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A KAPLANSKY THEOREM FOR JB*-ALGEBRAS

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ABSTRACT. We provide a new proof of a previously known result, namely every (not necessarily complete) algebra norm on a JB*-algebra generates a topology stronger than the one of the JB*-norm. As a consequence, if θ is a homomorphism of a JB*-algebra A into a Banach Jordan algebra B, then

(i) the range of θ is closed in B if θ is continuous,

(ii) θ is injective if and only if it is bounded below.

Introduction. A Jordan algebra is a nonassociative algebra A over the complex or real field in which the product satisfies $ab = ba$ and $(ab)a^2 = a(ba^2)$, $a, b \in A$. The Jordan triple product $\{abc\}$ is defined to be $(ab)c + a(bc) - (ac)b$, and for a in A, L_a denotes the operator of left multiplication by a.

A Banach Jordan algebra is a Jordan algebra A equipped with a complete norm $\| \cdot \|$, such that $\|ab\| \leq \|a\|\|b\|$, $a, b \in A$. A complex Banach Jordan algebra A with an involution $*$, such that $\|\{aa^*a\}\| = \|a\|^3$ for all a in A is called a JB*-algebra. It has been shown in [**18**] that in a JB*-algebra A the involution $*$ is an isometry, and every closed associative $*$-subalgebra of A is a C^*-algebra. this shows that the class of JB*-algebras coincides with the class of Jordan C^*-algebras introduced by Kaplansky in 1976, see [**17**]. For a JB*-algebra A, we denote by $C^*(a)$ the C^*-subalgebra of A generated by a self-adjoint element $a \in A$. If A is a C^*-algebra we define the Jordan product of two elements a, b in A by $a.b = (ab + ba)/2$. In terms of this product, A becomes a JB*-algebra. A closed linear $*$-subspace of a C^*-algebra B which is closed under the Jordan product is called a JC*-algebra. The theory of JB*-algebra is of capital importance in the theory of JB*-triples, and the classification of bounded symmetric domains in the complex Banach spaces, see [**6**] and [**9**].

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Kaplansky [**8**] proved that any algebra norm on $C(X)$, the C^*-algebra of real or complex valued continuous functions vanishing at infinity on a locally compact Hausdorff space X, dominates the usual supremum norm. This result was improved by S. Cleveland [**4**], showing that every (not necessarily complete) algebra norm on a C^*-algebra generates a topology stronger than the one of the C^*-norm. Recently, the subharmonic methods in [**1**] and [**14**] have been applied in [**5**] and [**15**], see also [**10**, Theorem 6.1.16] to give distinct but closely related proofs of Cleveland's result. Very recently the arguments in [**5**] and [**15**] have been applied in [**3**] and [**12**] to extend Cleveland's result to JB*-algebra. In this paper we prove Cleveland's theorem for JB*-algebra by purely classical methods, avoiding the application of subharmonic methods.

Suppose θ is a homomorphism of a Banach Jordan algebra A into a Banach Jordan algebra B. The range of θ is denoted by $R(\theta)$ and we define the separating space $\sigma_B(\theta)$ of θ in B by

$$\sigma_B(\theta) = \{b \in B \mid \exists \{a_n\} \subseteq A, a_n \to 0, \text{ and } \theta(a_n) \to b\}$$

and the separating space $\sigma_A(\theta)$ of θ in A is defined by $\sigma_A(\theta) = \theta^{-1}(\sigma_B(\theta))$. $\sigma_B(\theta)$ and $\sigma_A(\theta)$ are closed linear subspaces of B and A, respectively. $\sigma_A(\theta)$ is an ideal in A and $\sigma_B(\theta)$ is an ideal in $\overline{R(\theta)}$, the closure of $R(\theta)$ in B. By the closed graph theorem θ is continuous if and only if $\sigma_B(\theta) = \{0\}$. The same argument as in [**4**, page 1099] shows that the main boundedness theorem is valid for nonassociative complete normed algebras, that is, if A and B are nonassociative complete normed algebras, if θ is a homomorphism of A into B, and if $\{x_n\}$ and $\{y_n\}$ are sequences in A such that $x_n y_m = 0$, $n \neq m$, then

$$\operatorname{Sup}\left\{\frac{\|\theta(x_n y_n)\|}{\|x_n\|\|y_n\|} : n \in N\right\} < \infty.$$

2. The results.

Lemma 1. *Let A be a* JB**-algebra, and let a and b be positive elements in A such that $ab = 0$. Then $L_{a^n} L_{b^m} = L_{b^m} L_{a^n}$, $m, n \in N$.*

