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KINDS OF DERIVATIONS ON HILBERT C^*-MODULES AND THEIR OPERATOR ALGEBRAS

HOSSEIN SAIDI, ALI REZA JANFADA, AND MADJID MIRZAVAZIRI

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Abstract. Let $\mathcal{M}$ be a Hilbert C^*-module. A linear mapping $d:\mathcal{M}\to\mathcal{M}$ is called a derivation if $d(<x,y>z) = <dx,y>z + <x,dy>z + <x,y>dz$ for all $x,y,z\in\mathcal{M}$. We give some results for derivations and automatic continuity of them on $\mathcal{M}$. Also, we will characterize generalized derivations and strong higher derivations on the algebra of compact operators and adjointable operators of Hilbert C^*-modules, respectively.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be a C^*-algebra. A $pre-Hilbert\quad \mathcal{A}-module\quad \mathcal{M}$ is a left $\mathcal{A}$-module equipped with a sesquilinear form $<\cdot,\cdot>:\mathcal{M}\times\mathcal{M}\longrightarrow\mathcal{A}$ which satisfies the following axioms for all $x,y\in\mathcal{M}$ and $a\in\mathcal{A}$:
(1) $<x,x>\geq 0$;
(2) $<x,x>=0 \Longleftrightarrow x=0$;
(3) $<x,y>^*=<y,x>$;
(4) $<ax,y>=a<x,y>$.
For every $x\in\mathcal{M}$, set $\| x \|=\| <x,x> \|^{1/2}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to this norm is called a $Hilbert\quad \mathcal{A}-module$. For example, a complex Hilbert space H is a Hilbert C^*-module over the C^*-algebra of complex numbers or a C^*-algebra $\mathcal{A}$ is a Hilbert C^*-module over $\mathcal{A}$ by $<a,b>=ab^*$, for all $a,b\in\mathcal{A}$. A linear mapping $T:\mathcal{M}\to\mathcal{M}$ is called an $operator$ if T is continuous and $\mathcal{A}$-linear (i.e. $T(ax)=aT(x)$ for all $a\in\mathcal{A}$ and $x\in\mathcal{M}$). By $End(\mathcal{M})$, we denote the set of all operators on $\mathcal{M}$. A mapping $T:\mathcal{M}\to\mathcal{M}$ is called $adjointable$ if there exists a mapping $T^*:\mathcal{M}\to\mathcal{M}$ such that $<Tx,y>=<x,T^*y>$ for all $x,y\in\mathcal{M}$. As a well-known result, every adjointable mapping $T:\mathcal{M}\to\mathcal{M}$ is an operator. The set of all adjointable mappings on $\mathcal{M}$ is denoted by $End^*(\mathcal{M})$ which is a C^*-algebra under the usual operator norm. For $x,y\in\mathcal{M}$, define $\theta_{x,y}:\mathcal{M}\to\mathcal{M}$ by $\theta_{x,y}(z)=<z,y>x$, for all $z\in\mathcal{M}$. Clearly, $\theta_{x,y}\in End^*(\mathcal{M})$ with $\theta^*_{x,y}=\theta_{y,x}$.

