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Traveling Wave Solutions For Some Nonlinear $(N + 1)$-Dimensional Evolution Equations by Using $(G^{'}/G)$ and $(1/G^{'})$-Expansion Methods

M. Gachpazan *, M. Mortazavi
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
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Abstract: In this paper, with the aid of Maple, the $(G^{'}/G)$ and $(1/G^{'})$-expansion methods are applied to determine the exact solutions of $(N+1)$-dimensional generalized Boussinesq equation and $(N+1)$-dimensional sine-cosine-Gordon equation.These equations play a very important role in mathematical physics and engineering sciences. This methods are more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear PDEs arising in mathmatical physics.

Keywords: Exact traveling wave Solutions; Nonlinear PDEs; $(G^{'}/G)$ and $(1/G^{'})$-expanson methods

1 Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. It is well known that many nonlinear evolution equations are widely used to describe these complex phenomena. So, the powerful and efficient methods to find analytic solutions of nonlinear equations have drawn a lot of interest by many authers. In recent years, directly searching for traveling wave solutions of nonlinear evolution equations has become more and more attractive partly due to the availability of computer symbolic systems like Matlab, Maple, or Mathematica which allow us to perform some complicated and tedious algebraic calculations on a computer, as well as help us to find new exact solutions of nonlinear evolution equations. In the literature, there is awide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are: Backlund transformation method ([1, 2]), tanh-function method ([3, 4]), exp-function method [5], the painlere expansion method [6], symmetry method [7], the homogeneous balance method [8], the tri-function method ([9, 10]), similarity reductions method ([11, 12]), the sine-cosine method ([13, 14]), first integral method ([15, 16]), Hirota bilinear method [17], the F-expansion method [18], Homotopy perturbation method [19], and so on.

Wang et al firstly proposed a $(G^{'}/G)$-expansion method [20], then many diverse group of researchers extended this method by different names like extended further extended improved, generalized and improve $(G^{'}/G)$-expansion method ([21–31]) with different auxiliary equations. In the present paper, we will use the expansion method and $(1/G^{'})$-expansion method. The key idea of the original $(G^{'}/G)$-expansion method is that the exact solutions of nonlinear partial differential equations (PDEs) can be expressed by a polynomial in $(G^{'}/G)$, in which $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G^{''}(\xi) + \lambda G^{'}(\xi) + \mu G(\xi) = 0$, where λ and μ are constants. The main idea of $(1/G^{'})$-expansion method is that our solutions can be expressed by a polynomial $(1/G^{'})$ and $G = G(\xi)$ satisfies a second order linear ODE $G^{''}(\xi) + \lambda G^{'}(\xi) + \mu = 0$. For both method, the degree of the polynomials can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms in the given nonlinear PDEs. $(1/G^{'})$-expansion method has first been introduced by Yokus [32].

In this paper,we obtain the traveling wave solutions of high dimensional equations. The $(N + 1)$-dimensional equations such as generalized Boussinesq equation, sine-cosine-Gordon equation have significant applications in real world problems ([33], [34, 35]). The purpose of this paper is to obtain new traveling wave solutions of these $(N+1)$-dimensional equations by applying the $(G^{'}/G)$ and $(1/G^{'})$-exponsion method.

*Corresponding author. *E-mail address*: gachpazan@um.ac.ir

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2 Analysis of the two methods

The $(G^{'}/G)$-expansion method, the $(1/G^{'})$ method have been applied for a wide variety of nonlinear problems. The main features of the two methods will be reviewed briefly.

For both methods, we first use the wave variable $\xi = x + by - ct$ to carry a PDE in three independent variables

$$p(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{xt}, \ldots) = 0, \tag{1}$$

into an ODE

$$Q(u, u^{'}, u^{''}, u^{'''}, \ldots) = 0. \tag{2}$$

Equation (2) is then integrated as long as all terms contain derivatives, where integration constants considered zeros.

