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Exp-Kumaraswamy Distributions: Some Properties and Applications

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Abstract

In this paper, we propose and study exp-kumaraswamy distribution. Some of its properties are derived, including the density function, hazard rate function, quantile function, moments, skewness and kurtosis. A data set is used to illustrate an application of the proposed distribution. Also, we obtain a new distribution by transformation on exp-kumaraswamy distribution. New distribution is an alternative to skew-normal distribution. Basic properties of this new distribution, such as moment generating function, moments, skewness, kurtosis and maximum likelihood estimation are studied. Its applicability is illustrated by means of two real data sets.

Keywords: Log-exp-kumaraswamy distribution; Moments; Moment generating function; Kurtosis; Skewness.

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Introduction

Beta distribution is extremely versatile to model data restricted to any finite interval. This distribution has widespread applications in different areas. On the other hand, Kumaraswamy [1] pleads that beta distribution does not fit hydrological random variable well and proposed a new double bounded distribution named after him, say Kw distribution, as an alternative to beta distribution. The probability density function (pdf) and cumulative distribution function (cdf) of Kw distribution are as

$$f(x)=abx^{a-1}(1-x^{a})^{b-1}, 0<x<1, \quad (1)$$

and

$$F(x)=1-(1-x^{a})^{b}, \quad 0<x<1, \quad (2)$$

respectively, for $a,b>0$. Kw distribution has widely been used in hydrology and related areas [2-6]. According to Nadarajah [7], many papers in the hydrological literature have used this distribution because it is deemed as a better alternative to the beta distribution [8]. Jones [9] noted some similarities and differences between beta and Kw distributions. He emphasized several advantages of Kw distribution over the beta distribution: the normalizing constant is very simple; simple explicit formulae for the distribution and quantile functions which do not involve any special functions; a simple formula for random variate generation; and explicit formulae for moments of order statistics and L-moments. Further, according to Jones [9], the beta distribution has the following advantages over the Kw distribution: simpler formulae for moments and moment-generating function; a one-parameter sub-family of symmetric distributions; simpler moment estimation and more ways of generating the distribution via physical processes. Generalized Kw distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. Cordeiro and Castro [10] defined a new family of Kw generalized (Kw-G) distributions to extend several widely-known distributions such as the normal, Weibull, gamma and Gumbel distributions. Bourguignon et al. [11] Introduced Kumaraswamy Pareto (Kw-P) distribution and provided some structural properties of the proposed distribution including explicit expressions for the moments and generating function. Pascoa et al. [12] studied a four parameter lifetime distribution, so-called Kum-generalized gamma (KumGG) distribution which is a simple extension of the generalized gamma distribution. Cordeiro et al. [13] proposed a new four-parameter distribution called the Kumaraswamy generalized half-normal (Kw-GHN) distribution to extend thehalf-normal (HN) and generalized half-normal (GHN) distributions. Pescim et al. [14] defined a new family of Kummer beta generalized (KBG) distributions to extend several widely known distributions such as the normal, Weibull, gamma and Gumbel distributions. Nadarajah and Eljabri [15] proposed, simple generalization of GP distribution, Kumaraswamy GP (KumGP) distribution.

In this paper, we propose a generalization of Kw distribution. The generalization is motivated by the following general class,

$$F(x)=\begin{cases}\dfrac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}} & x\neq 0\\ G(x) & x=0\end{cases}, \quad (3)$$

where $\lambda\epsilon$R. The cdf $G(x)$ could be quite arbitrary and F is named exp-G distribution.

One major advantage of exp-G distribution is its flexibility for fitting a wide spectrum of real data sets. Barreto-Souza and Simas [16] obtained several mathematical properties of this class of distributions and discussed the two special cases: exp-Weibull and exp- beta distributions. Javanshiri et al. [17] introduced what is known as the exp-uniform distribution and provided closed form expressions for hazard function, density function and moments of order statistics and discussed the maximum likelihood estimation procedure, the asymptotic properties of estimations and various characterizations.

We introduce a three parameters distribution, exp-Kumaraswamy (exp-Kw) distribution, as an alternative to beta and Kw distributions. The pdf, cdf and hazard rate function(hrf) of exp-Kw distribution are as follows,

$$f(x)=\frac{ab\lambda x^{a-1}(1-x^{a})^{b-1}e^{\lambda(1-x^{a})^{b}}}{e^{\lambda}-1}, \quad 0<x<1, \quad (4)$$

$$F(x)=\frac{e^{\lambda}-e^{\lambda(1-x^{a})^{b}}}{e^{\lambda}-1}, \quad 0<x<1, \quad (5)$$

and

$$h(x)=\frac{ab\lambda x^{a-1}(1-x^{a})^{b-1}}{1-e^{-\lambda(1-x^{a})^{b}}}, \quad 0<x<1, \quad (6)$$

for $a,b>0,\lambda\neq 0$.

We provide three possible motivations for introducing exp-Kw distribution. First, exp-Kw distribution contains, as sub-models, truncated exponential distribution and truncated Weibull distribution. For $a=b=1$, Equation (4) gives truncated exponential distribution over (0,1). For

$b=1$, it yields truncated Weibull distribution over (0,1).

Second, in reliability and life testing experiments, many times data are modeled by finite range distributions [18]. The exp-Kw distribution, due to the flexibility of its hrf could be an important model in a variety of problems in survival analysis. Its hrf can be bathtub shaped, monotonically increasing and upside-down bathtub depending basically on the values of the parameters.

Our final and major motivation is based on a transformation.

Let $Y = \log\frac{X}{1-X}$, be a transformation from the unit interval to the whole real line. The distribution of Y is an interesting heavy-tail alternative to skew-normal distribution. This distribution has three parameters and its pdf is unimodal like skew-normal distribution. It has some advantages over skew-normal distribution. Its cdf, hrf and quantile function have closed form. Also, the ranges of skewness and kurtosis for this new distribution are larger than skew-normal distribution. So, the new distribution could be more appropriate for fitting skew and heavy tailed data.

Shape of exp-Kw distribution

Note from (4) that $f(x) \sim x^{a-1}$ as $x \to 0$, $f(x) \sim (1-x^a)^{b-1}$ as $x \to 1$ and,

- if $a>1$, $b>1$ and $\lambda \in R$, then $f(x)$ is unimodal.
- if $a<1$, $b<1$ and $\lambda \in R$, then $f(x)$ is uniantimodal.
- if $a>1$, $b\leq 1$ and $\lambda \in R$, then $f(x)$ is increasing or unimodal.
- if $a\leq 1$, $b>1$ and $\lambda \in R$, then $f(x)$ is decreasing or unimodal.
- if $a=b=1$ and $\lambda>0$, then $f(x)$ is decreasing.
- if $a=b=1$ and $\lambda<0$, then $f(x)$ is increasing.

Note from (6) that $h(x) \sim x^{a-1}$ as $x \to 0$, $h(x) \sim \frac{(1-x^a)^{b-1}}{1-e^{-\lambda(1-x^a)^b}}$ as $x \to 1$.

Also from (6) we see that $h(x)$ can be bathtub shaped, monotonically increasing and upside-down bathtub. Plots of the pdf (4) and hrf (6) for some values a, b and λ are given in Figures 1 and 2.

