Probabilistic fuzzy systems, expressions and approaches

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Abstract: In this paper, we aim to provide an overall perspective to the various expressions and approaches regarding probabilistic fuzzy systems. We begin with providing an introduction to the different kinds of uncertainties and associated tools that address them. A general description about the principle of modelling in stochastic environments is then represented. Two different expressions on probabilistic fuzzy sets are subsequently discussed. Moreover three distinct probabilistic fuzzy approaches facing stochastic problems are described and the advantages and disadvantages related to each approach are also illustrated. Several simulations and comparative analysis are also provided to clarify the mentioned concepts.

Keywords: Probabilistic, fuzzy, deterministic, uncertainty, modeling

1 Introduction

Mankind has always tried to obtain a better perception of the evolving world in which he lives. The more comprehensively identifying the effective aspects of a given phenomenon, the more accurate our perception about it will be. However, man's knowledge and perceptional abilities are clearly limited, and so a comprehensive assessment about all aspects of a phenomenon can never be achieved. The concept of *uncertainty* helps address this lack of ability and knowledge regarding a complete description of the universe. We generally face two kinds of uncertainties in the real world. The first is stochastic uncertainty that addresses the random nature of a phenomenon such as the probability of the occurrence or non-occurrence of an event in the future; while the second is deterministic uncertainty such as one that addresses the ambiguity in information or a lack of precision [2], [15].

2 TOOLS COVERING UNCERTAINTIES

While probability theory covers stochastic uncertainty, fuzzy logic provides a platform covering deterministic uncertainty either by *precisiating* the ambiguities such as in type-I fuzzy systems, or by *managing* them such as in type-II fuzzy systems [2], [15]. Such systems cover human knowledge, linguistic variables and provide general approximation of functions [18], [21], [23]. More specifically, type-II fuzzy systems aim to further cover the deterministic aspect of uncertainty by providing appropriate tools to manage it [10], [12], [14], [19].

The relation between probability and fuzzy logic has frequently been considered. There is general agreement that, in spite of their fundamental differences, these two concepts are complimentary and synergistic, rather than inconsistent and competitive [3], [22], [24]. One example where this synergism comes to fruition is *probabilistic fuzzy systems* that cover both kinds of uncertainties in a unifying framework to reach a more realistic and comprehensive model of the real world [11], [13], [14], [16]. We should consider that the concept of *fuzzy probability* introduced by Zadeh is completely different with probabilistic fuzzy. Fuzzy probability is the expression of probability values by fuzzy numbers instead of crisp numbers [25].

According to the particular aspect being considered, probabilistic fuzzy systems can be decomposed into several forms such as probabilistic fuzzy sets, rules, reasoning and logic, as illustrated in Fig. 1 [2]. Moreover, a comparison of uncertainty coverage capability in probabilistic, fuzzy and probabilistic fuzzy systems is represented in Table I.

In the following, we first review the basic principle of stochastic modeling and probabilistic fuzzy sets in Section 3 and 4, respectively. Two different expressions of probabilistic fuzzy sets are discussed in Section 4. Then in Section 5, three different approaches to probabilistic fuzzy systems are discussed. Comparative simulations are then provided in Section 6. Finally, conclusions are drawn in Section 7.

3 PRINCIPLE OF STOCHASTIC MODELLING

Generally, the modelling of systems can be considered in two different approaches: deterministic modelling and stochastic modelling. Deterministic modelling offers a deterministic mapping between input and output spaces, while stochastic modelling represents a probabilistic relation between them [20].

Stochastic modelling can be expressed in discrete and continuous spaces called distributed stochastically uncertain models (DSU) and continuous stochastically uncertain models (CSU) [15], [16].

Fig. 1. Different combinations of fuzzy and probability approaches [2]

TABLE I
A COMPARISON EXPRESSING HOW DIFFERENT SYSTEMS COVER DIFFERENT TYPES OF UNCERTAINTIES

Uncertainty type	Probabilistic systems	Fuzzy systems	Probabilistic Fuzzy systems
Deterministic	-	+	+
Stochastic	+	-	+

4 PROBABILISTIC FUZZY SETS

For several reasons, such as the subjective nature of fuzzy sets itself, there may be the uncertainty while assigning a membership function to a fuzzy variable. If this uncertainty is because of a stochastic nature, we would be best inspired to utilize the concept of probabilistic fuzzy sets. Probabilistic fuzzy sets are an extension of fuzzy sets assigning different membership function values with different probabilities to a fuzzy variable [16], [17]. Here, two distinct probabilistic fuzzy set expressions (PFSE) are described as in the following.