2.1 The $(G^{'}/G)$-expansion method

Suppose that the solutin of equation (2) can be expressed by a polynomial in $(G^{'}/G)$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (G^{'}/G)^i, \tag{3}$$

while $G = G(\xi)$ satisfies the second order linear differential equation in the form

$$G^{''}(\xi) + \lambda G^{'}(\xi) + \mu G(\xi) = 0, \tag{4}$$

where $a_i, (i = 0, 1, \ldots, m)$, λ and μ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest derivative terms and the nonlinear terms appearing in equation (2).

Substituting equation (3) into equation (2) and using equation (4), solving the resulting system of algebraic equations by using the computerized symbolic calculations to obtain all possible values of the parameters λ, μ, a_i, b, and c.

2.2 The $(1/G^{'})$-expansion method

Step1. Suppose that the solution of ODE (2) can be expressed by a polynomial $(1/G^{'})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (1/G^{'}(\xi))^i, \tag{5}$$

where $G = G(\xi)$ satisfies the second order LODE

$$G^{''}(\xi) + \lambda G^{'}(\xi) + \mu = 0, \tag{6}$$

where $a_i (i = 0, 1, \ldots, m)$, λ and μ are constants to be determined later and the positive integer m is homogeneous balance number.
Step2. The solution of the differential equation (6) is

$$G(\xi) = -\frac{\mu\xi}{\lambda} + c_1 e^{-\lambda\xi} + c_2. \tag{7}$$

Then

$$\frac{1}{G^{'}(\xi)} = \frac{\lambda}{-\mu + \lambda c_1 [cosh(\lambda\xi) - sinh(\lambda\xi)]}, \tag{8}$$

can be written.
Step3. By substituting (5) into (2) and using second order linear ODE (6), the left-hand side of (2) can be converted into a polynomial in terms of $(1/G^{'})$. Equating each coefficient of the polynomial to zero yields a system of algebraic equations and solving the algebraic equations by Maple we obtain a_i, c, λ, b and μ constants.

3 Applications of the $(G^{'}/G)$ and $(1/G^{'})$-expansion method

3.1 solution of $(N+1)$-dimensional generalized Boussinesq equation

3.1.1 Using the $(G^{'}/G)$-expansion method

The $(N+1)$-dimensional generalized Boussinesq equation is given by [35]

$$u_{tt} = u_{xx} + \lambda(u^n)_{xx} + u_{xxxx} + \sum_{i=1}^{N-1} u_{y_i y_i}, \tag{9}$$

where $\lambda \neq 0$ is constant and $N > 1$ is an integer. To solve equation (9), consider the wave transformation

$$u(x, y_1, y_2, \ldots, y_{N-1}, t) = u(\xi), \quad \xi = k(x + \sum_{i=1}^{N-1} y_i - ct), \tag{10}$$

where k, c are constants that to be determined later.

By using the transformation ξ, (10), equation (9) can be coverted to follwing ODE:

$$(N - c^2)u^{''} + \lambda(u^n)^{''} + k^2 u^{(4)} = 0, \tag{11}$$

Integrating equation (11) with respect to ξ and ignoring constants of integration we obtain

$$(N - c^2)u^{'} + \lambda(u^n)^{'} + k^2 u^{(3)} = 0. \tag{12}$$

By balancing $u^{(3)}$ and $(u^n)^{'}$ we get

$$m + 3 = nm + 1 \Rightarrow m = \frac{2}{n-1}. \tag{13}$$

To get a closed form analytic solution, the parameter m should be integer. A transformation formula

$$u = v^{\frac{1}{n-1}}, \tag{14}$$

should be used to achieve our goal. This in turn transforms (12) to

$$\begin{aligned}&(N - c^2)(n-1)^2 v^2 v^{'} + \lambda n(n-1)^2 v^3 v^{'} + k^2(n-2)(2n-3)(v^{'})^3 \\ &+ 2k^2(n-1)(2-n)vv^{'}v^{''} + k^2(n-1)^2 v^2 v^{'''} = 0.\end{aligned} \tag{15}$$