Quantile functions are in widespread use in general statistics, the quantile function of exp-Kw distribution is

$$Q(u) = (1-(1+\frac{\log(1-u(1-e^{-\lambda}))}{\lambda})^{\frac{1}{b}})^{\frac{1}{a}} . \qquad (7)$$

So, exp-Kw distribution is easily simulated by $X=Q(U)$, where U is an uniform random variable on the unit (0,1).

Assume that X is an exp-Kw random variable with parameters λ, a and b. We sumarize the relationship between exp-Kw distribution and some related distributions as follows,

- if a=b=1, then X has truncated exponential distribution on (0,1) with parameter λ.
- If b=1, then X has truncated distribution on (0,1) with parameters a and λ.
- X^r has exp-Kw distribution with parameters $\frac{a}{r}$, b and λ.
- $(1-(1-X^a)^b)^{\frac{1}{a}}$ has truncated Weibull distribution on (0,1) with parameters a and λ.
- If b=1, then $\frac{1}{X}$ has truncated Freche distribution on (1,+∞) with parameters a and λ.
- $\frac{1}{(1-(1-X^a)^b)^{\frac{1}{a}}}$ has truncated Fréchet distribution on (1,+∞) with parameters a, b and λ.
- If $\lambda=1$, then $-\log(1-(1-X^a)^b)$ has truncated Gumbel distribution on (0,+∞) with parameters 0 and 1.

Measures based on quantiles

The effect of the parameters on skewness and kurtosis can be considered based on quantile measures. Bowley skewness [19] and the Moors kurtosis [20] are defined as

$$B=\frac{Q(\frac{3}{4})+Q(\frac{1}{4})-2Q(\frac{1}{2})}{Q(\frac{3}{4})-Q(\frac{1}{4})}, \tag{8}$$

and

$$M=\frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})} \tag{9}$$

respectively. The above measures exist even for distributions without moments. We plot the measures B and M for exp-Kw distribution for some a and b, **as** functions of λ in Figure 3. These plots show that the Bowley skewness is increasing with λ (for any values of a and b), is decreasing whit a (for any values of λ and b) and is increasing with b (forany values of λ and a). In contrast, the Moors kurtosis first decreases to the minimum value and then increases, with λ (for some values of a and b).

The Bowley skewness can take positive and negative values. Table 1 gives intervals for the parameter λ (for some values of a and b), when the Bowley skewness is positive and negative. Table 2 gives the parameter λ (for some values of a and b) that give the minimum and maximum Moors kurtosis.

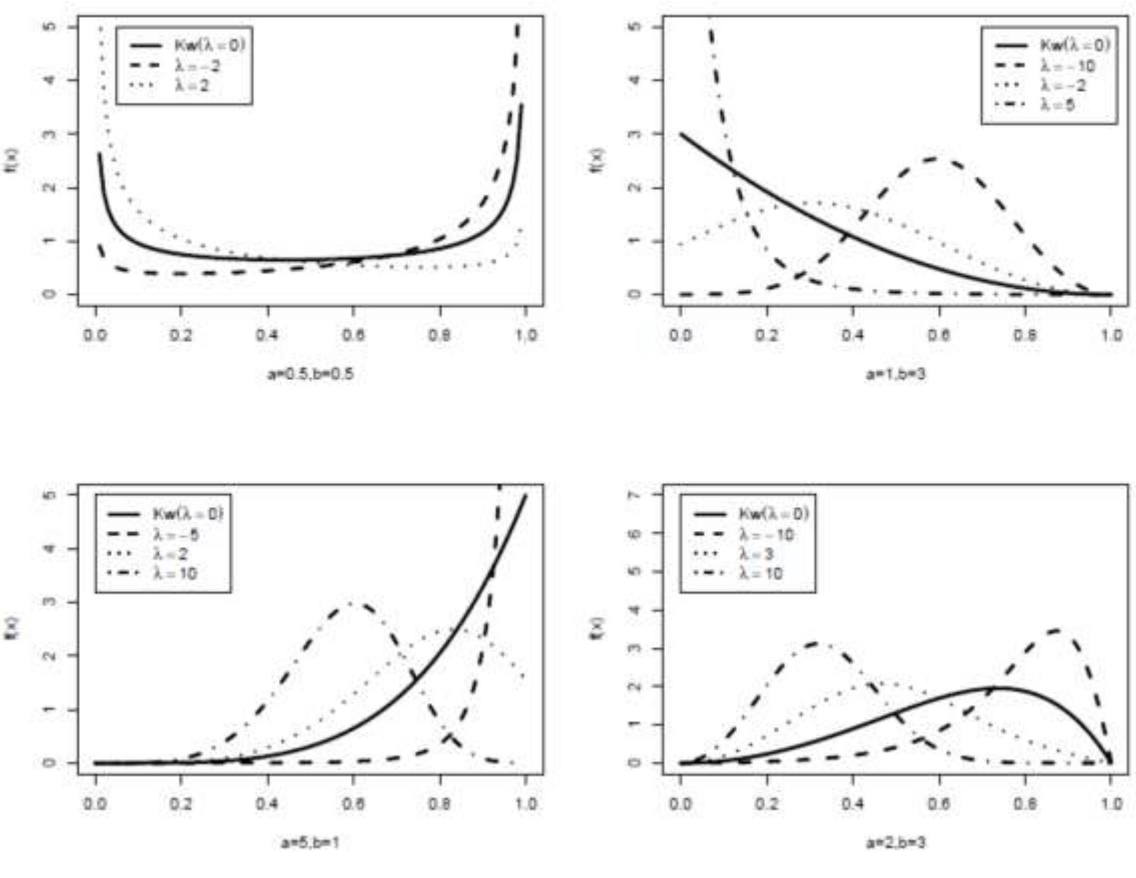

Figure 1. Pdf plots for some values of a, b and λ.