4.1 First expression of probabilistic fuzzy sets (PFSE1)

This expression of probabilistic fuzzy sets can be visualized in a three-dimensional coordinate system as $(x, w, \mu(x,w))$ where x is the input variable, w is an arbitrary random variable covering the stochastic characteristic of probabilistic fuzzy set and $\mu(x,w)$ is the probabilistic membership function value. The mentioned expression of fuzzy sets not only clearly offers the possibility of separating the two kinds of uncertainties but it also provides the facility of probability and possibility relation expression. Though the value of probabilistic membership functions can obviously be achieved as $\mu(x,w)$, the probability values of probabilistic fuzzy values cannot be directly obtained via this expression of probabilistic fuzzy sets [4], [5], [17].

4.2 Second expression of probabilistic fuzzy sets (PFSE2)

The second expression of probabilistic fuzzy sets offers a more comprehensive expression and in spite of the previous expression, the probability of probabilistic fuzzy membership function (PFMF) values can be directly achieved by the latter expression. The second expression can also be displayed in a three-dimensional coordinate system as $(x,u(x),\Pr(x,u(x)))$ where x is the input variable, $u(x)$ is a special random variable ($u(x_i)$ is a random variable essentially covering all probable values of the probabilistic membership function when the input variable is x_i) and $\Pr(x,u(x))$ represents the probability of being the probabilistic fuzzy membership function value equal to $u(x)$ [26]. A sample discrete probabilistic fuzzy set has been illustrated in Fig. 2.

A brief comparison between two represented expressions about probabilistic fuzzy sets is represented in Table II.

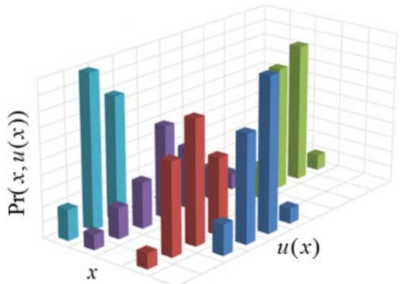

Fig. 2. A sample discrete probabilistic fuzzy set (PFSE2)

TABLE II
COMPARISON BETWEEN PROBABILISTIC FUZZY SET EXPRESSIONS

Expression	Simplicity	Direct Calculation
$(x,w,\mu(x,w))$	Simple	PFMFs Possibility
$(x,u(x),\Pr(x,u(x)))$	Complex	PFMFs Probability

5 PROBABILISTIC FUZZY SYSTEMS

In this section we comprehensively investigate three distinct probabilistic fuzzy system approaches (PFSA). The first approach is based on a special vision regarding fuzzy rule-bases. The second approach construction is based on the first expression of probabilistic fuzzy sets (PFSE1) and the third approach is constructed according to the second expression of probabilistic fuzzy sets (PFSE2).

5.1 First approach regarding probabilistic fuzzy systems (PFSA1)

This approach introduced by M-R Akbarzadeh and A-H Meghdadi in 2001, is based on an innovative vision regarding fuzzy rule-bases [17]. PFSA1 not only offers a simple and understandable approach resulting in low process burden, but it also covers both kinds of uncertainties concurrently. The general form of PFSA1 rule-base structure for a multi-input single-output (MISO) system is represented in (1) (each multi-input multi-output system can be represented by some MISO systems) [2], [15], [16], [17]:

$$\begin{aligned}
&\textit{Rule } r: \ \text{if} \ \ x_1 \textit{ is } MF_1^r \ \& \ldots \& \ x_n \textit{ is } MF_n^r \\
&\qquad \text{then} \ \ y \text{ is } \mathrm{OMF}_1^r \text{ with probability } \mathrm{P}_1^r \\
&\qquad\qquad \& \ldots \\
&\qquad\qquad \& \, y \text{ is } \mathrm{OMF}_1^r \text{ with probability } \mathrm{P}_m^r \\
&\textit{where} \ \sum_{j=1}^{m} P_j^r = 1
\end{aligned} \tag{1}$$

All principle of operations in PFSA1 but selection of consequent part of rules in reasoning procedure are similar to regular fuzzy systems. Selection of consequent part of a rule during PFSA1 reasoning process is simply done by a roulette-wheel and has been illustrated in Fig. 3 [15], [17].

Another valuable aspect of PFSA1 is providing a platform expressing the relation between probability and possibility. Though probability and possibility are completely two distinct concepts, they are not irrelevant. Dubois and Prade have comprehensively investigated the relation between probability and possibility [7], [8], [9]. Generally, we can argue that the high level of possibility does not necessarily result in the high level of probability, even though the impossibility for an event, essentially causes that event being improbable [1].