Note that $\theta_{x,y}$ is quite different from rank one projections in Hilbert spaces. For example we can not infer $x = 0$ or $y = 0$ from $\theta_{x,y} = 0$. We denote by $\mathcal{K}(\mathcal{M})$ the closed linear span of $\{\theta_{x,y} : x, y \in \mathcal{M}\}$. The elements of $\mathcal{K}(\mathcal{M})$ are called *compact operators*. This concept of compact operators is different from compact operators in the usual sense. However, this concept coincides with the concept of usual compact operators when we choose a Hilbert space as a Hilbert C^*-module. Set $I = \text{span}\{< x, y >: x, y \in \mathcal{M}\}$. It is easy to see that I is a $*$-bi-ideal of $\mathcal{A}$. An important class of Hilbert C^*-modules are *full* modules. A Hilbert C^*-module $\mathcal{M}$ is called full if $\overline{I} = \mathcal{A}$, where $\overline{I}$ is the norm closure of I in $\mathcal{A}$. For example, $\mathcal{A}$ is a full $\mathcal{A}$-module. It is well-known that the derivations on Banach algebras are the generators of certain dynamical systems. A linear mapping $\phi : \mathcal{M} \to \mathcal{M}$ is called a *homomorphism* if $\phi(< x, y > z) =< \phi x, \phi y > \phi z$ for all $x, y, z \in \mathcal{M}$. A dynamical system on $\mathcal{M}$ is a strongly continuous one-parameter family $(u_t)_{t\in\mathbb{R}}$ of homomorphisms. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a *derivation* if $d(< x, y > z) =< dx, y > z+ < x, dy > z+ < x, y > dz$ for all $x, y, z \in \mathcal{M}$, see [1] and [2]. In [1], Abbaspour and Skeide proved that a C_0-group $u = (u_t)_{t\in\mathbb{R}}$ is a dynamical system if and only if its generator is a derivation and every derivation on full Hilbert C^*-module $\mathcal{M}$ is a generalized derivation i.e. there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ax) = \delta(a)x + ad(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}$. Also, they proved that every derivation on full Hilbert C^*-modules extends as a $*$-derivation to the linking algebra. In this paper, we consider derivations on Hilbert C^*-modules and give some results about adjointable derivations and automatic continuity of them.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A *σ-derivation* is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. If $\sigma = I$, where I is the identity operator on $\mathcal{A}$, then d is a derivation. A generalized derivation on $\mathcal{A}$ is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. In [7], P. Li, D. Han and W. S. Tang proved that every derivation on $End^*(\mathcal{M})$ is inner if $\mathcal{A}$ is commutative and unital. In section 3, we will characterize generalized derivations on $\mathcal{K}(\mathcal{M})$ without commutativity condition. Suppose that $\{d_n\}_{n=0}^{\infty}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$. It's called a *higher derivation* if $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in \mathcal{A}$ and all $n \geqslant 0$. If $d_0 = I$, $\{d_n\}_{n=0}^{\infty}$ is called a *strong higher derivation*. Let δ be a derivation on $\mathcal{A}$ and define the sequence $\{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$ by $d_0 = I$ and $d_n = \dfrac{\delta^n}{n!}$ for every $n \geqslant 1$. By Leibniz rule, $\{d_n\}_{n=0}^{\infty}$ is a higher derivation on $\mathcal{A}$. Higher derivations were introduced by Hasse and Schmidt [4] and algebraists sometimes call them Hasse-Schmidt derivations. For a higher derivation obviously, d_0 is a homomorphism and d_1 is a d_0-derivation in the sense of [11]. Therefore, higher derivations are the generalizations of homomorphisms and derivations. In [12], higher derivations are applied to study generic solving of higher differential equations. For more information about higher derivations and its applications see [5], [6], [9] and [10]. The last author in [10], characterized the strong higher derivations in terms of derivations. In section 4 we give a characterization of higher derivation on $End^*(\mathcal{M})$ with use of elements whose product is in $\mathcal{K}(M)$.

2. Derivations on Hilbert C^*-modules

Let $\mathcal{M}$ be a Hilbert C^*-module. Recall that a linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a *derivation* if

$$d(<x,y>z) = <dx,y>z + <x,dy>z + <x,y>dz$$

for all $x,y,z \in \mathcal{M}$. Note that if $d : \mathcal{M} \to \mathcal{M}$ is an adjointable map with $d^* = -d$, then d is a derivation. But the converse is not true. For example suppose that H is a Hilbert space. Set $u_0 \in B(H)$ such that $u^* = -u$ and u is not in the center of $B(H)$. Define $d : B(H) \to B(H)$ by $d(v) = u_0 v - v u_0$ for every $v \in B(H)$. It is easy to see that d is a derivation on $B(H)$ as a $B(H)$-module but d is not adjointable. Otherwise, d is $\mathcal{A}$-linear and Therefore,

$$u_0 vv - vvu_0 = d(vv) = vd(v) = vu_0 v - vvu_0$$

for every $v \in B(H)$. This implies that u_0 is in the center of $B(H)$, which is a contradiction. Let M be a full Hilbert C^*-module. Note that if there exists $a \in \mathcal{A}$ such that $ax = o$ for every $x \in \mathcal{M}$, then $a = o$. Therefore, we have the following theorem:

Theorem 1. *Let $\mathcal{M}$ be a full Hilbert C^*-module. Then $d \in End^*(\mathcal{M})$ is a derivation if and only if $d^* = -d$.*

Proof. Suppose that $d \in End^*(\mathcal{M})$ is a derivation. Then $(<dx,y> + <x,dy>)z = 0$ for all $x,y,z \in \mathcal{M}$. Hence $d^* = -d$. The converse is trivial. □