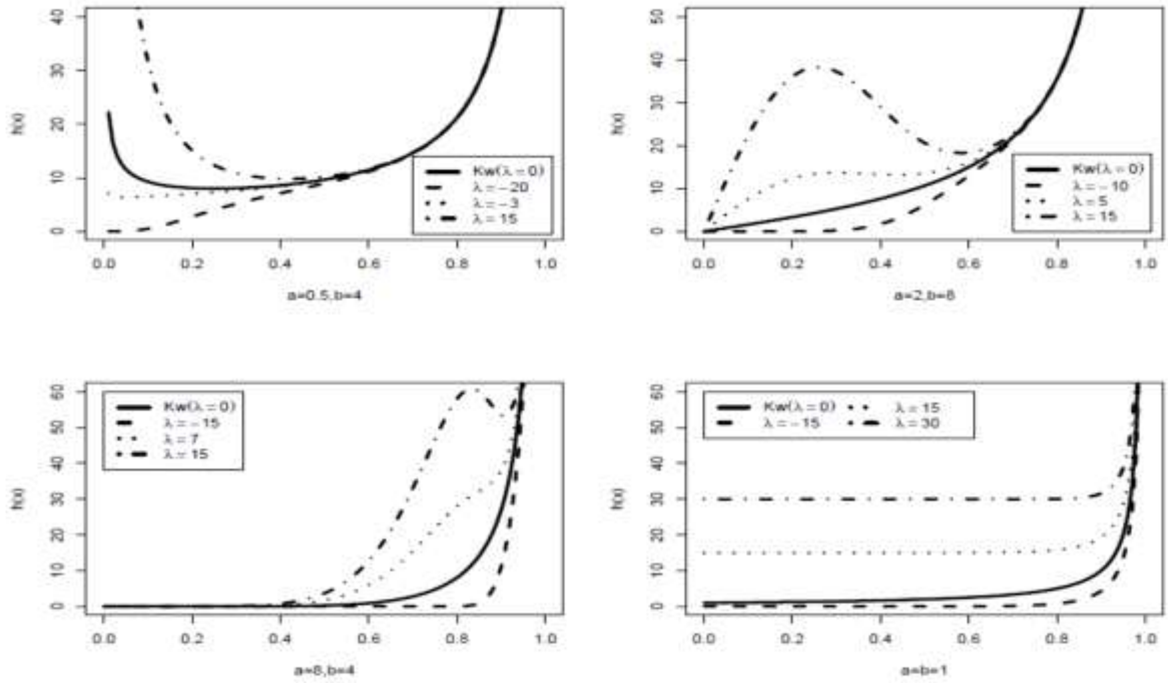

Figure 2. Hrf plots for some values of a,band λ.

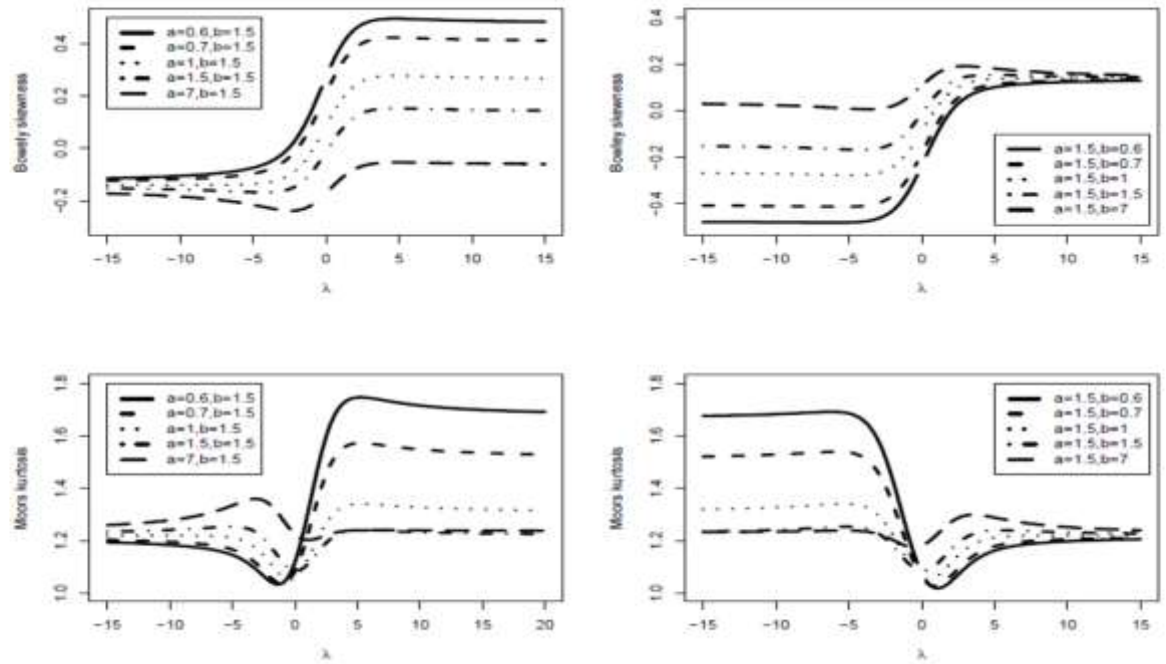

Figure 3. The Bowely skewness and Moors kurtosis of exp-Kw distribution as functions of λ for some values of a and b.

Table 1. Intervals for the parameter (for fixed a and b) for positive and negative B.

Some values of a and b	B<0	B>0
a=1.5,b=0.6	(-∞,2.04)	(2.04,+∞)
a=1.5,b=1	(-∞,0.87)	(0.87,+∞)
a=1.5,b=7	ø	(-∞,+∞)
a=0.6,b=1.5	(-∞,-2.60)	(-2.60,+∞)
a=1,b=1.5	(-∞,-0.87)	(-0.87,+∞)
a=7,b=1.5	(-∞,+∞)	ø

Table 2. Values of the parameter λ (for fixed a and b) for which M has minimum or maximum value.

Some values of a and b	Minimum value for M	Maximum value for M
a=1.5,b=0.6	1.08	-6.19
a=1.5,b=1	0.43	-5.31
a=1.5,b=7	-0.53	3.51
a=0.6,b=1.5	-1.34	5.22
a=1,b=1.5	-0.43	5.31
a=7,b=1.5	1.04	-3.12

Moments

Consider Y and X random variables with G and exp-G distributions, respectively. The r-th moment of X can be expressed in terms of $E(Y^r G(Y)^j)$ for j=0,1,… as defined by Barreto-Souza and Simas [16].

$$E(X^r) = \frac{\lambda}{1-e^{-\lambda}} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} E(Y^r G(Y)^j). \tag{10}$$

From (10) the r-th moment of exp-Kw distribution is

$$E(X^r) = \frac{\lambda}{1-e^{-\lambda}} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \int_0^1 (1-(1-u)^{\frac{1}{b}})^{\frac{r}{a}} u^j du. \tag{11}$$

Now we give a similar alternative expression to (11)

$$E(X^r) = \int_0^1 \frac{ab\lambda x^{r+a-1}(1-x^a)^{b-1} e^{\lambda(1-x^a)^b}}{e^{\lambda}-1} dx \tag{12}$$

Expanding $e^{\lambda(1-x^a)^b}$ in Maclaurin's series yields

$$E(X^r) = \frac{1}{e^{\lambda}-1} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} E(Y_k^r), \tag{13}$$

where Y_k has Kw distribution with parameters a and b(k+1). Hence we have

$$E(X^r) = \frac{\lambda b}{e^{\lambda}-1} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} B(1+\frac{r}{a}, b(k+1)) \tag{14}$$

The skewness and kurtosis measures can be calculated using the following relationships, respectively

$$S(X) = \frac{E(X-E(X))^3}{(E(X-E(X))^2)^{\frac{3}{2}}},$$

$$K(X) = \frac{E(X-E(X))^4}{(E(X-E(X))^2)^2} \tag{15}$$

Figure 4 shows skewness and kurtosis for some choices of a and b as a function of λ.

Estimation parameters of exp-Kw distribution

Suppose $x_1,\dots,x_n$ constitute a random sample from an exp-Kw distribution with density (4). The log-likelihood function is given by

$$\ell(\lambda,a,b) = n\log(a) + n\log(b) + n\log(\frac{\lambda}{e^{\lambda}-1}) + (a-1)\sum_{i=1}^{n}\log(x_i) + (b-1)\sum_{i=1}^{n}\log(1-x_i^a) + \lambda\sum_{i=1}^{n}(1-x_i^a)^b.$$

It follows that the maximum likelihood estimators (MLEs) are the solution of the equations:

$$n(\frac{1}{\lambda} - \frac{1}{1-e^{-\lambda}}) + \sum_{i=0}^{n}(1-x_i^a)^b = 0,$$

$$n + \sum_{i=1}^{n} \frac{\log(x_i^a)}{1-x_i^a}(1-bx_i^a(1+\lambda(1-x_i^a)^b) = 0,$$

$$n + b\sum_{i=1}^{n}\log(1-x_i^a)(1+\lambda(1-x_i^a)^b = 0,$$

The above equations can be solved numerically by using R software [22].

