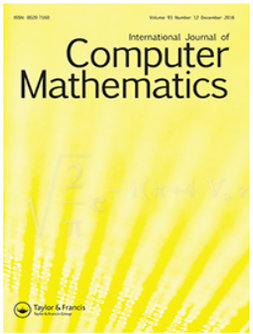

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A new solution method for stochastic differential equations via collocation approach

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The objective of this paper is to propose a novel solution method for Itô stochastic differential equations (SDEs). It is discussed that how the SDEs could numerically be solved as matrix problems. To improve the accuracy of this technique in contrast to the existing solvers, some non-uniform grids of points for discretizations along the time direction are applied. Finally, the high accuracy of approximated solutions in this way are illustrated by several experiments.

Keywords: stochastic differential equations; collocation method; Lamperti transform; orthogonal polynomials

2010 AMS Subject Classifications: 65C30; 60H35; 65L60

1. Preliminaries

Mathematical modelling of numerous real-world phenomena associated with random noisy perturbations is not possible through the idea of ordinary differential equations (ODEs). As an alternative, such problems are frequently modelled by taking into account the notion of stochastic differential equations (SDEs). This makes the model to become more realistic. Some instances are in molecular biology, population dynamics, theoretical physics and (mathematical) finance [12].

Let us consider the following Itô SDE:

$$\mathrm{d}X(t) = a(t, X(t))\,\mathrm{d}t + b(t, X(t))\,\mathrm{d}W(t), \quad 0 \le t \le T, \tag{1}$$

with the given initial condition $X(0) = X_0$.

Models based on the SDE (1) are computationally much more challenging than deterministic ones. The high computational cost for numerical simulations of stochastic models arising in applications motivated the search for finding efficient numerical tools [1,20]. Some typical

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solvers to this purpose are the classic Euler–Maruyama (EM) method [14]

$$X_{i+1} = X_i + a(t_i, X_i)\Delta t_i + b(t_i, X_i)\Delta W_i, \tag{2}$$

where $\Delta t_i = t_{i+1} - t_i$ and $\Delta W_i = W_{t_{i+1}} - W_{t_i}$, and Milstein method (MM) [14]

$$X_{i+1} = X_i + a(t_i, X_i)\Delta t_i + b(t_i, X_i)\Delta W_i + \frac{1}{2} b(t_i, X_i)\frac{\partial b}{\partial x}(t_i, X_i)(\Delta W_i^2 - \Delta t_i). \tag{3}$$

It should be reminded that by a solution of Equation (1), a probability space with an increasing family of σ-fields of $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is considered, while a family of stochastic processes $\{X(t), W(t)\}$ defined on it such that [12]

(1) with probability one, $X(t)$ and $W(t)$ are continuous in t and $W(0) = 0$;
(2) $X(t)$ and $W(t)$ are $\mathcal{F}_t$-measurable;
(3) $W(t)$ is a $\mathcal{F}_t$-martingale such that

$$\mathbb{E}\left[\frac{(W(t) - W(s))^2}{\mathcal{F}_s}\right] = t - s, \quad t \geq s; \tag{4}$$

(4) $X(t)$ and $W(t)$ satisfy

$$X(t) = X(0) + \int_0^t a(s, X(s))\,\mathrm{d}s + \int_0^t b(s, X(s))\,\mathrm{d}W(s), \tag{5}$$

where the integral by $\mathrm{d}W(s)$ is understood in the sense of Itô (stochastic) integral.

The explicit numerical methods (2) and (3) based on equidistant grid of nodes for solving stiff SDEs can entail severe restrictions on the step size Δt due to their finite regions of mean-square stability, and subsequently various implicit schemes have been appeared in the literature to overcome this limitation [18]. It is well known that for problems which are stiff in both the deterministic and stochastic components, the implicit Euler method is not yet appropriate and much more complicated methods such as split-step methods [7] were introduced which suffer of high computational burden.

One way to reduce the computational cost of an approximation algorithm is to use adaptive time-stepping schemes to advance the numerical solution. When the ODEs are dealt with, step-size adapting strategies have been proved to be essential in generating optimal algorithms. There are rich strategies to adapt algorithms for ODEs, but little is known for SDEs [10].

The primary goal of this work is to distribute the grid of points so as to reduce the computational load while the accuracy of the approximations at the end point T increases. Our approach is different to the adaptation of nodes described in [10] or the discussions of stochastic finite elements in [6]. In fact, it is attempted to find the approximated solution for the SDE (1) in the strong sense via transforming the whole procedure into a matrix problem. This technique reduces solving Itô SDEs to those of solving a system of nonlinear or linear algebraic equations, and thus greatly simplifying the problem.

