Paper No. 1.7

ON HAWAIIAN GROUPS OF POINTED SPACES

Ameneh Babaee[1] and Behrooz Mashayekhy[2] and Hanieh Mirebrahimi[3]
[1] e-mail: am.babaee88@gmail.com,
[2] e-mail: bmashf@um.ac.ir,
[3] e-mail: h_mirebrahimi@um.ac.ir,
[1,2,3] Department of Pure Mathematics, Centre of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran.

Abstract

In this paper we investigate some behaviours of Hawaiian groups of pointed spaces with respect to their base points. It is known that Hawaiian groups are not independent to the choice of the points in general. We present some conditions of spaces whose Hawaiian groups are independent of the choice of base point.

Keywords: Hawaiian Earring, n-dimensional Hawaiian Earring, Homotopy group, Hawaiian group.

1 Introduction and Motivation

One-dimensional Hawaiian Earring $\mathbb{H}^1$ is defined to be the union of circles in the Euclidean plane $\mathbb{R}^2$ with center $(1/n, 0)$ and radius $1/n$ for $n = 1, 2, 3, \ldots$, equipped with the subspace topology.

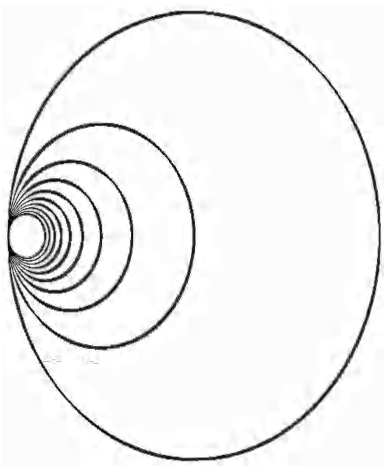


Figure 1: One-dimensional Hawaiian Earring.

In 2000, Hawaiian Earring extended to higher dimension. Eda et. al. [2] defined the n-dimensional Hawaiian Earring, $n \in \mathbb{N}$, as the following subspace of the $(n+1)$-Euclidean space $\mathbb{R}^{(n+1)}$

$$\mathbb{H}^n = \{(r_0, r_1, ..., r_n) \in \mathbb{R}^{(n+1)} \mid (r_0 - 1/k)^2 + \sum_{i=1}^{n} r_i^2 = (1/k)^2, k \in \mathbb{N}\},$$

which is the union of n-dimensional spheres S_k^n, with center $(1/k, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ and radius $1/k$ for $k = 1, 2, 3, \ldots$. Here $\theta = (0, 0, ..., 0)$ is regarded as the base point of $\mathbb{H}^n$.

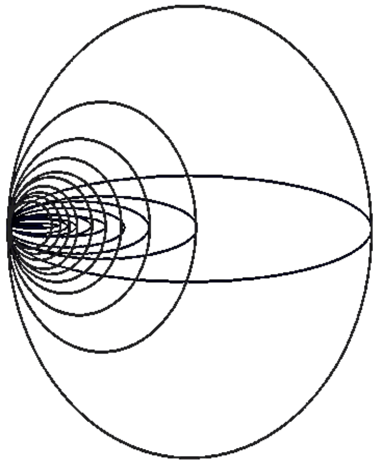


Figure 2: Two-dimensional Hawaiian Earring.

In 2006, Karimov et. al. [3], using n-dimensional Hawaiian Earring, for $n \in \mathbb{N}$, defined a new notion, the n-th Hawaiian group of a pointed space (X, x_0) to be the set of all pointed homotopy classes $[f]$, where $f : (\mathbb{H}^n, \theta) \to (X, x_0)$ is continuous, with a group operation which comes naturally from the operation of n-th homotopy group denoted by $\mathscr{H}_n(X, x_0)$. This correspondence induces a covariant functor $\mathscr{H}_n : hTop_* \to Groups$, from the pointed homotopy category, $hTop_*$, to the category of all groups, $Groups$, for $n \geq 1$.

There exists a natural relation between the Hawaiian groups and the homotopy groups of a pointed space (X, x_0). Karimov et. al. defines homomorphism $\varphi : \mathscr{H}_n(X, x_0) \to \prod_{k \in \mathbb{N}} \pi_n(X, x_0)$, with the rule $\varphi([f]) = ([f|_{S_1^n}], [f|_{S_2^n}], [f|_{S_3^n}], \ldots)$. The authors proved that the homomorphism φ can be a monomorphism in some senses and they present an inverse for it [1].

In [3] also some advantages of Hawaiian group functor is presented rather than other famous functors such as homotopy, homology and cohomology functors. In particular, there exists a contractible space with non-trivial 1-Hawaiian group.

Example 1.1 ([3]). Let $C(\mathbb{H}^1)$ be the cone over $\mathbb{H}^1$ which is contactible space, hence all its homotopy, homology and cohomology groups are trivial, but it is showed that $\mathscr{H}_1(C(\mathbb{H}^1), \theta)$ is uncountable [3].