Proof. First we show that $L_a L_b = L_b L_a$. Since $ab = 0$ and $a \geq 0$, we have $(a^{1/2})^2 b = 0$; therefore, by [**2**, Lemma 3-2], $a^{1/2} b = 0$ and by [**7**, Equation 2-42] for each $x, y \in A$, we have

$$\{xa^{1/2}y\}b = \{(xb)a^{1/2}y\} + \{xa^{1/2}(yb)\} - \{x(a^{1/2}b)y\}.$$

Taking $y = a^{1/2}$, it follows that $L_a L_b = L_b L_a$. Now for $m \in N$, by [**2**, Corollary 3-3(iii)], $a^m b = 0$; hence, $L_{a^m} L_b = L_b L_{a^m}$ since a^m and b are positive. The above argument, replacing a and b by b and a^m, respectively, shows that $b^n a^m = 0$ for all $n \in N$. Therefore, $L_{a^m} L_{b^n} = L_{b^n} L_{a^m}$, $m, n \in N$. □

As a consequence, if a and b are as above, and if $f \in C^*(a)$ and $g \in C^*(b)$, then $L_f L_g = L_g L_f$ and $fg = 0$.

Lemma 2. *If θ is a homomorphism of a* JB**-algebra A into a Banach Jordan algebra B and $\{x_n\}$ is a sequence of positive elements in $\sigma_A(\theta)$ such that $x_n x_m = 0$, $n \neq m$, then, except for a finite number of values of n, we have $\theta(x_n)^4 = 0$.*

Proof. Suppose that $\theta(x_n)^4 \neq 0$ for infinitely many n. Replacing by a subsequence, if necessary, we may assume $\theta(x_n)^4 \neq 0$ for all positive integers n. Since $x_n \in \sigma_A(\theta)$, there is a sequence $\{a_{nk}\}$ in A such that $\lim_{k\to\infty} a_{nk} = 0$ and $\lim_{k\to\infty} \theta(a_{nk}) = \theta(x_n)$. Thus, $\lim_{k\to\infty}\{x_n a_{nk} x_n\} = 0$, $n \in N$. We have

$$\begin{aligned}\lim_{k\to\infty} \theta(x_n\{x_n a_{nk} x_n\}) &= \lim_{k\to\infty} \theta(x_n)\{\theta(x_n)\theta(a_{nk})\theta(x_n)\}\\ &= \theta(x_n)\{\theta(x_n)\theta(x_n)\theta(x_n)\}\\ &= \theta(x_n)^4 \neq 0.\end{aligned}$$

Therefore, $\lim_{k\to\infty} \|\theta(x_n\{x_n a_{nk} x_n\})\|/\|\{x_n a_{nk} x_n\}\| = \infty$, $n \in N$. For each positive integer n, pick $l(n)$ such that

$$\frac{\|\theta(x_n)\{x_n a_{nl(n)} x_n\}\|}{\|\{x_n a_{nl(n)} x_n\}\|} > n\|x_n\|.$$

Put $y_n = \{x_n a_{nl(n)} x_n\}$. Since $x_n, x_m \geq 0$ and $x_n x_m = 0$, $n \neq m$, then by Lemma 1, we have $L_{x_n} L_{x_m} = L_{x_m} L_{x_n}$. Therefore $x_m y_n = 0$, $n \neq m$, and $\|\theta(x_n y_n)\|/\|x_n\|\|y_n\| > n$, $n \in N$. By the main boundedness theorem, this is a contradiction. □

Theorem 3. *Suppose θ is a homomorphism of a* JB**-algebra A into a Banach Jordan algebra B. Then*

(i) *$R(\theta)$ is closed in B if θ is continuous,*

(ii) *If θ is injective and $R(\theta)$ is dense in B, then the map ϕ of A into $B/\sigma_B(\theta)$ defined by $\phi(a) = \theta(a) + \sigma_B(\theta)$ is a continuous surjective homomorphism.*

Proof. (i) Set $\operatorname{Ker}\theta = \theta^{-1}(\{0\})$ and $A^0 = A/\ker(\theta)$, then $\operatorname{Ker}(\theta)$ is a closed $*$-ideal and A^0 is a JB*-algebra [**11, 17**]. Define $\theta^0 : A^0 \to B$ by $\theta^0(a + \operatorname{Ker}\theta) = \theta(a)$. Then θ^0 is an injective homomorphism, $R(\theta^0) = R(\theta)$ and $\|\theta^0\| = \|\theta\|$. Let x be an element in A^0. Consider the C^*-algebra generated by xx^*, then by [**11**, Proposition 2.2] and the Kaplansky theorem for commutative C^*-algebras [**8**], we have

$$\begin{aligned}\|x\|^2 &\le 2\|xx^*\| \le 2\|\theta^0(xx^*)\| \\ &= 2\|\theta^0(x)\theta^0(x^*)\| \\ &\le 2\|\theta^0\|\|\theta^0(x)\|\|x\|.\end{aligned}$$

Hence $\|x\| \le 2\|\theta^0\|\|\theta^0(x)\|$, $x \in A^0$. It follows that $R(\theta)$ is closed.

