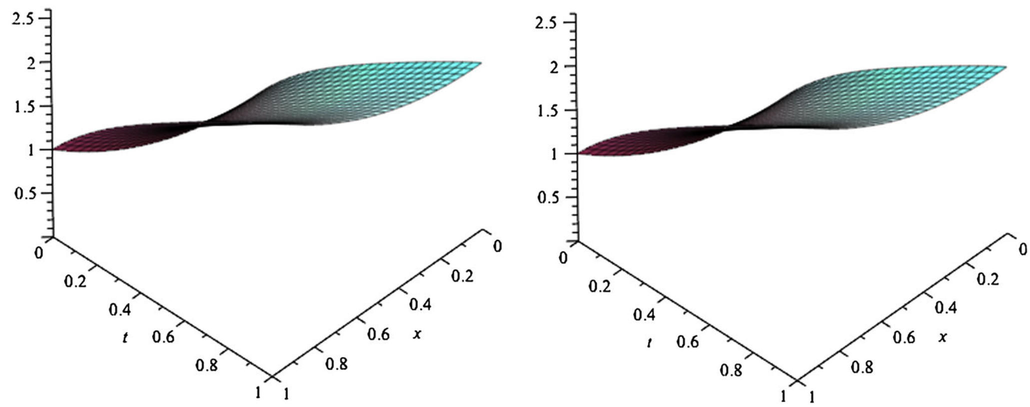

Figure 2. Plot of the approximate solution by GSM with $N = 4, M = 2, \lambda = 0.8$ (left), and exact solution (right) for Example 4.3 when $\alpha = 0.4$.

Table IV. Comparison of absolute error for $\alpha = 0.5, t = 0.5$, and $\lambda = \alpha$ for Example 4.3.

	Wavelet method	Sinc–Legendre method(n = 7)	
x	$m = 64$	$m = 15$	$m = 25$
0.1	1.210 e-03	6.994 e-05	6.462 e-06
0.2	1.259 e-03	1.721 e-04	1.578 e-05
0.3	1.865 e-03	2.472 e-04	2.272 e-05
0.4	7.412 e-03	2.912 e-04	2.674 e-05
0.5	1.000 e-06	3.004 e-04	2.759 e-05
0.6	7.460 e-03	2.760 e-04	2.534 e-05
0.7	1.724 e-03	2.213 e-04	2.035 e-05
0.8	4.990 e-03	1.440 e-04	1.320 e-05
0.9	1.678 e-02	5.026 e-05	4.653 e-06

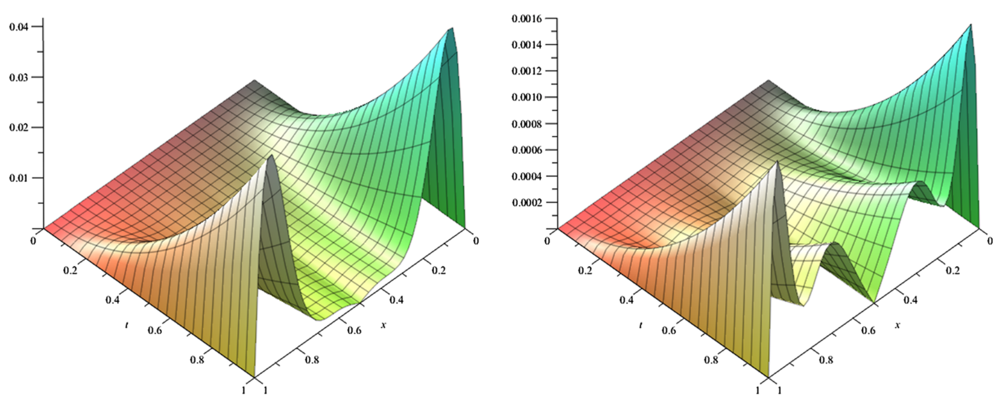

Figure 3. Plot of the absolute error of GSM approximate solution with $N = 2, \alpha = 0.1$ and $M = 5$ (left), $M = 7$ (right) for Example 4.4.

Example 4.4

Consider the following time-fractional diffusion equation:

$$D_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), 0 < x < 1, \quad 0 < t \le 1, \quad 0 < \alpha \le 1, \tag{4.1}$$

where

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x),$$

with the initial condition $u(0,x) = 0$ and the boundary conditions $u(0,t) = u(1,t) = 0$.

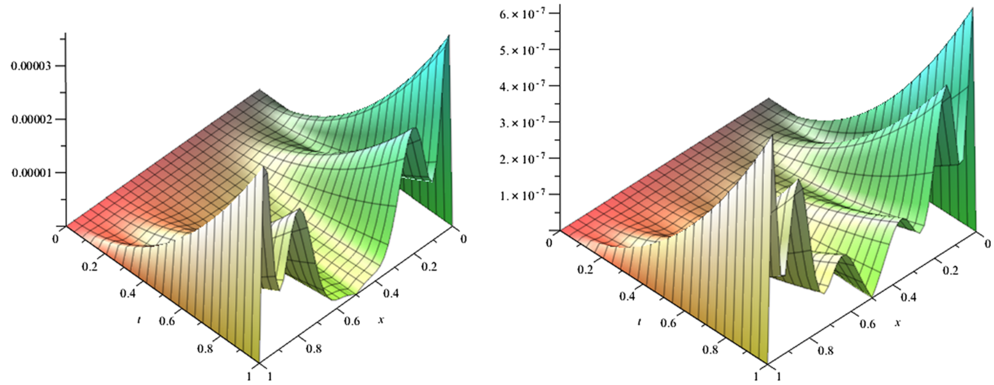

Figure 4. Plot of the absolute error of GSM approximate solution with $N = 2, \alpha = 0.1$ and $M = 9$ (left), $M = 11$ (right) for Example 4.4.

The exact solution to this problem is [16]

$$f(x,t) = t^2 \sin(2\pi x).$$

We solved the problem by applying the GSM when $\alpha = 0.1$. To examine the dependence of errors on the discretization parameters M and N, in Figures 3 and 4, we plotted the absolute error function $|u(x,t) - u^{(\lambda)}_{M,N}(x,t)|$ obtained by the present method with $N = 2$, $\lambda = \alpha$ and for different values of M. One can see from Figures 3 and 4, accurate results even by using $N = 2$ and no longer for M. On comparison, authors of [30] have applied the Sinc–Legendre collocation method for solving (4.1) with $n = 8$ and $m = 10, 15, 20$ and $m = 25$. For $n = 8$ and $m = 25$, they obtained 10^{-5} accuracy as the maximum value of absolute error of the numerical solution. Note that our method has been reached to 10^{-7} accuracy with $N = 2$ and $M = 11$.

5. Conclusion

In the aforementioned discussion, we applied the Gegenbauer collocation spectral method to solve the time-fractional convection–diffusion equation with variable coefficients. This method utilizes a stable procedure to implement and yields the desired accuracy. The basis functions have three different properties: easy computation, rapid convergence, and completeness, which means that any solution can be presented to arbitrarily high accuracy by taking the truncation N to be sufficiently large. Finally, accuracy and rapidity of the proposed method are illustrated by some examples.

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