The authors [1, Theorem 2.13], presented the structure of Hawaiian groups of any cone as follows.

Theorem 1.2 ([1]). *Let CX denote the cone over a space X, then*

$$\mathscr{H}_n(CX,\tilde{x}_t) \cong \frac{\mathscr{H}_n(X,x_0)}{\prod^{w}_{i\in\mathbb{N}}\pi_n(X,x_0)},$$

when $\tilde{x}_t = [x_0,t]$ and $t \neq 1$. It is known that if $t = 1$, then $\tilde{x}_t$ is the vertex of the cone CX. So $(CX,\tilde{x}_1)$ is pointed homotopy equivalent to a point whose Hawaiian groups are trivial.

Also, this functor can help us to get some local properties of spaces. In fact, if X has a countable local basis at x_0, then countability of the n-Hawaiian group $\mathscr{H}_n(X,x_0)$ implies n-locally simply connectedness of X at x_0 (see [3, Theorem 2]). By n-locally simply connectedness of X at x_0, we mean each open neighbourhood U of x_0 contains an open neighbourhood V of x_0 such that the homomorphism $\pi_n(V,x_0) \to \pi_n(U,x_0)$ induced by the inclusion is trivial.

In the following Theorem, the authors give some equivalent conditions for n-locally simply connectedness.

Theorem 1.3 ([1]). *Let (X,x_0) be a first countable pointed space and $n \geq 1$, then the following statements are equivalent.*
(i) X is n-locally simply connected at x_0.
(ii) $\varphi : \mathscr{H}_n(X,x_0) \to \prod^{w}\pi_n(X,x_0)$ is an isomorphism.
(iii) $\mathscr{H}_n(C(X),\hat{x}_t)$ is trivial, where $C(X)$ is the cone over X, $t \neq 1$ and $\hat{x}_t = (x_0,t)$.

By the above fact, this functor has useful advantages than homotopy group. For instance, it is known that in a path connected space, all homotopy groups are independent of the choice of points, but there exist some examples of path connected spaces with non-isomorphic Hawaiian group at several points (see Example 1.1).

Consequently, if two points $x_1,x_2 \in X$ satisfy $\mathscr{H}_n(X,x_1) \ncong \mathscr{H}_n(X,x_2)$, then there is no pointed homotopy equivalence between (X,x_1) and (X,x_2).

In this paper, we establish some conditions under which the Hawaiian groups of (X,x_1) and (X,x_2) with distinct points $x_1,x_2 \in X$ are isomorphic.

2 Main Results

Definition 2.1. Let (X,x_0) and (Y,y_0) be two pointed topological spaces. We say that (X,x_0) and (Y,y_0) are semi-locally $\mathscr{H}_n$-isomorphic, if there exist open neighbourhoods U of x_0 and V of y_0 such that

$$\mathscr{H}_n(U,x_0) \cong \mathscr{H}_n(V,y_0).$$

Remark 2.2. Let (X,x_0) and (Y,y_0) be two pointed spaces and let there exist neighbourhoods U of x_0, V of y_0 and a pointed homotopy equivalence $f : (U,x_0) \to (V,y_0)$. Then f induces isomorphism $\mathscr{H}_n(f) : \mathscr{H}_n(U,x_0) \to \mathscr{H}_n(V,y_0)$ and so (X,x_0) and (Y,y_0) are semi-locally $\mathscr{H}_n$-isomorphic.

Remark 2.3. By [1], for $n \geq 2$ and $\theta' \neq \theta$, we have

$$\mathscr{H}_n(\mathbb{H}^n, \theta) \ncong \mathscr{H}_n(\mathbb{H}^n, \theta').$$

So there is no pointed homotopy equivalence from $(\mathbb{H}^n, \theta)$ to $(\mathbb{H}^n, \theta')$, for $\theta' \neq \theta$. Because we know that $\mathscr{H}_n : hTop_* \to Groups$ is a functor and hence if two spaces (X, x_0) and (Y, y_0) are pointed homotopy equivalent, then $\mathscr{H}_n(X, x_0) \cong \mathscr{H}_n(Y, y_0)$.

This fact has a more strong version for homotopy groups; if two spaces (X, x_0) and (Y, y_0) are freely homotopy equivalent, then

$$\pi_n(X, x_0) \cong \pi_n(Y, y_0),$$

but it is not authentic for Hawaiian groups. In the following theorem, we intend to have a similar result.