5.2 Second approach regarding probabilistic fuzzy systems (PFSA2)

The second approach regarding probabilistic

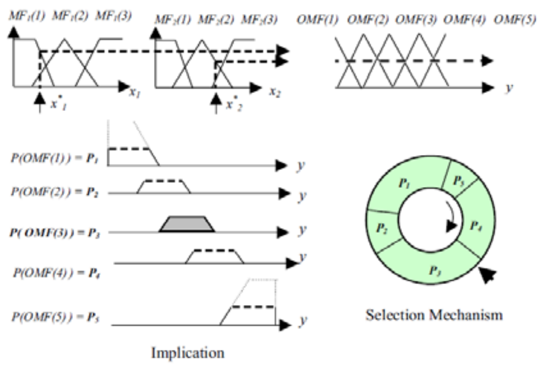

Fig. 3. FPSA1 reasoning process

fuzzy systems is based on the first expression of probabilistic fuzzy sets (FPSE1) [4]. PFSA2 offers a rule-base structure which both antecedent and consequent parts of rules include separate probability values illustrated in (2). Thanks to the set-related approach offered in PFSA2, probabilistic fuzzy relations can be achieved using Cartesian product as represented in (3) [4], [5].

$$\begin{aligned} &\text{if } \mathrm{A_i} \text{ and } \mathrm{B_k} \text{ then } \mathrm{C_{ik}} \\ &\mathrm{A_i} \in \mathrm{P(X)}, \mathrm{Bi} \in \mathrm{P(Y)}, \mathrm{C_{ik}} \in \mathrm{P(U)} \end{aligned} \tag{2}$$

$$\mathrm{R_{ik}} = (A_i \times B_k) \times C_{ik} \tag{3}$$

In this approach, we essentially need a previous knowledge about the statistical characteristics of antecedent parts. Moreover, the estimation process of consequent cumulative density functions according to antecedent statistical characteristics is very complex. Mentioned disadvantages, result in considerable limits in PFSA2 applications. In spite of the proposed disadvantages, PFSA2 provides the ability of solving probabilistic fuzzy problems in two distinct domains; fuzzy domain and probability domain, separately but concurrently [4], [5], [6].

5.3 Third approach regarding probabilistic fuzzy systems (PFSA3)

The third approach facing probabilistic fuzzy systems is constructed on the second expression of probabilistic fuzzy sets (PFSE2) [14]. Consider $(U, \wp, \mathrm{P})$ as a probability space where U is the sample space, $\wp$ is a sigma-algebra covering all considered events and P is the probability function. The union and intersection of two probabilistic fuzzy sets (PFSE2) are defined as (4) and (5) [14].

$$\begin{aligned} &\tilde{A} \cup \tilde{B} \equiv \bigcup_{x \in X} (U_{\tilde{A} \vee \tilde{B}}, \wp_{\tilde{A} \vee \tilde{B}}, P) \\ &P(E) = P(E_{\tilde{A}}).P(E_{\tilde{B}}) \geq 0 \;\; ; \; (\mathrm{P}(\mathrm{U}_{\tilde{A} \vee \tilde{B}}) = 1) \end{aligned} \tag{4}$$

$$\begin{aligned} &\tilde{A} \cap \tilde{B} \equiv \bigcap_{x \in X} (U_{\tilde{A} \wedge \tilde{B}}, \wp_{\tilde{A} \wedge \tilde{B}}, P) \\ &P(E) = P(E_{\tilde{A}}).P(E_{\tilde{B}}) \geq 0 \;\; ; \; (\mathrm{P}(\mathrm{U}_{\tilde{A} \wedge \tilde{B}}) = 1) \end{aligned} \tag{5}$$

For a specified input variable x and its related membership function value $u \in [0,1]$, the probability fuzzy set $\tilde{A}_x$ can be expressed by a probability space $(U_x, \wp, \mathrm{P})$ where U_x is a set including all probable events ($u \in [0,1]$), $\wp$ is a sigma-algebra and P is a probability function defined on $\wp$ satisfying following equation [14]:

$$P(E_i) \geq 0, \quad P(\sum E_i) = \sum P(E_i), \quad P(U_x) = 1 \tag{6}$$

$\mathrm{E_i}$ is an event corresponding to $u = u_i \subseteq [0,1]$ $(i = 1,..., S)$ which u_i is membership function value. Therefore, the main probabilistic fuzzy set covering all input values can be represented as the union of a finite number of sub-probability spaces.