The presented solution method is a collocation method since it uses zeros of orthogonal polynomials as the grid points. Hence, the grid of points are not anymore equidistant. In this way, using the schemes such as EM and MM is tangled since Δt_i must be obtained for each partition. To circumvent on this pitfall and the drawback of low accuracy for EM and MM, the notion of differentiation matrices (DMs) is applied [3,5]. It is known that the concepts of DMs are both theoretically and numerically useful well beyond the realm of (computational) methods derived from orthogonal polynomials.

It is stated that the proposed approach is an extension over the work given by Huang and Zhang [8]. To be more precise, this work extends the idea of spectral collocation for SDEs by further concentrations on the fact that SDEs should be considered in the very weak sense in terms of defining the derivative and the use of more general and efficient orthogonal polynomials to obtain the non-equidistant grid of points and their DMs. In addition, it is shown that how such an approach is efficient for systems of SDEs and Itô–Volterra integral equations.

Consequently, the main contribution of this paper is in extending the use of an adapted grid of points via the (spectral) collocation points obtained mainly from the well-known general orthogonal polynomials, such as shifted Legendre or Gegenbauer polynomials. Moreover, a spectral accuracy can be attained in the proposed approach, which means that by increasing the number of non-equidistant collocation points, an engrossing increase in the accuracy of the approximated solutions is derived and observed.

After these introductory descriptions, the remaining sections are organized in what follows. Section 2 is devoted to the application of collocation approach for SDEs in a matrix form. Some pitfalls are discussed and resolved. In Section 3, the zeros of orthogonal polynomials are taken into consideration so as to build the DMs. These non-equidistant partitions are very useful so as to increase the accuracy of our approximated values. In this way, the proposed approach is a spectral collocation method for SDEs. Experiments are given in Section 4 along with comparisons to clearly put on show the efficiency of the solution method. Finally, some concluding remarks and open problems are furnished in Section 5.

2. Collocation method for SDEs

Let us first consider a set of nodes in the interval $[0, T]$, as the collocation points. Next, the idea of solving (1) via the DM is to first interpolate a polynomial $p(t)$ on these points. And then, by taking the derivative of the polynomial and evaluating it at the collocation points we obtain the DM, D_N, which plays the role of operator $\mathrm{d}/\mathrm{d}t$.

To be more precise, let us write Equation (1) as follows:

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{a(t, X(t))\,\mathrm{d}t}{\mathrm{d}t} + \frac{b(t, X(t))\,\mathrm{d}W(t)}{\mathrm{d}t}, \tag{6}$$

which simplifies as

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = a(t, X(t)) + b(t, X(t))\frac{\mathrm{d}}{\mathrm{d}t}W(t). \tag{7}$$

It is of great import to put into words that this representation of the model is somewhat formal, that is, the pure white noise terms $\mathrm{d}W/\mathrm{d}t$ need to be interpreted in an extremely *weak sense* (in terms of defining the derivative). Both Equations (1) and (7) are a same SDE, but in different notations.

Remark 1 To be more precise, here a proper meaning for the operator $\mathrm{d}/\mathrm{d}t$ is some kind of weak derivative in a suitable Hilbert space defined for the Itô processes.

Now, replacing the operator $\mathrm{d}/\mathrm{d}t$ with the DM, one may have

$$D_N X(t) = a(t, X(t)) + b(t, X(t))D_N W(t). \tag{8}$$

It is reminded that the DM can be defined as follows [4]. Assume that $\{t_j\}_{j=0}^{N}$ be the set of $N + 1$ collocation points on the integration interval $[0, T]$. Furthermore, $p(t)$ is considered to

be the unique polynomial of degree N with $p(0) = 0$ and $p(t_j) = x_j$, $1 \le j \le N$, and the vector $\mathcal{X} = (x_0, \ldots, x_N)^*$ is the unknown solution corresponding to a sample path of the SDE (1). If it is set that

$$\varpi_j = p'(t_j), \tag{9}$$

then using $\varpi = (\varpi_0, \ldots, \varpi_N)^*$, the DM can be constructed by

$$\varpi = D_N \mathcal{X}. \tag{10}$$

Some difficulties emerged before proceeding further. First, D_N is always singular, and second, the product of DM into the Wiener term W produces inaccuracy. This is because of the fact that the collocation methods are mostly useful for problems having smooth solutions. To remedy on these shortcomings, first the initial condition of the original SDE (1) is used, since X_0 is known always at the time $t_0 = 0$. This implies that the first column and row of D_N have no effects on the obtained approximated values of the nodal points.