Theorem 2.4. *Let (X, x_0) and (Y, y_0) be two semi-locally $\mathscr{H}_n$-isomorphic pointed spaces, and also let $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$, then*

$$\mathscr{H}_n(X, x_0) \cong \mathscr{H}_n(Y, y_0).$$

Sketch of proof. Let U and V be neighbourhoods of x_0 and y_0 respectively, and let $h : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ and $f : \mathscr{H}_n(U, x_0) \to \mathscr{H}_n(V, y_0)$ be isomorphisms. Now let $[\alpha] \in \mathscr{H}_n(X, x_0)$, since $\alpha : (\mathbb{H}^n, \theta) \to (X, x_0)$ is continuous, there exists $K \in \mathbb{N}$ such that if $k \geq K$, then $Im(\alpha|_{S_k^n}) \subseteq U$. Let K be the minimum integer for all elements of $[\alpha]$. Define $\psi : \mathscr{H}_n(X, x_0) \to \mathscr{H}_n(Y, y_0)$ by $\psi([\alpha]) = [\beta]$, in which $\beta|_{S_k^n} = \beta_k$ such that β_k is an element of class $h([\beta|_{S_k^n}])$ for $k < K$ and $\beta|_{\tilde{\bigvee}_{k \geq K} S_K^n} = \gamma|_{\tilde{\bigvee}_{k \geq K} S_K^n}$ when $\gamma \in f([\alpha|_{\tilde{\bigvee}_{k \geq K} S_K^n}])$. By [1, Lemma 2.2], β is continuous and ψ is an isomorphism. □

Corollary 2.5. *Let X and Y be freely homotopy equivalent and (X, x_0) and (Y, y_0) are semi-locally pointed homotopy equivalent, then*

$$\mathscr{H}_n(X, x_0) \cong \mathscr{H}_n(Y, y_0).$$

Proof. It is known that two free homotopy equivalent spaces have isomorphic homotopy groups, so by Theorem 2.4 the result holds. □

Example 2.6. Let $C\mathbb{H}^n$ be the cone over the n-dimensional Hawaiian Earring. By [1], we can conclude that

$$\mathscr{H}_n(C\mathbb{H}^n, \theta) \ncong \mathscr{H}_n(C\mathbb{H}^n, *).$$

So there is no pointed homotopy equivalent neighbourhoods of θ and $*$.

Corollary 2.7. *If (X, x_0) and (Y, y_0) are semi-locally pointed homotopy equivalent, then*

$$\mathscr{H}_n(CX, \tilde{x}_t) \cong \mathscr{H}_n(CY, \tilde{y}_t),$$

in which $\tilde{x}_t = [x_0, t]$ and $\tilde{y}_t = [y_0, t']$, $t, t' \neq 1$.

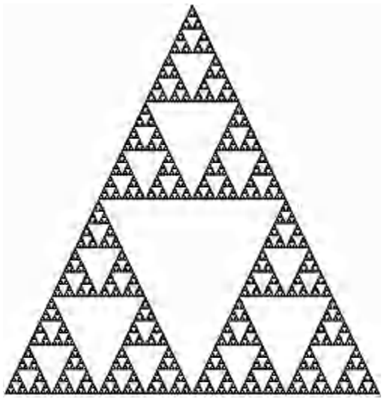
Proof. We know that the cone over any space X is contractible, and so $\pi_n(CX)$ is trivial. Consequently, Theorem 2.4 gives the result. □

Corollary 2.8. *Suppose x_0, x_1 are two points of space X with a path from x_0 to x_1, so that x_0 and x_1 have pointed homotopy equivalent open neighbourhoods. Then*

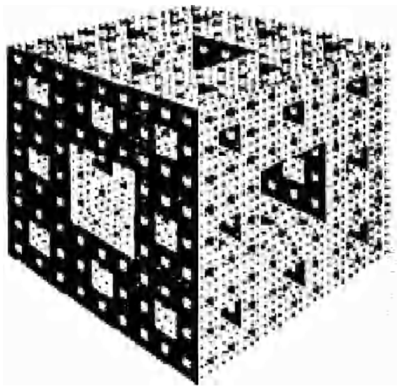
$$\mathscr{H}_n(X, x_0) \cong \mathscr{H}_n(X, x_1).$$

Proof. We know that if X is path connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for all distinct points $x_0, x_1 \in X$. So by Theorem 2.4, we conclude the result. □

Remark 2.9. As an example, if X is a self-similar path connected space such as Seipeińsky gasket or Menger sponge, then for all $n \in \mathbb{N}$, $\mathscr{H}_n(X)$ is independent of the choice of the base point.



Seipeińsky gasket



Menger sponge

References

[1] A. Babaee, B. Mashayekhy and H. Mirebrahimi, *On Hawaiian groups of some topological spaces*, Topology Appl. 159 (2012), no. 8, 2043–2051.

[2] K. Eda and K. Kawamura, *Homotopy and homology groups of the n_dimensional Hawaiian Earring*, Fundamenta Mathematicae 165 (2000), no. 1, 17–28.

[3] U.H. Karimov and D. Repovš, *Hawaiian groups of topological spaces* (Russian), Uspekhi. Mat. Nauk. 61 (2006), no. 5 (371), 185–186; transl. in Russian Math. Surv. 61 (2006), no. 5, 987–989.