Applications of exp-Kw distribution

This section contains an application of exp-Kw distribution to real data. The data are shape measurements of 48 rock samples from a petroleum reservoir. Data were extracted from BP Research, image analysis by Ronit Katz, U. Oxford. The data set is:

0.0903296, 0.148622, 0.183312, 0.117063, 0.122417, 0.167045, 0.189651, 0.164127, 0.203654, 0.162394, 0.150944, 0.148141, 0.228595, 0.231623, 0.172567, 0.153481, 0.204314, 0.262727, 0.200071, 0.144810, 0.113852, 0.291029, 0.240077, 0.161865, 0.280887, 0.179455, 0.191802, 0.133083, 0.225214, 0.341273, 0.311646, 0.276016, 0.197653, 0.326635, 0.154192, 0.276016, 0.176969, 0.438712, 0.163586, 0.253832, 0.328641, 0.230081, 0.464125, 0.420477, 0.200744, 0.262651, 0.182453, 0.200447.

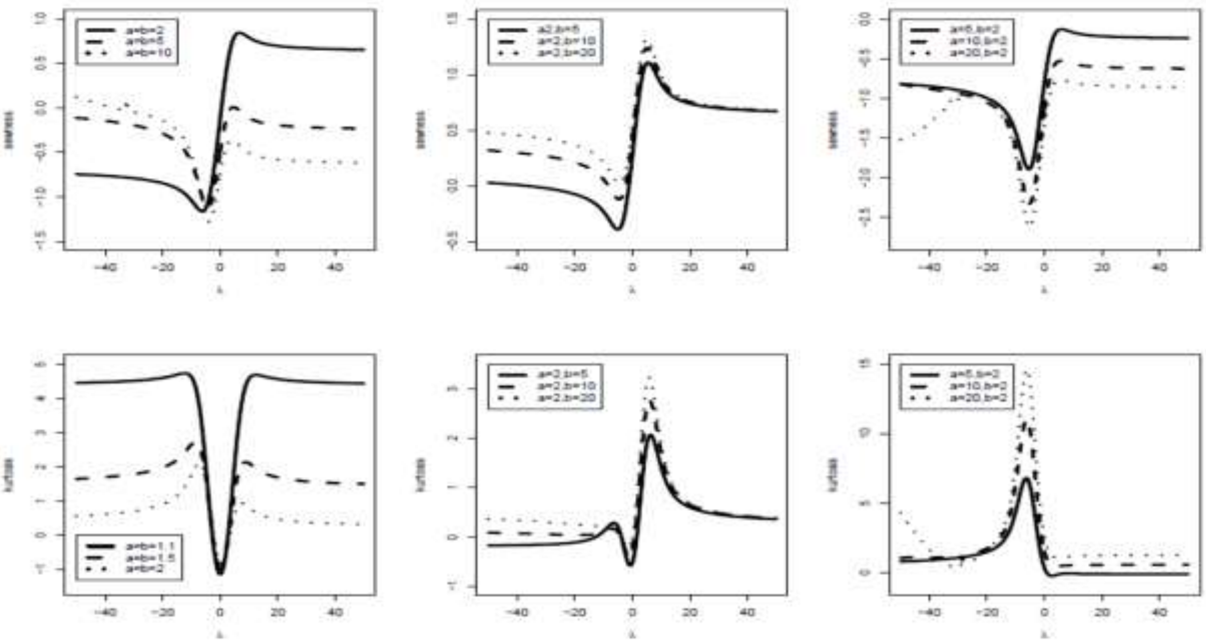

Figure 4. The skewness and kurtosis of exp-Kw distribution as functions of λ for somevalues ofa and b.

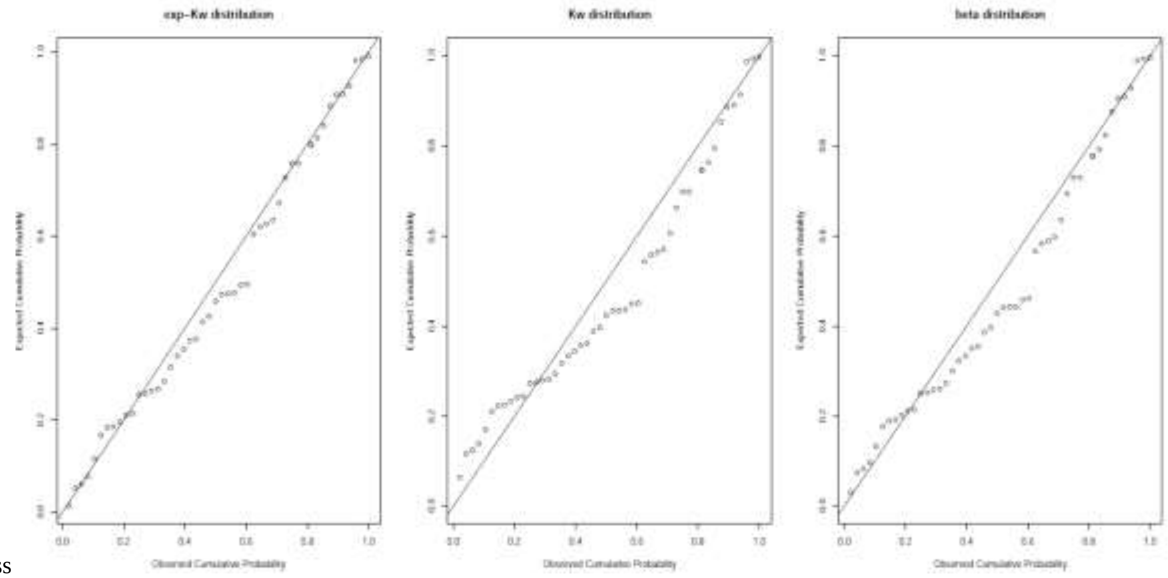

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Figure 5. Probability plots for the fit of the exp-Kw distribution, Kw distribution and beta distribution.

Table 3. MLEs of parameters, $\hat{\ell}$, p-values of Kolmogotoov-Smirnov test, AICc and BIC.