Accordingly, a non-singular matrix $\bar{D}_N$ has now been attained as the DM, and subsequently, Equation (8) can be rewritten as follows; while for simplicity in notations, we set $\mathcal{X} = (x_1, \ldots, x_N)^*$ from now on:

$$\mathcal{X} = \bar{D}_N^{-1} a(t, \mathcal{X}) + \bar{D}_N^{-1} b(t, \mathcal{X}) \bar{D}_N W, \tag{11}$$

where W is the vector consisting of Wiener increments over a path. The DM D_N is of the dimensions $(N + 1) \times (N + 1)$, while $\bar{D}_N$ is of the size $N \times N$.

The second shortcoming has still its negative effect on Equation (11) specially if one wishes to cope with nonlinear or multidimensional SDEs. To overwhelm this flaw, the eminent finding of Lamperti in [13] is recalled. Actually, he introduced key concepts related to what is now referred to as self-similar processes.

The Lamperti transformation

$$Y(t) = F(X(t)) = \int_z^{X(t)} \frac{1}{b(u)} \mathrm{d}u \tag{12}$$

defines a one-to-one correspondence between stationary processes on the real line and self-similar processes on the real half-line. Although dating back to 1962, this fundamental result has recently been discussed again in [2], with emphasis on variations which can be made on the initial formulation.

Here z is any arbitrary value in the state space of X. Indeed, the process $Y(t)$ solves the SDE

$$\mathrm{d}Y(t) = \bar{a}(t, Y(t))\, \mathrm{d}t + \mathrm{d}W(t), \tag{13}$$

with the initial condition $Y(0) = F(X(0))$, whereas

$$\bar{a}(t, y) = \frac{a(t, F^{-1}(y))}{b(F^{-1}(y))} - \frac{1}{2} b_x(F^{-1}(y)). \tag{14}$$

One may use the Itô formula to attain Equation (14). SDEs with a twice continuously differentiable drift term with respect to x and continuously differentiable with respect to t, while the diffusion term only depends on x and continuously differentiable can always be transformed into one with a unitary diffusion coefficient by applying the Lamperti transform. For further discussion, one may consult [9].

Hence by considering that the drift and diffusion terms of Equation (1) are sufficiently differentiable which is a mild condition in contrast to the higher order methods for SDEs, we can resolve the second drawback and propose the following matrix problem for solving Equation (1):

$$\mathcal{Y} = \bar{D}_N^{-1}\bar{a}(t, \mathcal{Y}) + W, \tag{15}$$

wherein $\bar{a}(t, y)$ is defined by Equation (14).

Note that $\mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_N)^*$ is calculated via the matrix problem (15) and the inverse of the Lamperti transformation results in $\mathcal{X} = F^{-1}(\mathcal{Y})$, which is the final strong solution of Equation (1). For general SDEs, that is, the non-autonomous ones whose drift part is dependent on the time variable t, the Lamperti transformation and its inverse could not be mostly put to use. Therefore, an active topic of further future research is centred in finding some other transformations that could bypass this imperfection.

In addition, to vanish the initial condition, one could also set $Y(t) \leftarrow X(t) - X_0$. Therefore, the final collocation approach for Equation (1) could be written in what follows:

$$\mathcal{Y} = \bar{D}_N^{-1}\bar{a}(t, \mathcal{Y} + X_0) + W. \tag{16}$$

Now, a reliable tool becomes available, which is known as the collocation method for solving SDEs.

3. Non-equidistant partitions

The most important part in the solution method (16) is the construction of the DM for the non-equidistant grid of nodes (in the direction of time). Our approach is to attain as high as possible of accuracy by consuming as low as possible of nodal points. Toward this purpose, the spectral approach [17] is considered, that is, by considering the zeros of orthogonal polynomials on the shifted interval $[0, T]$.

It should here be pointed out that high-degree polynomial interpolation $p(t)$ has an unuseful behaviour for equidistant nodes because of the famous Runge phenomenon and the general fact that for any node density function, one can design a continuous function such that the max-norm error tends to infinity as the number of nodes is increased [4].

This bad behaviour is not always justified, since interpolation using the zeros of orthogonal polynomials as nodes features no Runge phenomenon. The low Lebesgue constant $\mathrm{O}(\log(N))$ ensures that, even for the most artificially constructed functions (e.g. *those that are not continuous at any point*), high-degree interpolation still leads to a polynomial that is not too far off from the optimal one.

In the collocation approach, orthogonal polynomials of degree $N-1$ possessing exactly $N-1$ real zeros in the shifted interval $[0, T]$ are applied, [15]. Then, initial and end points of the interval $[0, T]$ are considered as the first and last nodes, so as to construct a non-equidistant grid of $N+1$ nodes. This implies that a polynomial of degree N passing through these nodes in order to construct the DM is within reach [19].