Balancing $v^{'}v^3$ and $v^2 v^{'''}$, we find

$$3m + m + 1 = 2m + m + 3 \Rightarrow m = 2. \tag{16}$$

Consequently from (3) we get

$$v(\xi) = a_0 + a_1(\frac{G^{'}(\xi)}{G(\xi)}) + a_2(\frac{G^{'}(\xi)}{G(\xi)})^2. \tag{17}$$

Substituting (17) into equation (15), collecting the coefficients of each power of $(\frac{G^{'}}{G})$, and solve the system of algebraic equations using Maple, we obtain the set of solution:

$$\begin{aligned}a_0 =& -\frac{4\mu k^2(2n^2 - 4n + 3)}{n\lambda(n-1)^2} \quad , a_1 = -\frac{4k^2(2n^2 - 4n + 3)}{n(n-1)^2} \quad , a_2 = -\frac{4\mu k^2(2n^2 - 4n + 3)}{n\lambda(n-1)^2}, \\ k =& k, \quad c = \frac{1}{n-1}\sqrt{(n-1)^2(N + k^2\lambda^2 - 4k^2\mu) + (-n+2)(-4k^2\mu + k^2\lambda^2}.\end{aligned}$$

On solving equation (4), we deduce after some reduction that

$$\frac{G'}{G}=\begin{cases}\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{c_1cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{c_1sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}\right)-\frac{\lambda}{2}, & if\lambda^2-4\mu>0,\\ \frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-c_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+c_2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{c_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+c_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)-\frac{\lambda}{2}, & if\lambda^2-4\mu<0,\\ \frac{c_2}{c_2\xi+c_1}-\frac{\lambda}{2} & ,if\lambda^2-4\mu=0,\end{cases} \tag{18}$$

where c_1 and c_2 are arbitrary constants.

case1. If $\sqrt{\lambda^2-4\mu}>0$, then we have the hyperbolic solution

$$u_1(x,\mathbf{y},t)=\left[\frac{-4k^2(2n^2-4n+3)}{n(n-1)^2}(\frac{\mu}{\lambda}-\frac{\lambda}{4})\left(1-(\frac{c_1cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{c_1sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)})^2\right)\right]^{\frac{1}{n-1}}. \tag{19}$$

case2. If $\sqrt{\lambda^2-4\mu}<0$, then we have the trigonometric solution

$$u_2(x,\mathbf{y},t)=\left[\frac{-4k^2(2n^2-4n+3)}{n(n-1)^2}(\frac{\mu}{\lambda}-\frac{\lambda}{4})\left(1+(\frac{-c_1\sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)+c_2\cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)}{c_1\sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)+c_2\cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)})^2\right)\right]^{\frac{1}{n-1}}. \tag{20}$$

case3. If $\lambda^2-4\mu=0$ then we have the rational solution

$$u_3(x,\mathbf{y},t)=\left[\frac{-4k^2(2n^2-4n+3)}{n(n-1)^2}\left(-\frac{\lambda}{2}+\frac{1}{\lambda}(\mu+(\frac{c_2}{c_2\xi+c_1})^2)\right)\right]^{\frac{1}{n-1}}. \tag{21}$$

In particular, if we set $c_2=0, c_1\neq 0, \lambda>0$ and $\mu=0$, in equation (19), then we get

$$u_4(x,\mathbf{y},t)=\left[\frac{-4k^2(2n^2-4n+3)}{n(n-1)^2}\frac{\lambda}{4}csch^2(\frac{\lambda}{2}\xi)\right]^{\frac{1}{n-1}}, \tag{22}$$

where $\mathbf{y}=y_1,y_2,\ldots,y_{N-1}$ and
$\xi=k\left(x+\sum_{i=1}^{N-1}y_i-\frac{1}{n-1}\sqrt{(n-1)^2(N+k^2\lambda^2-4k^2\mu)+(-n+2)(-4k^2\mu+k^2\lambda^2t}\right)$.