Model	Estimated parameters	$\hat{\ell}$	p-value	AICc	BIC
Exp-Kw(λ, a, b)	$\hat{\lambda} = -32.71, \hat{a} = 0.67, \hat{b} = 9.15$	57.82	0.63	-109.64	-104.03
Kw(a,b)	$\hat{a} = 2.71,\ \hat{b} = 44.04$	52.49	0.21	-100.98	-97.24
Beta (α, β)	$\hat{a} = 5.94$, $\hat{\beta} = 21.20$	55.60	0.28	-107.20	103.46

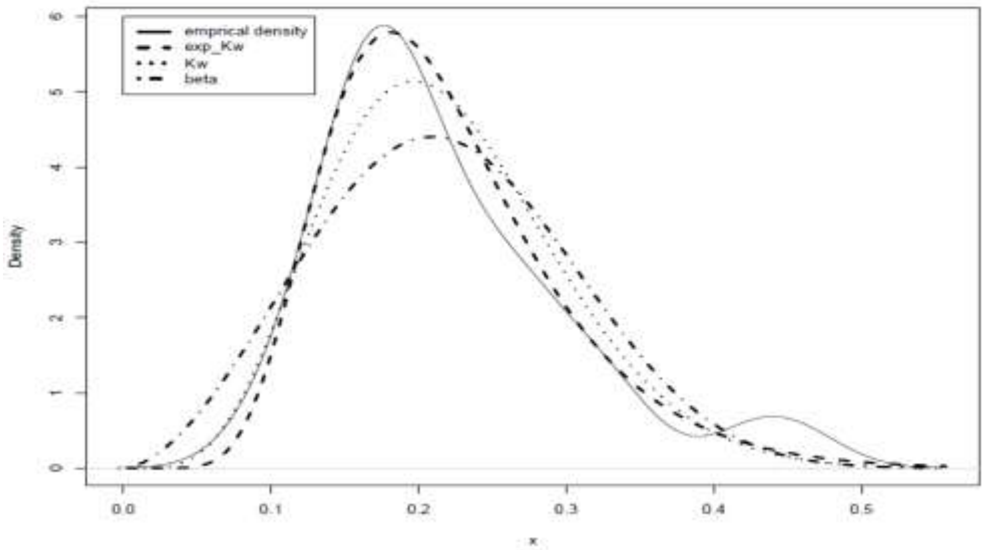

Figure 6. Histogram and estimated pdfs for the shape measurements of 48 rock samplesfrom a petroleum reservoir.

Now, we fit exp-Kw distribution, Kw distribution and beta distribution to the data. The MLEs of the parameters and the maximized log-likelihood ($\hat{\ell}$) for this distributions are computed.

The results of goodness of fit tests based on Kolmogorov-Smirnov test, the evaluation of the corrected Akaike information criterion (AICc) [23,24] and Bayesian information criterion (BIC) [25] are shown in Table 3. From Table 3, we see that the exp-Kw distribution is a better fit, judging on the basis of p-values, AICcs and BICs. The probability plots are given in Figure 5 and the histogram of the data is shown in Figure 6 along with the estimated densities of the exp-Kw, Kw and beta distributions. Apparently, the exp-Kw distribution gives the best fit.

Log-exp-Kw distribution

Transformation $Y = \log \frac{X}{1-X}$ yields log-exp-Kw distribution as an alternative to skew-normal distribution with pdf, cdf and hrf respectively as

$$g(y) = \frac{ab\lambda e^{-y}((1+e^{-y})^a - 1)^{b-1} e^{\lambda(1-(\frac{1}{1+e^{-y}})^a)^b}}{(e^{\lambda}-1)(1+e^{-y})^{ab+1}} \tag{16}$$

$$G(y) = \frac{1-e^{-\lambda} e^{\lambda(1-(\frac{1}{1+e^{-y}})^a)^b}}{1-e^{-y}}, \tag{17}$$

$$h(y) = \frac{ab\lambda e^{-y}((1+e^{-y})^a - 1)^{b-1} e^{\lambda(1-(\frac{1}{1+c^{-y}})^a)^b}}{(e^{\lambda(1-(\frac{1}{1+e^{-y}})^a)b} - 1)(1+e^{-y})^{ab+1}} \tag{18}$$

for a,b>0, $\lambda \neq 0$. Plots of the pdf (16) and hrf (18) for some special values of a,b and λ are given in Figures 7 and 8. The quantile function corresponding to (16) is