Now, we concisely recall three significant orthogonal polynomials that we will use in the next section.

3.1 *Chebyshev polynomials of the first kind*

Series of Chebyshev polynomials are often used in making numerical approximations to functions. The Chebyshev polynomials of the first kind $T_n(x)$ (also known as Tchebyshev) in the

interval $[-1, 1]$ are defined by

$$T_n(\cos\theta) = \cos(n\theta). \tag{17}$$

They are normalized so that $T_n(1) = 1$ and read the orthogonality relation

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}\,\mathrm{d}x = 0, \tag{18}$$

for $m \neq n$. The $T_n(x)$ also satisfy an orthogonality relation under summation at discrete points in x corresponding to the roots of $T_n(x)$.

3.2 *Legendre polynomials*

Legendre polynomials $L_n(x)$ in the interval $[-1, 1]$ arise in the study of systems with three-dimensional spherical symmetry. They read the ODE

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \tag{19}$$

and the orthogonality relation $\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = 0$ for $m \neq n$. Note that $L_n(x)$ reads Bonnet's recursion formula

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x). \tag{20}$$

3.3 *Gegenbauer polynomials*

The Gegenbauer polynomials $C_n^{(m)}(x)$ are orthogonal on the interval $[-1, 1]$ with weight function $(1-x^2)^{m-1/2}$, corresponding to integration over a unit hypersphere. Let $C_n^{(m)}(x)$ satisfies the ODE

$$(1-x^2)y'' - (2m+1)xy' + n(n+2m)y = 0, \tag{21}$$

then we use the zeros, when $m = \frac{3}{2}$ as our collocation points in this work. Note that $C_n^{(m)}(x)$ reads recursion formula

$$C_n^{(m)}(x) = \frac{1}{n}[2x(n+m-1)C_{n-1}^{(m)}(x) - (n+2m-2)C_{n-2}^{(m)}(x)]. \tag{22}$$

Gegenbauer polynomials $C_n^{(m)}(x)$ can be viewed as generalizations of the Legendre polynomials to systems with $(m+2)$-dimensional spherical symmetry. They are sometimes known as ultra-spherical polynomials.

To clearly show the advantages of such non-equidistant nodes, in Figures 1 and 2, the distribution of zeros of above orthogonal polynomials is provided along with the equidistant distribution on the interval $[-1, 1]$ by increasing the number of nodes in the interval. They manifest that accurate approximations of the function derivatives can be done at the end point using the zeros of orthogonal polynomials. Therefore, by reducing the computational load, one can simply increase the accuracy of approximations at the end points.

It is required to point out that by increasing the number of nodes in the interval $[0, T]$ using the shifted orthogonal polynomials, Equations (17)–(22), the condition numbers of DMs increase, but this growth is not at all at a level to make the process ill-conditioned. Figure 3 plots the condition numbers of DMs for two different non-equidistant partitions chosen by Tchebyshev and Legendre nodes on $[0, 3]$. They show that the matrix problem will remain well conditioned by increasing the nodes normally.

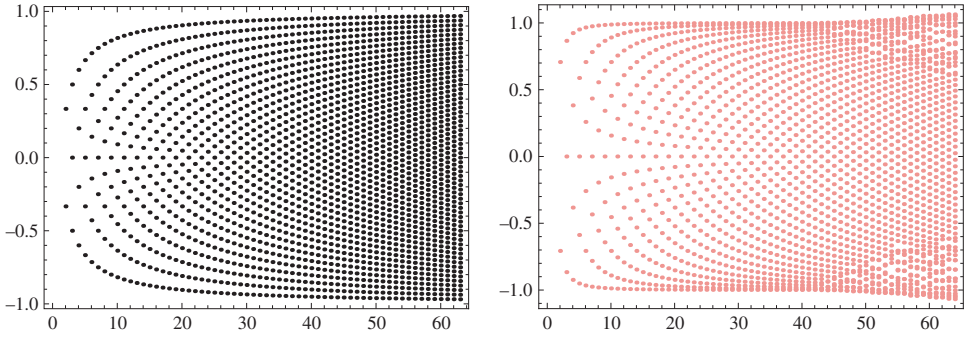

Figure 1. Distribution of equidistant nodes (left) and non-equidistant Chebyshev nodes of the first kind (right).