(ii) It is easy to see that ϕ is a well-defined homomorphism with dense range, by [**16**, Lemma 1.3] ϕ is continuous and, by part (i), ϕ is surjective. $\square$

Theorem 4. *Suppose θ is an injective homomorphism of a* JB**-algebra A into a Banach Jordan algebra B. Then there exists a constant $M > 0$ such that $\|x\| \le M\|\theta(x)\|$, $x \in A$.*

Proof. We may replace B by the closure of $R(\theta)$ and assume that $R(\theta)$ is dense in B. The map ϕ is a continuous surjective homomorphism by Theorem 3. It is enough to show that ϕ is injective, since then the inverse map ϕ^{-1} is continuous by the open mapping theorem and hence there is a constant M such that

$$\|x\| \le M\|\phi(x)\| = M\|\theta(x) + \sigma_B(\theta)\| \le M\|\theta(x)\|, \quad x \in A.$$

Now suppose $\phi(a) = 0$ for some nonzero element a in A. By the definition of ϕ, a lies in $\sigma_A(\theta)$ and we can assume without loss of generality that $a > 0$, since $\sigma_A(\theta)$ is a closed ideal and therefore it is self-adjoint by [**11**]. If $SP(a)$ denotes the spectrum of a, then $SP(a)$ is finite, since otherwise the same argument as in [**4**, Theorem 5.1] shows the existence of a sequence $\{x_n\}$ of nonzero positive elements of $\sigma_A(\theta)$ such that $x_n x_m = 0$, $n \neq m$, so $\theta(x_n)^4 = 0$ for all but a finite number of values of n, by Lemma 2, and so $x_n = 0$ for all but a finite number of values of n, since θ is injective and $x_n \geq 0$ for all n. From the finiteness of $SP(a)$ and the spectral theory, there is a nonzero projection $p \in \sigma_A(\theta)$. Thus, $\theta(p) \in \sigma_B(\theta)$ and $\theta(p)^2 = \theta(p) \neq 0$. This is a contradiction since $\sigma_B(\theta)$ contains no nonzero idempotents [**13**]. Therefore ϕ is injective. $\square$

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REFERENCES

1. B. Aupetit, *The uniqueness of the complete norm topology in Banach algebras and Banach Jordan algebras*, J. Funct. Anal. **47** (1982), 7–25.

2. M. Bataglia, *Annihilators in* JB-*algebras*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 317–323.

3. A. Bensebah, *Weakness of the topology of a* JB*-*algebra*, Canad. Math. Bull. **35** (1992), 449–454.

4. S.B. Cleveland, *Homomorphisms of non-commutative* $*$-*algebras*, Pacific J. Math. **13** (1963), 1097–1109.

5. H.G. Dales, *On norms on algebras*, Proc. Centre Math. Anal. Austral. Nat. Univ. **21** (1989), 61–69.

6. Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for* JB*-*triples*, Duke Math. J. **53** (1986), 139–148.

7. H. Hanche Olsen and E. Stormer, *Jordan operator algebras*, Pitman, 1984.

8. I. Kaplansky, *Normed algebras*, Duke Math. J. **16** (1949), 399–418.

9. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503–529.

10. T.W. Palmer, *Banach algebras and the general theory of* $*$-*algebras*, Vol. I. *Algebras and Banach algebras*, Cambridge University Press, Cambridge, 1994.

11. R. Paya, J. Perez and A. Rodriguez, *Non-commutative Jordan* C^*-*algebras*, Manuscripta Math. **37** (1982), 87–120.

12. J. Perez, L. Rico and A. Rodriguez, *Full subalgebras of Jordan-Banach algebras and algebra norm on* JB*-*algebras*, Proc. Amer. Math. Soc. **121** (1994), 1133–1143.

13. P.S. Putter and B. Yood, *Banach Jordan* $*$*-algebras*, Proc. London Math. Soc. (2) **41** (1980), 21–44.

14. T.J. Ransford, *A short proof of Johnson's uniqueness of norm theorem*, Bull. London Math. Soc. **21** (1989), 487–488.

15. A. Rodriguez, *Automatic continuity with application to* C^**-algebras*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 345–347.

16. A.M. Sinclair, *Automatic continuity of linear operators*, London Math. Soc. Lecture Note Ser. **21** (1976).

17. J.D.M. Wright, *Jordan* C^**-algebras*, Michigan Math. J. **24** (1977), 291–301.

18. M.A. Youngson, *A Vidav theorem for Banach Jordan algebras*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 263–272.

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