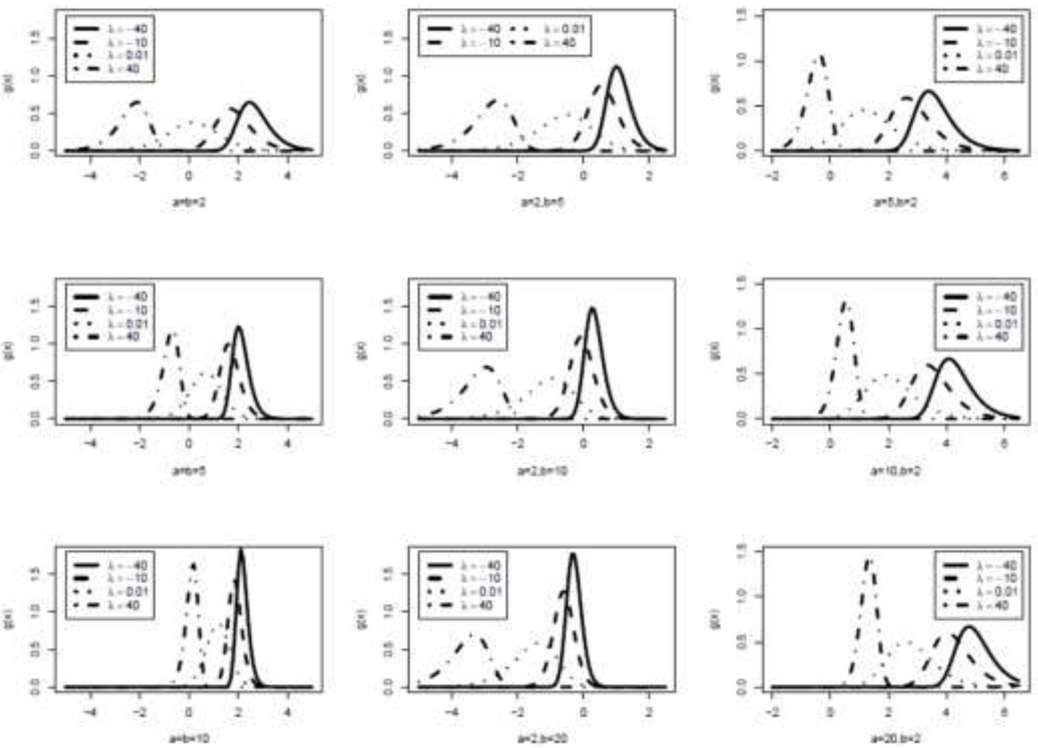

Figure 7. Plots of log-exp-Kw pdf for some values of a,b and λ

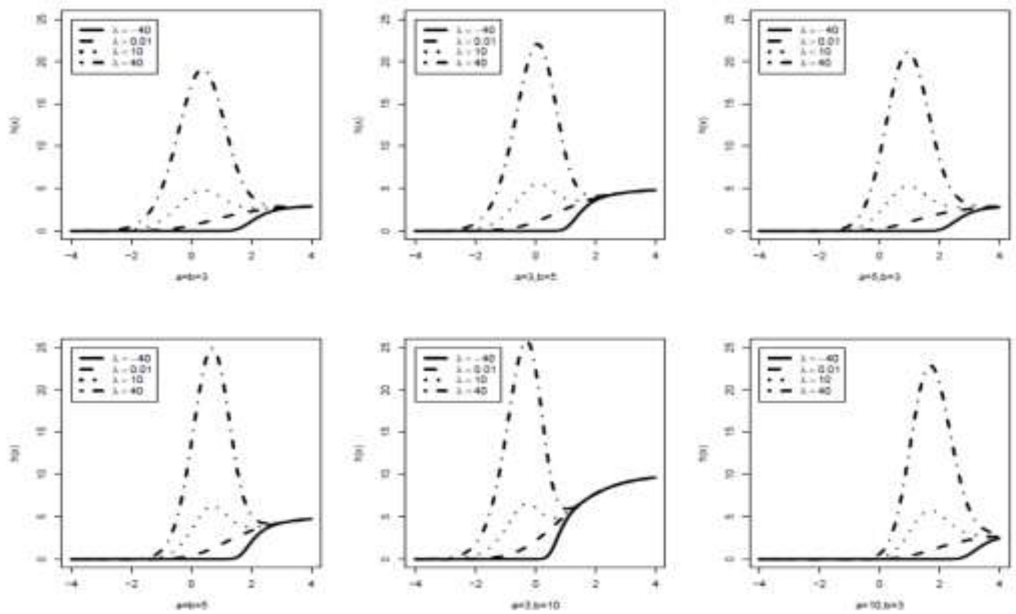

Figure 8. Plots of log-exp-Kw hrf for some values of a,b and λ

$$Q(u)=\log\left(\frac{(1-(\frac{\log(1-(1-e^{\lambda})u}{\lambda}+1)^{\frac{1}{b}})^{\frac{1}{a}}}{1-(1-(\frac{\log(1-(1-e^{-\lambda})u)}{\lambda}+1)^{\frac{1}{a}}}\right) \qquad (19)$$

Simulation of log-exp-Kw distribution is easy by Y = Q(U), where U is a uniform over (0,1).

Expansions for moment generating function and moments

The mgf of log-exp-Kw distribution is given by

$$M(t)=\frac{ab\lambda}{e^{\lambda}-1}\int_0^1 x^{t+a-1}(1-x)^{-t}(1-x^a)^{b-1}e^{\lambda(1-x^a)^b}dx. \qquad (20)$$

Expanding $e^{\lambda(1-x^a)^b}$ in Maclaurins series yields

$$M(t)=\frac{ab\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\int_0^1 x^{t+a-1}(1-x)^{-t}(1-x^a)^{b(k+1)-1}dx \qquad (21)$$

Now, we expand $(1-x^a)^{b(k+1)-1}$ in Maclaurin's series

$$M(t)=\frac{ab\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{+\infty}(-1)^j\binom{b(k+1)-1}{j}\int_0^1 x^{t+a+aj-1}(1-x)^{-t}dx. \qquad (22)$$

The above expression shows that the mgf of log-exp-Kw distribution exist if t<1. So, for t<1 we have

$$M(t)=\frac{ab\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{+\infty}(-1)^j\binom{b(k+1)-1}{j}B(t+a+aj,i-t)\ , \qquad (23)$$

The r-th moment of the log-exp-Kw distribution can be obtained froms $E(Y^r)=\frac{d^r M_Y(t)}{dt^r}|_t=0$. So, we have

$$E(Y^r)=\frac{ab\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{+\infty}(-1)^j\binom{b(k+1)-1}{j}\frac{\partial^r B(t+a+aj,i-t)}{\partial t^r}|_{t=0}\,. \tag{24}$$

In particular, the first four moments of Y are as follows

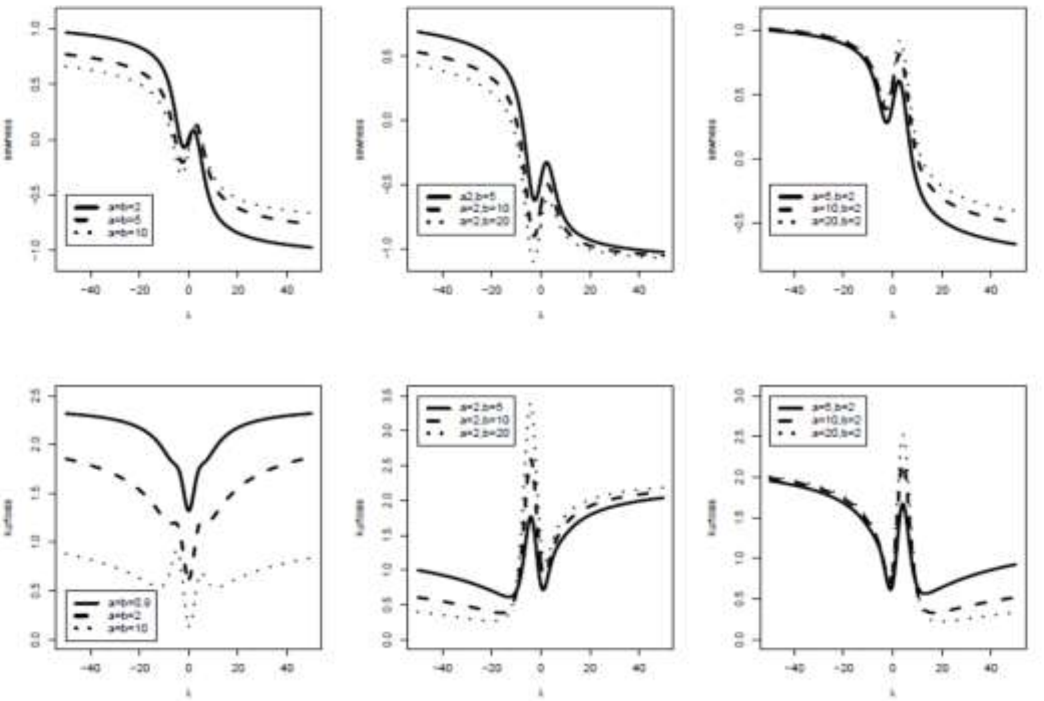

Figure 9. Skewness and kurtosis of log-exp-Kw distribtion for some choices of a and b asfunction of λ

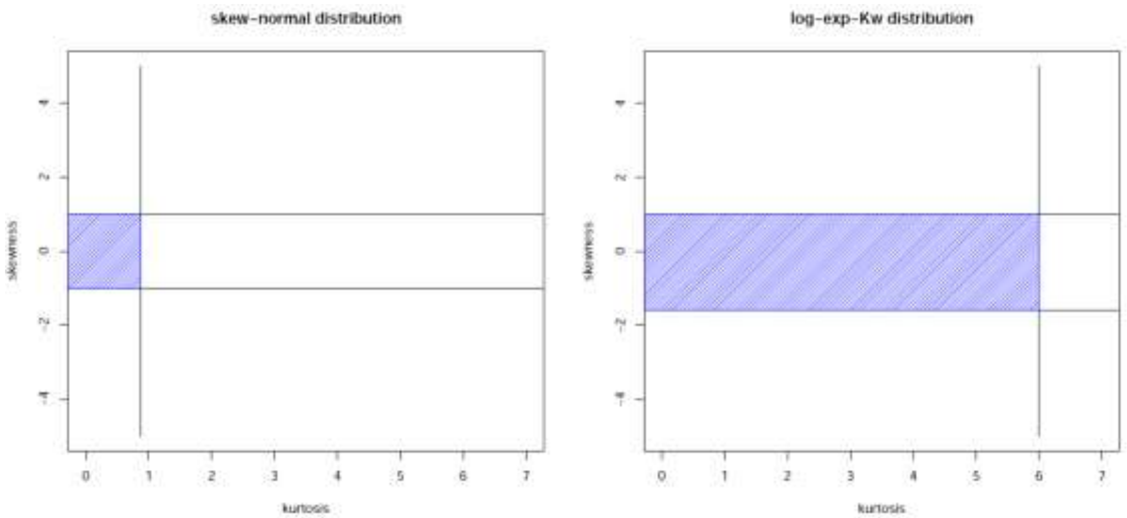

Figure 10. The ranges of skewness and kurtosis of the log-exp-Kw distribution and skew-normal distribution.