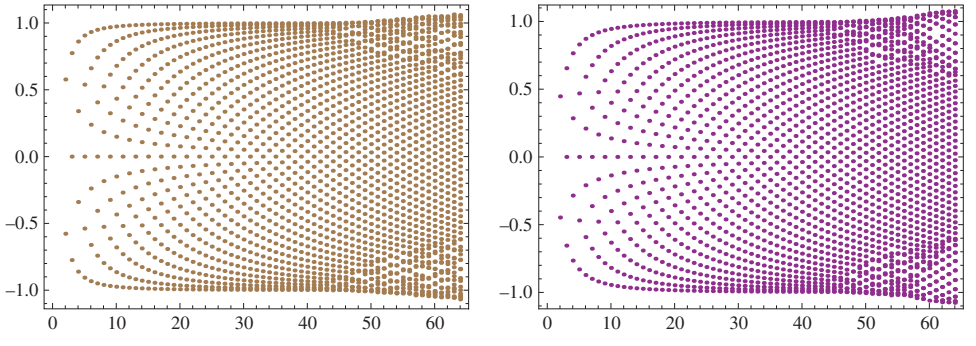

Figure 2. Distribution of non-equidistant Legendre nodes (left) and non-equidistant Gegenbauer nodes (right).

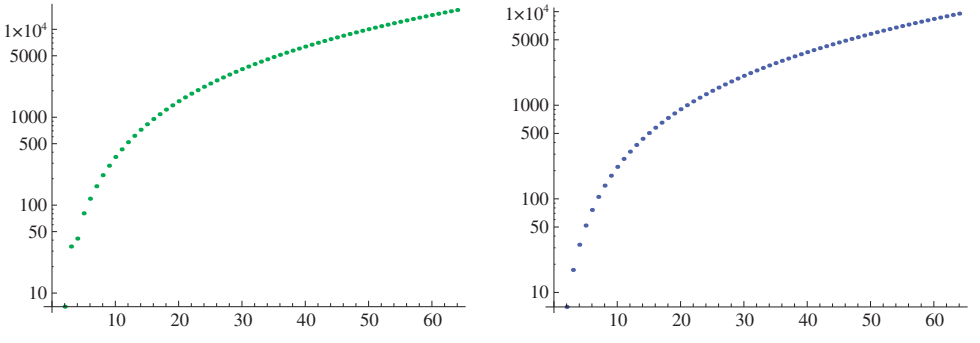

Figure 3. The growth of condition numbers for the DMs by applying the Tchebyshev nodes (left) and the Legendre nodes (right).

At this moment, it is recalled that a time discrete approximation $\mathcal{Y} = (\mathcal{Y}(t))$ converges strongly with order p to the true solution if there exist a constant C, such that

$$\mathbb{E}(\|\mathcal{Y}(t) - X(t)\|) \leq C\Delta t^p, \tag{23}$$

as $\Delta t \to 0$.

Here global approximations for $\mathrm{d}/\mathrm{d}t$ with spectral accuracy is attained (errors decreasing exponentially with N) using the discussed DMs. Note also that DMs from global spectral approximants are dense. In contrast to the finite difference scheme, which possesses slow convergence

rate, for example, second-order finite differences yield

$$\mathbb{E}(\|\mathcal{Y}(t) - X(t)\|) \sim O(\Delta t^2) \sim \frac{1}{N^2}; \tag{24}$$

the discussed (spectral) collocation approach produces the following well-known error bound [4]:

$$\mathbb{E}(\|\mathcal{Y}(t) - X(t)\|) \sim \exp(-\alpha N), \quad \alpha > 0. \tag{25}$$

The computational complexity of the presented collocation approach for the given SDEs is simplified in three steps. First is the process of transforming an SDE into a corresponding one with additive noise term. As explained before, this phase does not take any special effort using the Lamperti transform. Second, it is important to construct the DMs for the collocation points. Here the most crucial factor is that DMs can be derived and saved one time for the grid points and there is no need of consuming much time to construct them for every run. In fact, DMs are not dependent on the problem itself but dependent on the grid nodes [4]. And third, the most eminent phase is the procedure of solving the system of linear or nonlinear algebraic equations. Such systems could be solved numerically fast by the state-of-the-art solvers in recent computer algebra systems such as Mathematica.

4. Numerical experiments

The reliability of the collocation approach using the spectral nodes discussed in Sections 2 and 3 is now put into test. Our results with those of EM, MM and the nodal points of Chebyshev–Gauss–Lobatto (zeros of Chebyshev's polynomials of the second kind) given in [8] denoted by Huang Method (HM) are compared. The numerical schemes are denoted by CM, LM and GM for Tchebyshev, Legendre and Gegenbauer nodes, respectively.