A set of non-zero elements $\{x_i\}_{i\in I} \subseteq \mathcal{M}$ is called a standard basis for $\mathcal{M}$ if the reconstruction formula $x = \sum_{i\in I} <x,x_i> x_i$ holds for every $x \in \mathcal{M}$. Let

$$L_n(\mathcal{A}) = \{(a_1,a_2,\cdots,a_n) : a_i \in \mathcal{A}, 1 \le i \le n\}.$$

Then $L_n(\mathcal{A})$ a Hilbert C^*-module over C^*-algebra $\mathcal{A}$ with module product
$a(a_1,a_2,\cdots,a_n) = (aa_1,aa_2,\cdots,aa_n)$ and inner product

$$<(a_1,a_2,\cdots,a_n),(b_1,b_2,\cdots,b_n)> = a_1 b_1^* + a_2 b_2^* + \cdots + a_n b_n^*$$

for every $a \in \mathcal{A}$ and $(a_1,a_2,\cdots,a_n),(b_1,b_2,\cdots,b_n) \in L_n(\mathcal{A})$, see [8]. If $\mathcal{A}$ is unital then $L_n(\mathcal{A})$ has standard basis $\{e_i\}_{i=1}^n$ such that
$e_i = (0,0,\cdots,1_{i \cdot th},\cdots,0)$ and $1 \le i \le n$. In [3], D. Bakic and proved that every Hilbert C^*-module over the C^*-algebra of the compact operators possesses a standard basis.

Theorem 2. *Let $\mathcal{M}$ have a standard basis and $d \in End^*(\mathcal{M})$. Then d is a derivation if and only if $d^* = -d$.*

Proof. Let $\{x_i\}_{i\in I}$ be a standard basis for $\mathcal{M}$ and $d \in End^*(\mathcal{M})$ be a derivation. Then $d(x) = \sum_{i\in I} < x, x_i > dx_i$. On the other hand,

$$\begin{aligned} dx &= \sum_{i\in I} < dx, x_i > x_i + \sum_{i\in I} < x, dx_i > x_i + \sum_{i\in I} < x, x_i > dx_i \\ &= dx + \sum_{i\in I} < d^*x, x_i > x_i + \sum_{i\in I} < x, x_i > dx_i \\ &= dx + d^*x + dx. \end{aligned}$$

So, $d^* = -d$. □

Lemma 1. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then there exist $x_1, \cdots, x_n \in \mathcal{M}$ such that $\sum_{i=1}^{n} < x_i, x_i >= 1$.*

Proof. See [8]. □

A Hilbert C^*-module $\mathcal{M}$ over C^*-algebra $\mathcal{A}$ is called simple if the only closed submodules of $\mathcal{M}$ over A are $\{0\}$ and $\mathcal{M}$. For example, let H be a Hilbert space and $\mathcal{K}(H)$ denotes the the algebra of compact operator on H. Then $\mathcal{K}(H)$ is a simple Hilbert C^*-module over itself.

Theorem 3. *Let $\mathcal{M}$ be a full and simple Hilbert C^*-module over the unital C^*-algebra $\mathcal{A}$ and d be a derivation on $\mathcal{M}$ with closed range. Then d is continuous or surjective.*

Proof. Define the separating space S(d)=$\{y \in \mathcal{M} : \exists\{x_n\} \to 0$ in $\mathcal{M}$ such that $dx_n \to y\}$. As a well-known result $S(d)$ is a closed subspace of $\mathcal{M}$. By lemma 1, there exist $x_1, \cdots, x_m$ such that $\sum_{i=1}^{m} < x_i, x_i >= 1$. Therefore, $a = \sum_{i=1}^{m} < ax_i, x_i >$ for all $a \in \mathcal{A}$. For $z \in S(d)$ there exists a sequence $z_n \to 0$ such that that $dz_n \to z$. Hence

$$d(az_n) = \sum_{i=1}^{m} < adx_i, x_i > z_n + \sum_{i=1}^{m} < ax_i, dx_i > z_n + \sum_{i=1}^{m} a < x_i, x_i > dz_n \to az. \tag{2.1}$$