3.1.2 Using the $(\frac{1}{G'})$ expansion method

From equation (16) we know the balance number is $m=2$, so if subsitute this number into (5), our solution can be expressed as follows:

$$v(\xi)=a_0+a_1(\frac{1}{G'})+a_2(\frac{1}{G'})^2. \tag{23}$$

If we Substitute into ODE (15) using equation (6) and collecting all terms with the same power of $(\frac{1}{G'})$. Setting each coefficients of this polynomial to zero yields a system of algebraic equation which can be solved by Maple to find following results:

$$a_0 = 0, a_1 = -\frac{4\mu k^2(2n^2-4n+3)}{n(n-1)^2}, \quad a_2 = -\frac{4\lambda k^2(2n^2-4n+3)}{n(n-1)^2}, \quad k = k$$
$$c = \frac{1}{n-1}(\sqrt{N(n-1)^2 + k^2\lambda^2(n^2-3n+3)}), \quad \mu = \lambda, \quad \lambda = \lambda. \tag{24}$$

By substituting (24) into (23) and using (7) and (8), we obtain

$$u(x, \mathbf{y}, t) = -\frac{4\lambda k^2(2n^2-4n+3)}{n(n-1)^2}\left[\frac{1}{-1+c_1[cosh(\lambda\xi) - sinh(\lambda\xi)]} + \frac{1}{(-1+c_1[cosh(\lambda\xi) - sinh(\lambda\xi)])^2}\right]^{\frac{1}{n-1}}, \tag{25}$$

where $\xi = k\left(x + \sum_{i=1}^{N-1} y_i - (\frac{1}{n-1}(\sqrt{N(n-1)^2 + k^2\lambda^2(n^2-3n+3)}))t\right)$.

3.2 solution of $(N+1)$-dimensional sine-cosine-Gordon equation

3.2.1 Using the $(G^{'}/G)$-expansion method

The $(N+1)$-dimensional sine-cosine-Gordon equation is given by [21]

$$\sum_{i=1}^{N} u_{x_i x_i} - u_{tt} - \alpha\cos(u) - \beta\sin(2u) = 0, \tag{26}$$

where β, α are a nonzero constant.

To study the traveling wave solutions of equation (26), we take the following transformation

$$u(x_1, x_2, \dots, x_N, t) = u(\xi), \quad \xi = k(\sum_{i=1}^{N} x_i - ct), \tag{27}$$

where $k, c \neq 0$ are constants. By using (27) and (26) is converted into an ODE

$$k^2(N-c^2)u^{''} - \alpha\cos(u) - \beta\sin(2u) = 0. \tag{28}$$

We introduce the following transformation $u = 2\tan^{-1}(v), (v = \tan\frac{u}{2} \in (-\infty, \infty))$, then we have

$$u^{''} = \frac{2(v^{''} + v^{''}v^2 - 2(v^{'})^2 v)}{(1+v^2)^2}, \quad \cos(u) = \frac{1-v^2}{1+v^2}, \quad \sin(2u) = \frac{4v(1-v^2}{(1+v^2)^2}. \tag{29}$$

Substituting these transformations (29) in equation (28), we can rewrite the $(N+1)$-dimensional sine-cosine-Gordon equation (26) in the following form

$$2k^2(N-c^2)(1+v^2)v^{''} - 4k^2(N-c^2)v(v^{'})^2 + (v^2-1)(\alpha v^2 + 4\beta v + \alpha) = 0. \tag{30}$$

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (30), we get $m = 2$. Thus, we have

$$v(\xi) = a_0 + a_1(\frac{G^{'}(\xi)}{G(\xi)}) + a_2(\frac{G^{'}(\xi)}{G(\xi)})^2, \tag{31}$$

where a_0, a_1, a_2 are constants to be determined later. Substituting equation (31) and equation (4) into (30), collecting the coefficients of each power of $(G^{'}/G)$, and solve the resulting system of algebraic equations to find the sets of solution:

$$a_0 = -\frac{\lambda^2}{-\lambda^2+4\mu}, \quad a_1 = -\frac{4\lambda}{-\lambda^2+4\mu}, \quad a_2 = -\frac{4}{-\lambda^2+4\mu},$$
$$c = c, \quad \alpha = \alpha, \quad \beta = -\alpha, \quad k = \sqrt{-\frac{\alpha}{-N\lambda^2 + 4N\mu + c^2\lambda^2 - 4c^2\mu}}. \tag{32}$$