The computer algebra system Mathematica Version 10 is employed [11,16]. To compare the methods from the strong point of view, the mean values of the (absolute) errors $E(T) = \|\mathcal{X}(T) - X(T)\|$ at the end point T (for $k = 100$ realizations) are reported. We used 2^8 partitions for reproducing the involved Wiener processes in the following experiments.

In order for easing up the implementation and convergence studies of the solution method (16), it is now rewritten as comes next

$$\mathcal{Y} - \bar{D}_N^{-1}\bar{a}(t, \mathcal{Y} + X_0) - W = 0. \tag{26}$$

This always yields a linear or nonlinear system of algebraic equations. Therefore, the whole procedure of solving Equation (1) has been reduced to a matrix problem. Note that since we are using the zeros of orthogonal polynomials as the nodes, so the size of the system(s) will not be generally high, while the approximate values possess high accuracies in the end points as will be seen here.

Example 4.1 Let us solve the Itô SDE

$$\begin{aligned} \mathrm{d}X(t) &= \exp(t)\,\mathrm{d}t + \mathrm{d}W(t), \\ X(0) &= 0, \end{aligned} \tag{27}$$

with the exact solution $X(t) = \exp(t) - 1 + W(t)$, wherein $0 \le t \le 2$.

Table 1. Comparisons of mean results for Example 4.1.

$N\backslash$ Error	EM	MM	HM	CM	LM	GM
2^1	2.67077	2.67077	9.524×10^{-1}	9.245×10^{-1}	9.136×10^{-1}	9.524×10^{-1}
2^2	1.46471	1.46471	8.00×10^{-3}	1.166×10^{-2}	1.779×10^{-4}	1.357×10^{-2}
2^3	5.76539×10^{-1}	5.76539×10^{-1}	1.375×10^{-7}	4.836×10^{-8}	1.154×10^{-14}	3.443×10^{-7}
2^4	3.90999×10^{-1}	3.90999×10^{-1}	2.664×10^{-14}	4.440×10^{-15}	–	4.235×10^{-15}
2^5	1.97578×10^{-1}	1.97578×10^{-1}	–	–	–	–

Table 2. Comparisons of mean results for Example 4.2.

$N\backslash$Error	EM	MM	HM	CM	LM	GM
2^3	5.743×10^{-1}	2.195×10^{-3}	3.552×10^{-15}	2.664×10^{-15}	1.998×10^{-15}	3.552×10^{-15}
2^4	4.353×10^{-1}	1.613×10^{-3}	–	–	–	–
2^5	4.158×10^{-1}	5.098×10^{-4}	–	–	–	–

In this experiment, neither the Lamperti transformation nor incorporating the initial condition into the SDE is required, because the diffusion term is already 1 and the initial setting is $X(0) = 0$.

The results for this experiment are put together in Table 1. They plainly show that the spectral collocation method, specially LM, possesses numerical results of high accuracies even for small number of partitions. In Table 1, the notation '–' is used, when the previous number of partitions yielded a very accurate approximation and, therefore, further decreasing in the step sizes is not anymore needed. Furthermore, since the diffusion term is one, the EM and the MM coincide and possess a similar rate of convergence 1.

Example 4.2 Consider the nonlinear SDE

$$\begin{aligned} \mathrm{d}X(t) &= \tfrac{1}{3}X(t)^{1/3}\,\mathrm{d}t + X(t)^{2/3}\,\mathrm{d}W(t), \\ X(0) &= 10, \end{aligned} \tag{28}$$

with the exact solution $X(t) = (X(0)^{1/3} + \frac{1}{3}W(t))^3$, wherein $0 \le t \le 1$.

In this experiment, one may apply the Lamperti transformation $Y(t) = F(X(t)) = 3X(t)^{1/3}$ so as to derive the corresponding SDE with constant diffusion term. In this way, the initial value for the new SDE is $Y(0) = 3(10^{1/3})$. We have summarized the results of this experiment in Table 2. They again manifest that the spectral collocation method is accurate.

Remark 2 Whenever the Lamperti transformation $Y(t) = F(X(t))$ eliminates the interactions between the state of the process and the increments of the Wiener process, this transformation method is welcomed because it reduces instability in the simulation process.

Stochastic integral equations (such as Volterra integral equations or integro-differential equations) arise in a wide range of problems such as the stochastic formulation of problems in reactor dynamics. In the following example, it would be illustrated that how the collocation approach could be used for solving a stochastic integral equation.

Example 4.3 Consider the following nonlinear stochastic Itô integral equation

$$X(t) = X_0 - \frac{a^2}{2}\int_0^t (\tanh(X(s)) \times \operatorname{sech}^2(X(s)))\,\mathrm{d}s + a\int_0^t \sec(X(s))\,\mathrm{d}W(s), \tag{29}$$

with $X(0) = \frac{1}{10}$ and the exact solution $X(t) = \operatorname{arcsinh}(aW(t) + \sinh(X(0)))$, wherein $0 \le t \le 2$.