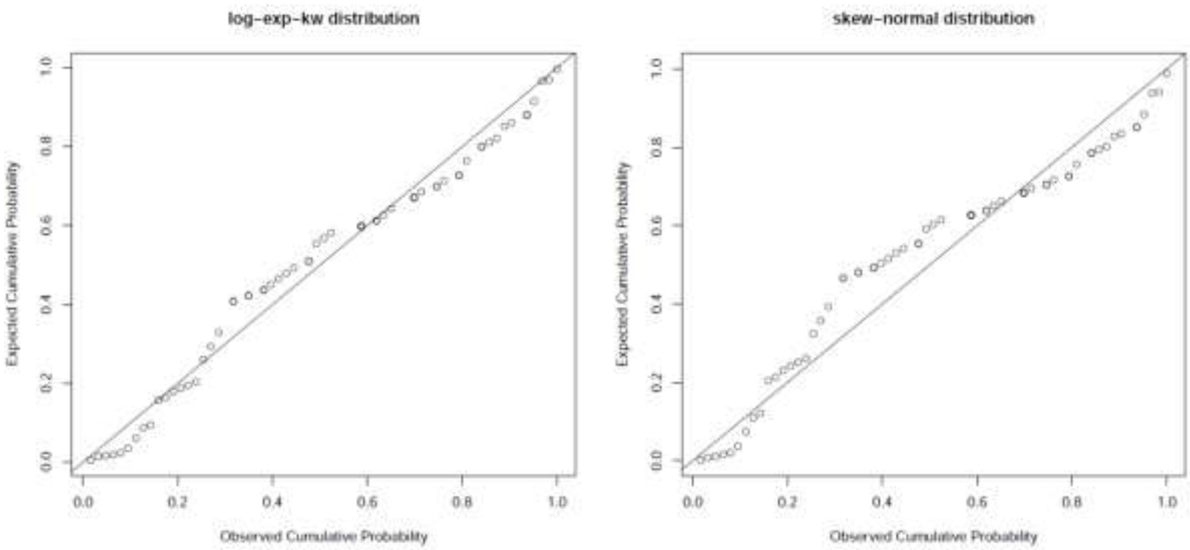

Figure 11. Probability plots for the fit of the log-exp-Kw distribution and skew-normal distribution.

$$E(Y)=\frac{b\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{+\infty}(-1)^j\frac{\Gamma(b(k+1))}{\Gamma(j+2)\Gamma(b(k+1)-j)}(\psi(aj+a)-\psi(1)),$$

$$E(Y^2)=\frac{b\lambda}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(b(k+1))}{\Gamma(j+2)\Gamma(b(k+1)-j)}(\psi(aj+a)-\psi(1))^2+(\psi'(aj+a)+\psi'(1)),$$

$$E(Y^3)=\frac{b\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(b(k+1))}{\Gamma(j+2)\Gamma(b(k+1)-j)}(\psi(aj+a)-\psi(1))^3+3(\psi(aj+a)-\psi(1))(\psi'(aj+a)+\psi'(1))+(\psi''(aj+a)+\psi''(1)),$$

and

$$E(Y^4)=\frac{b\lambda}{e^{\lambda}-1}\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}\sum_{j=0}^{+\infty}(-1)^j\frac{\Gamma(b(k+1))}{\Gamma(j+2)\Gamma(b(k+1)-j)}(\psi(aj+a)-\psi(1))^4+6(\psi(aj+a)-\psi(1))^2(\psi'(aj+a)+\psi'(1))+3(\psi'(aj+a)+\psi'(1))^2+4(\psi(aj+a)-\psi(1))(\psi''(aj+a)+\psi''(1))+(\psi'''(aj+a)-\psi'''(1)),$$

Where ψ(a) is $\frac{\Gamma(a)}{\Gamma(a)}=\frac{d\log\Gamma(x)}{dx}|_{x=a}$.

The skewness and kurtosis of Y can be obtained from (15). Plots of the skewness and kurtosis for some choices of a and b as functions of λ, are given in Figure 9. In Figure 10, we compare the ranges of kurtosis and skewness of log-exp-Kw distribution with those of skew-normal distribution, respectively. These plots indicate that the ranges of skewness and specially kurtosis in log-exp-Kw distribution are larger than those of skew-normal distribution.

Estimation of parameters of log-exp-Kw distribution

Here, we consider estimation by method of maximum likelihood. Suppose $y_1,\dots y_n$ constitute a random sample from (16), the log-likelihood function is

$$\ell(\lambda,a,b)=n\log(a)+n\log(b)+n\log(\frac{\lambda}{e^{\lambda}-1})-\sum_{i=1}^{n}y_i-(ab+1)\sum_{i=1}^{n}\log(1+e^{-y_i})+(b-1)\sum_{i=1}^{n}\log((1+e^{-y_i})^a-1)+\lambda\sum_{i=1}^{n}(1-\frac{1}{(1+e^{-y_i})^a})^b.$$

It follows that the MLEs are the solution of the equations:

$$n(\frac{1}{\lambda}-\frac{1}{1-e^{-\lambda}})+\sum_{i=0}^{n}(1-\frac{1}{t_i^a})^b=0,$$

$$n+a\sum_{i=1}^{n}\log(t_i)(\frac{b-t_i^a}{t_i^a-1}+\frac{b\lambda(t_i^a-1)^{b-1}}{t_i^{ab}})=0,$$

$$n+b\sum_{i=1}^{n}(1+\lambda(1-\frac{1}{t_i^a})^b)\times\log(1-\frac{1}{t_i^a})=0,$$

Where $t_i=\frac{1}{1+e^{-y_1}}$ for i=1,...,n. the above equations can be solved numerically by using R software [22].

Applications of log-exp-Kw distribution

In this section we fit log-exp-Kw distribution to real data sets. We provide two examples.

Example 1: The data is obtained from Smith and Naylor [26] represent the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. The data set is: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2,0.74, 1.04, 1.27, 1.39,1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62,1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84,0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

Now, we fit log-exp-Kw distribution and skew-normal distribution to the data. The MLEs of the parameters and the $\hat{\ell}$ for this distributions are computed.The results of goodness of fit tests based on Kolmogorov-Smirnov test, the evaluationof the AIC_c and BIC are shown in Table 4. From Table 4, we see that the log-exp-Kw distribution is a better fit, judging on the basis of p-values, AIC_cs and BICs. The probability plots given in Figure 11, also show that log-exp-Kw distribution gives a better fit than skew-normal distribution.

Example 2: The data is obtained from Nichols and Padgett [21] and it represents the breaking stress of carbon fibres (in Gba), (n = 100). Log-exp-Kw distribution and skew-normal distribution are fitted to the data set. The results of Kolmogorov-Smirnov test, AIC_c and BIC are reported in Table 5.