This implies that $S(d)$ is a submodule of $\mathcal{M}$. Since $\mathcal{M}$ is simple, $S(d) = \{0\}$ or $S(d) = \mathcal{M}$. If $S(d) = \{0\}$, by closed graph theorem, d is continuous. If $S(d) = \mathcal{M}$ by (2.1), $A\mathcal{M} \subseteq \overline{Im(d)}$. Since $\mathcal{A}$ is unital $\mathcal{A}\mathcal{M} = \mathcal{M}$. Therefore, $\overline{Im(d)} = Im(d) = \mathcal{M}$ and T is surjective. □

Lemma 2. *Let $\mathcal{M}$ be a Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then $I\mathcal{M} = \mathcal{M}$.*

Proof. Clearly, $I\mathcal{M} \subseteq \mathcal{M}$. let $z \in \mathcal{M}$ and set

$$x = \lim_{n\to\infty} (\frac{1}{n} + < z, z >^{1/3})^{-1} z.$$

One can see that $z =< x, x > x$ and therefore, $I\mathcal{M} = \mathcal{M}$. For more detail see [8]. □

Theorem 4. *Let $\mathcal{M}$ be a Hilbert C^*-module over unital C^* algebra $\mathcal{A}$. Suppose that $\mathcal{M}$ is a simple I-module and d be a derivation on $\mathcal{M}$ with closed range. Then d is continuous or surjective.*

Proof. Let $a \in I$ and $z \in S(d)$. So there exist a sequence

$$z_n \to 0, \quad x_1, x_2, \cdots, x_m, \quad y_1, y_2, \cdots, y_m \in \mathcal{M}$$

for some $m \in \mathbb{N}$ such that $dz_n \to z$ and $a = \sum_{i=1}^{m} < x_i, y_i >$. But

$$d(az_n) = \sum_{i=1}^{m} < dx_i, y_i > z_n + \sum_{i=1}^{m} < x_i, dy_i > z_n + \sum_{i=1}^{m} adz_n \to az. \tag{2.2}$$

This implies that $S(d)$ is a submodule of $\mathcal{M}$. Therefore, $S(d) = \{0\}$ or $S(d) = \mathcal{M}$. If $S(d) = \{0\}$, by closed graph theorem, d is continuous. If $S(d) = \mathcal{M}$, by (2.2), $I\mathcal{M} \subseteq \overline{Im(d)}$. Therefore, by lemma 2, $Im(d) = \mathcal{M}$ and T is surjective. □

3. Characterization of generalized derivations on the algebra of compact operators

Let $\mathcal{A}$ be an algebra. Recall that a derivation on $\mathcal{A}$ is a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in \mathcal{A}$. A generalized derivation on $\mathcal{A}$ is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. Recall that a linear mapping $\Pi : \mathcal{A} \to \mathcal{A}$ is called a left multiplier if $\Pi(ab) = \Pi(a)b$ for all $a, b \in \mathcal{A}$. For a generalized derivation d, set $\Pi = d - \delta$. One can easily see that Π is a left multiplier. let $d : \mathcal{A} \to \mathcal{A}$ de a linear mapping. As a well-known result d is a generalized derivation if and only if there exist a derivation $\delta : \mathcal{A} \to \mathcal{A}$ and left multiplier $\Pi : \mathcal{A} \to A$ such that $d = \delta + \Pi$.

Theorem 5. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then linear mapping $\Pi : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ is a left multiplier if and only if there exists $T \in End(\mathcal{M})$ such that $\Pi(A) = TA$ for all $A \in \mathcal{K}(\mathcal{M})$.*

Proof. Let $T \in End(\mathcal{M})$. Define $\Pi : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ by $\Pi(A) = TA$, for every $A \in \mathcal{K}(\mathcal{M})$. Clearly Π is a left multiplier. Conversely, since $\mathcal{M}$ is full, by lemma 1, there exist $x_1, \cdots, x_n$ such that $\sum_{i=1}^{i=n} < x_i, x_i >= 1$. Define $T : \mathcal{M} \to \mathcal{M}$ by

$$T(x) = \sum_{i=1}^{n} \Pi(\theta_{x,x_i})x_i,$$

for every $x \in \mathcal{M}$. For every $A \in \mathcal{K}(\mathcal{M})$ we have

$$TA(x) = \sum_{i=1}^{n} \Pi(\theta_{Ax,x_i})x_i = \sum_{i=1}^{n} \Pi(A\theta_{x,x_i})x_i = \sum_{i=1}^{n} \Pi(A)(\theta_{x,x_i})x_i$$