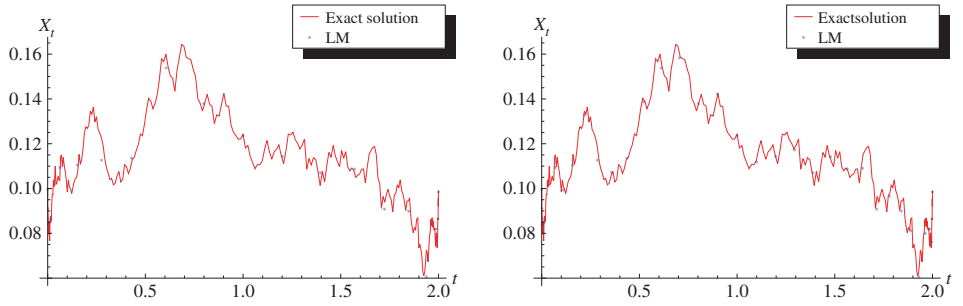

Figure 4. The Legendre spectral collocation method (LM) using $N = 2^4$ (left) and $N = 2^5$ (right) versus the exact solution.

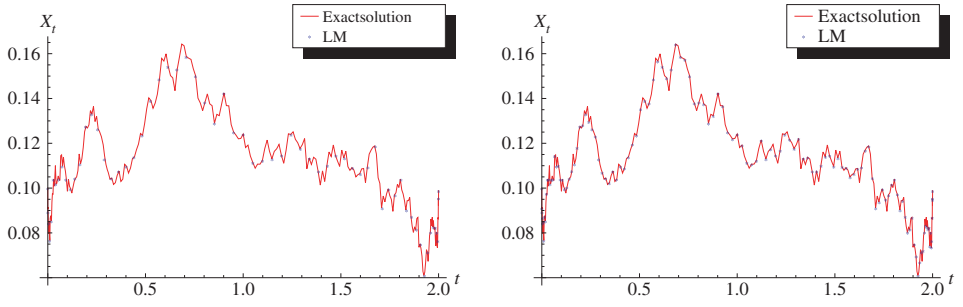

Figure 5. The Legendre spectral collocation method (LM) using $N = 2^6$ (left) and $N = 2^7$ (right) versus the exact solution.

The integral equation (29) can be solved by considering its corresponding differential form. In addition, by using Equation (12), we acquire $Y(t) = (1/a)(-\sinh(\frac{1}{10}) + \sinh X(t))$, while the inverse of the Lamperti transformation would be $\operatorname{arcsinh}(aY(t) + \sinh(\frac{1}{10}))$. Only in this example and since Tables 1 and 2 revealed the efficiency of LM, the results of LM over one sample path with $a = \frac{1}{20}$ are plotted. We do this to clearly reveal that how the spectral collocation methods (even with small number of partitions) are quite fascinating at the end points. The results are illustrated in Figures 4 and 5. They put on show the successful behaviour of LM for solving Equation (29).

Remark 3 When compared with the existing methods, it is essential to assert that a clear advantage of the proposed collocation approach lies in the accuracy of the final approximated solutions. In point of fact, transforming the whole SDE into a linear or nonlinear systems of algebraic equations intelligibly simplifies the process of solving SDEs in contrast to the typical solvers such as EM, MM or even Tamed Euler and stochastic Runge-Kutta.

Note that as observed above the new solution methods can also be applied for solving many other practical problems, such as the Black–Scholes SDE and the Heston model, to obtain spectral accuracies for showing the application prospects.

Example 4.4 We take into account the following two-dimensional linear Itô SDE:

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + BX(t)\,\mathrm{d}W(t), \quad 0 \le t \le T = 1, \tag{30}$$

where

$$A = \begin{pmatrix} -20 & 20 \\ 20 & -20 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{31}$$

while $X(t) = (X^{(1)}(t), X^{(2)}(t))^*$ and $W(t)$ is a standard Wiener process. The explicit solution of Equation (30) is known to be [12]

$$X(t) = \exp\left((A - \tfrac{1}{2}B^2)t + BW(t)\right)X(0), \tag{32}$$

where $X(0) = (X^{(1)}(0), X^{(2)}(0))^* = (1, 2)^*$.