From Table 5, we see that in both cases, we cannot reject the null hypothesis that data are coming from log-exp-Kw distribution or skew-normal distribution. On the other hand, the values of AICc , BIC and the probability plots given in Figure 12 are almost the same for both models. So, we compare them based on hrfs.

Table 4. MLEs of parameters, $\hat{\ell}$, P-values of Kolmogorov-Smirnov test, AIC_c and BIC**.**

Model	**Estimated parameters**	$\hat{\ell}$	**p-value**	**AIC_c**	**BIC**
Log-exp-Kw (λ,a,b)	$\hat{\lambda} = -2.28, \hat{a} = 16.84, \hat{b} = 34.19$	-13.54	0.31	33.08	39.51
Skew-normal (ξ, ω^2, α)	$\hat{\xi} = 1.51, \hat{\omega}^2 = 0.32, \hat{\alpha} = 0.005$	-17.91	0.03	41.82	48.25

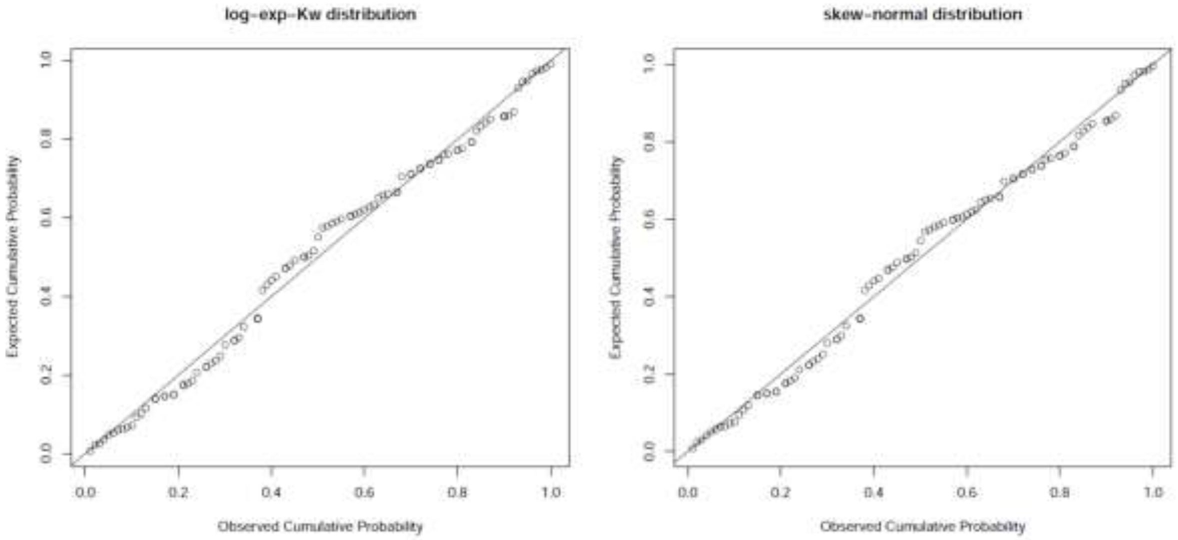

Figure 12. Probability plots for the fit of the new and skew-normal distribution to the data

Table 5. MLFs of parameters, $\hat{\ell}$, P-values of Kolmogorov-Smirnov test, AIC_cs and BIC_s

Model	**Estimated parameters**	$\hat{\ell}$	**p-value**	**AIC_c**	**BIC**
Log-exp-Kw (λ,a,b)	$\hat{\lambda} = -1.28, \hat{a} = 9.77, \hat{b} = 1.6$	-141.76	0.652	289.52	297.33
Skew-normal (ξ, ω^2, α)	$\hat{\xi} = 1.68, \hat{\omega}^2 = 1.38, \hat{\alpha} = 1.63$	-141.70	0.753	289.41	297.22

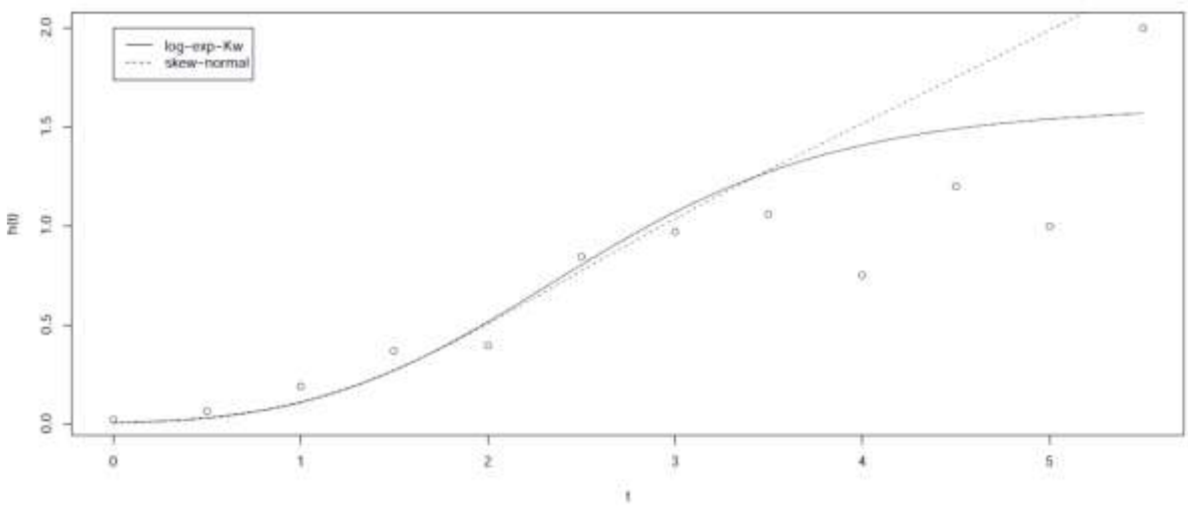

Figure 13. Estimated hrfs of the new and skew-normal distributions for the data set.

The plots of empirical Hrf of data and fitted hrfs are shown in Figure 13. This Figure indicates that the hrf of log-exp-Kw distribution has a better fit than that of skew-normal distribution. As a further check, we calculate the mean square difference of emprical hrf and estimated hrf for both distributions. This error is 0.09 for log-exp-Kw distribution and is 0.17 for skew-normal distribution. Hence, the log-exp-Kw distribution has a better fit to the data, in terms of hrf.

Results and Discussion

We studied exp-Kw distribution. We derived various properties of this distribution, including the shape of pdf and hrf, related distributions, moments, skewness, kurtosis and estimation of parameters. The exp-Kw distribution, due to the flexibility of its hrf could be an important model in a variety of problems in survival analysis. Also, a new distribution by transformation on exp-Kw distribution, as an alternative to skew-normal distribution, is obtained. This distribution has three parameters and its pdf is unimodal like skew-normal distribution. It has some advantages over skew-normal distribution. Its

Cdf, hrf and quantile function have closed form. Also, the ranges of skewness and kurtosis for this new distribution are larger than skew-normal distribution. So, the new distribution could be more appropriate for fitting skew and heavy tailed data. Application of distributions to real data sets was shown by three examples, showing that these distributions, can be used effectively in analysing data.

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