$$= \sum_{i=1}^{n} < x_i, x_i > \Pi(A)x = \Pi(A)x.$$

So $\Pi(A) = TA$. T is obviously a continuous linear mapping. To show that $T \in End(\mathcal{M})$ it's remain to show that T is $\mathcal{A}$-linear. Now suppose that $a \in \mathcal{A}$, $x \in \mathcal{M}$ and $A \in \mathcal{K}(\mathcal{M})$ We have,

$$\Pi(A)(ax) = TA(ax) = T(aA(x))$$

On the other hand

$$\Pi(A)(ax) = a\Pi(A)(a) = aTA(x)$$

and so $T(aA(x)) = aTA(x)$ for every $a \in \mathcal{A}$, $x \in \mathcal{M}$ and $A \in \mathcal{K}(\mathcal{M})$. Now lemma 2 implies that T is $\mathcal{A}$-linear. □

Definition 1. By $\mathcal{L}_0(\mathcal{M})$ we denote the set of all linear mapping A on $\mathcal{M}$ such that $AB - CA \in \mathcal{K}(\mathcal{M})$ for all $B, C \in \mathcal{K}(\mathcal{M})$.
Clearly $End^*(\mathcal{M}) \subset \mathcal{L}_0(\mathcal{M})$.

Theorem 6. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then linear mapping $\delta : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ is a derivation if and only if there exists $T \in \mathcal{L}_0(\mathcal{M})$ such that $\delta(A) = TA - AT$ for all $A \in \mathcal{K}(\mathcal{M})$.*

Proof. Let $T \in \mathcal{L}_0(\mathcal{M})$. Define $\delta : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ by $\delta(A) = TA - AT$, for every $A \in \mathcal{K}(\mathcal{M})$. Clearly, δ is a derivation. Conversely, since $\mathcal{M}$ is full by lemma 2 there exist $x_1, \cdots, x_n$ such that $\sum_{i=1}^{i=n} < x_i, x_i >= 1$. Define $T : \mathcal{M} \to \mathcal{M}$ by

$$T(x) = \sum_{i=1}^{i=n} \delta(\theta_{x,x_i})x_i,$$

for every $x \in \mathcal{M}$. For every $A \in \mathcal{K}(\mathcal{M})$ we have

$$\begin{aligned} TA(x) &= \sum_{i=1}^{n} \delta(\theta_{Ax,x_i})x_i \\ &= \sum_{i=1}^{n} \delta(A\theta_{x,x_i})x_i \\ &= \sum_{i=1}^{n} \delta(A)(\theta_{x,x_i})x_i + \sum_{i=1}^{n} A\delta(\theta_{x,x_i})x_i \\ &= \delta(A)x + AT(x). \end{aligned}$$

□

Theorem 7. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$ and $d : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ be a linear mapping. Then d is a generalized derivation if and only if there exist $T_1 \in \mathcal{L}_0(\mathcal{M})$ and $T_2 \in End(\mathcal{M})$ such that $d(A) = T_1 A - AT_1 + T_2 A$ for every $A \in \mathcal{K}(\mathcal{M})$.*

4. Characterization of higher derivation on the algebra of adjointable operators

Let $\mathcal{A}$ be an algebra and suppose that $\{d_n\}_{n=0}^{\infty}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$. It′s called a higher derivation if

$$d_n(ab) = \sum_{i=0}^{n} d_i(a) d_{n-i}(b) \tag{4.1}$$

for all $a, b \in \mathcal{A}$ and all $n \geqslant 0$. If $d_0 = I$, $\{d_n\}_{n=0}^{\infty}$ is called a strong higher derivation. If (4.1) holds for all $x, y \in \mathcal{A}$ and $n = 0, 1, 2, \ldots, m$, it is called a higher derivation of rank m. Now we are going to give a characterization of strong higher derivations in terms of operators whose product is compact.