The point is that although Equation (30) is linear, after incorporating the Lamperti transformation, we attain a nonlinear system of SDEs in what follows:

$$\begin{aligned} \mathrm{d}Y^{(1)}(t) &= -\frac{1 + 20(2 - 4\exp(-Y^{(1)}(t) + Y^{(2)}(t)))}{2}\mathrm{d}t + \mathrm{d}W(t), \\ \mathrm{d}Y^{(2)}(t) &= -\frac{1 - 20(-2 + \exp(-Y^{(2)}(t) + Y^{(1)}(t)))}{2}\mathrm{d}t + \mathrm{d}W(t). \end{aligned} \tag{33}$$

In general, when the SDE is a d-dimensional system, and we discretize the time interval into N partitions, then the resulting linear/nonlinear system is of dimensions $(\mathrm{d}N) \times (\mathrm{d}N)$. In this test, the values of the (absolute) error for one realization (produced by `SeedRandom[123]`) are reported in Tables 3 and 4. Here F stands for failure of the procedure to converge with even one digit of accuracy. Furthermore to compare the methods from the real-time property, it is necessary to fix a stopping termination $E(T) = \|X_N(T) - X(T)\| \leq 10^{-3}$ and see what the elapsed times would be. Accordingly, such a comparison is furnished in Table 5 for the mean over 50 realizations. The numerical results completely show the accuracy of the collocation methods, and specially LM.

The numerical results of this section demonstrate the effectiveness of the spectral collocation methods over similar methods and the existing well-known discretized schemes of EM and MM.

Table 3. Comparisons of mean results corresponding to $X^{(1)}(t)$ for Example 4.4.

N\Error	EM	MM	HM	LM
2^2	F	F	1.13709×10^{-2}	4.25589×10^{-3}
2^3	F	F	6.07888×10^{-3}	1.58385×10^{-3}
2^4	3.95364×10^{-1}	3.77124×10^{-1}	4.34987×10^{-4}	5.97974×10^{-7}
2^5	2.09386×10^{-1}	2.03398×10^{-1}	8.89447×10^{-7}	2.54441×10^{-8}
2^6	1.34173×10^{-1}	1.2045×10^{-1}	1.14034×10^{-11}	2.01228×10^{-15}

Table 4. Comparisons of mean results corresponding to $X^{(2)}(t)$ for Example 4.4.

N\Error	EM	MM	HM	LM
2^2	F	F	1.33546×10^{-2}	4.76235×10^{-3}
2^3	F	F	6.1784×10^{-3}	1.64304×10^{-3}
2^4	3.95362×10^{-1}	3.77796×10^{-1}	4.34397×10^{-4}	5.31125×10^{-6}
2^5	1.84843×10^{-1}	2.11571×10^{-1}	8.89338×10^{-7}	2.54686×10^{-8}
2^6	1.16868×10^{-1}	1.00901×10^{-1}	1.14034×10^{-11}	1.73472×10^{-15}

Table 5. Comparisons of mean results for the elapsed time in Example 4.4.

	EM	MM	HM	LM
Number of partitions	$N = 2^{14}$	$N = 2^{13}$	$N = 2^4$	$N = 2^4$
Mean of times	0.274316	0.189754	0.036102	0.034202

In fact, it has been shown that by applying the Lamperti transformation, we could reduce the instability of the SDE and obtain a corresponding SDE at which the noise is additive. Such a formulation allowed us to build a rapid numerical procedure via matrix formulations. In the meantime, we have obtained as high as possible of accuracy using the adaptive grid points by way of applying the roots of orthogonal polynomials. Tables 1–4 plainly disclosed the spectral accuracy of the proposed procedure in contrast to the existing discretization schemes. Finally, Table 5 uncovered that how the new procedure might be effective in terms of computational time in contrast to the fundamental schemes such as EM and MM.

5. Summary

Although there are some works in the literature so as to extend non-equidistant grid of points for SDEs, our work and discussions are apparently one of the first works that exploit this connection for the most important (shifted) orthogonal polynomials *in a systematic way*.

Here, a setting at which the initial condition vanished and the SDE transformed to a matrix problem using the Lamperti transformation had been applied. The Lamperti transformation is such that the multiplicative factor in front of the Wiener process no longer depends on the state of the process. This reduces the instability as well.

We have carried out several experiments and reported in Section 4. They overwhelmingly support the discussions of Sections 2 and 3 and nominate our general procedure for solving SDEs by incorporating them into some systems of linear or nonlinear algebraic equations.

Some open problems yet remained for further deep studies. An example is in the highest possible degree of interpolation for $p(t)$. Clearly, using lower degrees according to the approach of pseudo-spectral may result in great accuracies. Another future problem could concentrate on the extension of this approach by incorporating another time-stepping adaptation algorithm. These ideas may be considered for future works in this area.

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