Substituting these results into (31) and using (18) we obtain traveling wave solution
case1. If $\sqrt{\lambda^2-4\mu}>0$, then we have the hyperbolic solution

$$u_1(\mathbf{x},t)=2\tan^{-1}\left[\frac{c_1cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{c_1sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}\right]^2. \tag{33}$$

case2. If $\sqrt{\lambda^2-4\mu}<0$, then we have the trigonometric solution

$$u_2(\mathbf{x},t)=-2\tan^{-1}\left[\frac{-c_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+c_2\cos(\frac{1}{2}\sqrt{-4\mu-\lambda^2}\xi)}{c_1\cos(\frac{1}{2}\sqrt{-4\mu-\lambda^2}\xi)+c_2\sin(-4\mu-\frac{1}{2}\sqrt{\lambda^2}\xi)}\right]^2. \tag{34}$$

case3. If $\lambda^2-4\mu=0$, then we have the rational solution

$$u_3(\mathbf{x},t)=2\tan^{-1}\left[\frac{-4}{-\lambda^2+4\mu}(\frac{c_2}{c_2\xi+c_1})^2\right]. \tag{35}$$

In particular, if we set $c_2=0, c_1\neq 0$ and $\lambda>0, \mu=0$, in equation (33), then we get

$$u_4(\mathbf{x},t)=2\tan^{-1}\left[coth^2(\frac{\lambda}{2}\xi)\right], \tag{36}$$

where $\mathbf{x}=x_1,x_2,\ldots,x_N$ and $\xi=\sqrt{-\frac{\alpha}{-N\lambda^2+4N\mu+c^2\lambda^2-4c^2\mu}}(\sum_{i=1}^N x_i-ct)$.

3.2.2 Using the $(1/G^{'})$ expansion method

From (30), we know the balance number is $m=2$. Thus, the solutions of equation (30),according to equation (5) is

$$v(\xi)=a_0+a_1(\frac{1}{G'})+a_2(\frac{1}{G'})^2. \tag{37}$$

Substituting equation (37) and equation (6) into equation (30), collecting all terms whit the like powers of $(1/G^{'})$, setting them to zero, we get an over-determined system algebraic equations. Solving this over-determined system with the aid of symbolic computer software Maple, we have the following result.

$$\begin{aligned}a_0&=-1,\quad a_1=-4,\quad a_2=-4, c=c,\\ k&=\frac{1}{\mu}\sqrt{-\frac{\alpha}{-c^2+N}},\quad \alpha=\alpha,\quad \beta=\alpha,\lambda=\mu,\quad \mu=\mu.\end{aligned} \tag{38}$$

By substituting (38) into (37) and using (8) we obtain

$$u(\mathbf{x},t)=\tan^{-1}\left[-1+\frac{4}{1-c_1[cosh(\mu\xi)-sinh(\mu\xi)]}-\frac{4}{(1-c_1[cosh(\mu\xi)-sinh(\mu\xi)])^2}\right], \tag{39}$$

where $\mathbf{x}=x_1,x_2,\ldots,x_N$ and $\xi=\frac{1}{\mu}\sqrt{-\frac{\alpha}{-c^2+N}}(\sum_{i=1}^N x_i-ct)$.

4 Conclusion

In this paper, we focus our attention on the enhanced $(G/G^{'})$-expansion method and $(1/G^{'})$-expansion method to drive exact traveling wave solutions of the $(N+1)$-dimensional generalized Boussinesq and $(N+1)$-dimensional sine-cosine-gordon equation. We foresee that our results can be found potentially useful for applications in mathematical physics and engineering. All solutions in this paper have been found by aid of Maple packet progarm. Finally, this methods are productive, effective and well-built mathematical tool for solving nonlinear evolution equations.

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