Theorem 8. *Let $\mathcal{M}$ be a full Hilbert C^*-module over the unital C^*-algebra $\mathcal{A}$. Let $\{d_n : End^*(\mathcal{M}) \to End^*(\mathcal{M})\}_{n=o}^{\infty}$ be a sequence of linear mappings such that $d_0 = I$. Then $\{d_n\}_{n=o}^{\infty}$ is a strong higher derivation if and only if $d_n(AB) = \sum_{i=0}^{n} d_i(A) d_{n-i}(B)$ for all $A, B \in End^*(\mathcal{M})$ such that $AB \in \mathcal{K}(\mathcal{M})$ and all $n \geq 1$.*

Proof. By lemma 1, there exist $x_1, \cdots, x_n$ such that $\sum_1^n < x_i, x_i >= 1$. Let x_i for some $1 \leq i \leq n$, $x \in \mathcal{M}$, $A, B \in End^*(\mathcal{M})$, and $m \geq 1$ be arbitrary elements. Since $\mathcal{K}(\mathcal{M})$ is a two sided ideal in $End^*(\mathcal{M})$,

$$d_m(A\theta_{x,x_i}) = \sum_{i=0}^{m} d_i(A) d_{m-i}(\theta_{x,x_i})$$

and

$$d_m(AB\theta_{x,x_i}) = d_m(AB)\theta_{x,x_i} + \sum_{i=0}^{m-1} d_i(AB) d_{m-i}(\theta_{x,x_i}).$$

On the other hand,

$$d_m(AB\theta_{x,x_i}) = d_m(A)B\theta_{x,x_i} + \sum_{i=0}^{m-1} d_i(A) d_{m-i}(B\theta_{x,x_i}). \tag{4.2}$$

Take $m = 1$. By comparing these equalities, we obtain

$$d_1(AB)\theta_{x,x_i} = d_1(A)B\theta_{x,x_i} + Ad_1(B)\theta_{x,x_i}.$$

So

$$\begin{aligned} d_1(AB)x &= \sum_{i=1}^{n} d_1(AB) < x_i, x_i > x \\ &= \sum_{i=1}^{n} d_1(A)B < x_i, x_i > x + \sum_{i=1}^{n} Ad_1(B) < x_i, x_i > x \\ &= d_1(A)Bx + Ad_1(B)x. \end{aligned}$$

This implies that d_1 is a derivation. As an induction suppose that $\{d_0, d_1, \cdots, d_m\}$ is a higher derivation of rank m. By induction, we get

$$\begin{aligned} d_{m+1}(AB\theta_{x,x_i}) &= d_{m+1}(AB)\theta_{x,x_i} + \sum_{i=0}^{m} d_i(AB)d_{m+1-i}(\theta_{x,x_i}) \\ &= d_{m+1}(AB)\theta_{x,x_i} + \sum_{i=0}^{m}\sum_{j=0}^{i} d_j(A)d_{i-j}(B)d_{m+1-i}(\theta_{x,x_i}) \end{aligned}$$

and by (4.2),

$$\begin{aligned} d_{m+1}(AB\theta_{x,x_i}) &= d_{m+1}(A)B\theta_{x,x_i} + \sum_{i=0}^{m} d_i(A)d_{m+1-i}(B\theta_{x,x_i}) \\ &= d_m(A)B\theta_{x,x_i} + \sum_{i=0}^{m}\sum_{j=0}^{m-i} d_i(A)d_j(B)d_{m+1+i-j}(\theta_{x,x_i}). \end{aligned}$$

One can see that

$$\sum_{i=0}^{m}\sum_{j=0}^{i} d_j(A)d_{i-j}(B)d_{m+1-i}(\theta_{x_0,x_i}) = \sum_{i=0}^{m}\sum_{j=0}^{m-i} d_i(A)d_j(B)d_{m+1+i-j}(\theta_{x,x_i}).$$

Therefore,

$$d_{m+1}(AB)\theta_{x,x_i} = \sum_{i=0}^{m+1} d_i(A)d_{m+1-i}(B)\theta_{x,x_i}.$$

So

$$d_{m+1}(AB)x = \sum_{i=0}^{m+1} d_i(A)d_{m+1-i}(B)x.$$

will imply that $\{d_0, d_1, \cdots, d_{m+1}\}$ is a higher derivation of rank $m+1$. $\square$

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Authors' addresses

Hossein Saidi
University of Birjand, Department of Mathematics, P. O. Box 97175-615, Birjand, Iran
E-mail address: hosseinsaidi@birjand.ac.ir

Ali Reza Janfada
University of Birjand, Department of Mathematics, P. O. Box 97175-615, Birjand, Iran
E-mail address: ajanfada@birjand.ac.ir

Madjid Mirzavaziri
Ferdowsi University of Mashhad, Department of Pure Mathematics, P. O. Box 91775-1159, Mashhad, Iran
Current address: Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Mashhad, Iran
E-mail address: mirzavaziri@um.ac.ir