$$\tilde{A} \equiv \bigcup_{x \in X} (U_x, \wp, P) \tag{7}$$

PFSA3 rule-base structure is also based on probabilistic fuzzy sets as below:

$$\mathrm{i^{th}\ Rule: if\ x_1\ is\ \tilde{A}_{1,i}\ \&...\&\ x_j\ is\ \tilde{A}_{j,i}\ \ then\ \ y\ is\ \tilde{B}_i} \tag{8}$$

where $\tilde{A}_{j,i}(j = 1,..., J), \tilde{B}_i (= 1,..., I)$ are probabilistic fuzzy sets (PFSE2), J is the number of inputs and I is the number of rules [14]. Since PFSA3 is based on a probabilistic fuzzy set, probabilistic fuzzy rules can be expressed by the Cartesian product as below [14]:

$$\begin{aligned} &\tilde{R}_{\tilde{A}_{1,i} \times ... \times \tilde{A}_{J,i} \to \tilde{B}_i}(x, \tau) = \tilde{A}_{1,i} \cap ... \cap \tilde{A}_{J,i} \cap \tilde{B}_i \\ &\quad = \bigcup_{x \in X, \tau \in Y} (U_{\tilde{A}_{1,i} \cap ... \cap \tilde{A}_{J,i} \cap \tilde{B}_i}, \wp_{\tilde{A}_{1,i} \cap ... \cap \tilde{A}_{J,i} \cap \tilde{B}_i}, P) \end{aligned} \tag{9}$$

Moreover, fuzzification and defuzzification process applied in PFSA3 is considerably different with other proposed approaches and is comprehensively discussed in [14] and we do not consider them here for brevity. As it can be clearly seen, PFSA3 offers a very complex structure resulting in high process burden and application limitations.

5.4 Probabilistic fuzzy system approaches comparison

Having considered proposed points regarding different probabilistic fuzzy system approaches, we can briefly compare mentioned approaches in Table III. While PFSA1 offers low level of complexity and process burden, PFSA2 represents a more complex approach providing the possibility of separation of two kinds of uncertainties. PFSA3 demonstrates a great level of comprehensiveness at the expense of complexity increase.

6 SIMULATION

Since PFSA1 is more applicable in comparison with other approaches, in this section we represent an example regarding the prediction of Mackey-Glass chaotic time-series infected with noise (Fig. 4) based of PFSA1. Four *regressors* including $x(t-6)$, $x(t-12)$, $x(t-18)$ and $x(t-24)$ have been used to predict the present value of $x(t)$. 70% and 30% of data were used for train and test process respectively. A comparison regarding predicted result mean squared errors (MSE) obtained by regular fuzzy inference system (FIS), probabilistic fuzzy system (PFSA1) and PFSA1 integrated with probability and possibility relation (PPR) has been represented in Table IV. As it can be clearly seen, probabilistic fuzzy approach offers better results in stochastic environments.

7 CONCLUSION

In this paper different types of uncertainties were introduced and some different tools covering proposed kinds of uncertainties represented. We discussed, while fuzzy logic and probability theorem could only cover deterministic or stochastic uncertainty respectively, probabilistic fuzzy systems were able to support both kinds of uncertainties concurrently. Then some principles considered in stochastic modelling represented. We also proposed two different expressions about probabilistic fuzzy sets as PFSE1 and PFSE2.

TABLE III
A COMPARISON AMONG DIIFERENT PROBABILISTIC FUZZY APPROACHES

Approach	Complexity	Special feature
PFSA1	LOW	Simplicity and Applicability
PFSA2	HIGH	Separation of two kinds of uncertainties
PFSA3	VERY HIGH	Comprehensiveness

TABLE IV
MACKEY-GLASS TIME0SERIES PREDICTION ERROR

Approach	No-noise	$N(0,0.050^2)$	$N(0, 0.075^2)$
FIS MSE	0.0025	0.0077	0.0082
PFSA1 MSE	0.0056	0.0062	0.0079
PFSA1+PPR MSE	0.0050	0.0049	0.0049

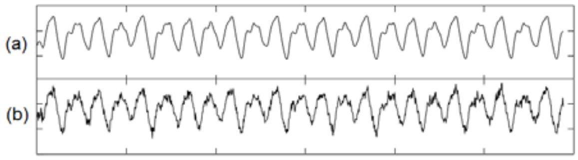

Fig. 4. (a) Mackey-Glass time serires (b) Mackey-Glass time serires infected with noise

PFSE1 provides direct access to possibility of PFMFs, while PFSE2 offers probability of PFMFs directly. After award, three distinct probabilistic fuzzy approaches facing stochastic problems were thoroughly described as PFSA1, PFSA2 and PFSA3. We discussed while PFSA1 provides a simple approach with low process burden. PFSA2 represents a more complex approach offering the possibility of separation of two kinds of uncertainties. Meanwhile, PFSA3 proposes a great comprehensiveness at the expense of increase in complexity. Finally, some simulation results represented and it was shown that probabilistic fuzzy systems offer good performances in stochastic environments.

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