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NEW $(\frac{F}{G})$-EXPANSION METHOD AND ITS APPLICATIONS TO NONLINEAR PDES IN MATHEMATICAL PHYSICS

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Abstract. In this work, the new $(\frac{F}{G})$ -expansion method is proposed for obtaining traveling wave solutions of non linear evolution equations. This method is more powerful than the method $(\frac{G'}{G})$ -expansion method. The efficiency of the method is demonstrated on a variety of nonlinear PDEs such as convection-diffusion equation, Zoomeron equation. As a result, more traveling wave solutions are obtained including not only all the known solutions but also the computation burden is greatly decreased compared with the existing method. Abundant exact traveling wave solutions of these equations are expressed by the hyperbolic functions the trigonometric functions. Also it is shown that the proposed method is efficient for solving nonlinear evolution equations in mathematical physics and in engineering.

1. Introduction

Nonlinear partial differential equations $(NLPDEs)$ have been widely applied in many branches of applied sciences such as fluids dynamics, bio-mechanics, chemical physics, particle physics, quantum field theory, optical fibers and plasma physics etc. So, the theory of nonlinear dispersive wave motion has recently undergone much study. Today there are several and ever in creasing number of papers that are being published in this area of research ([1]-[30]). The solutions of nonlinear equations play a crucial role in applied mathematics and physics, because; solutions of nonlinear partial differential equations provide a very significant contribution to people about the exact solutions of nonlinear evolution equations have been established and developed, such as the tanh-coth function expansion ([1]-[4]), the solitary wave ansatz method ([5]-[8]), Lie symmetry analysis [9], the sub-ODE method [10], exp-function method [11, 12], the homogeneous balance method [13], the first integral method [14, 15], the simplest equation method [16, 17] and so on. But there is no unified method that can be used to deal with all types of nonlinear avolution equations.

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Recently, Wang et al. [18] interoduced a new direct method called the $(\frac{G^{'}}{G})$-expansion method to look for traveling wave solutions of nonlinear evolution equations. One of the most effective straightforward method to construct exact solutions of PDEs is the $(\frac{G^{'}}{G})$-expansion method ([19]-[21]). Motivated by work in [18], the main purpose of this paper is to introduce a new technique called $(\frac{F}{G})$-expansion method is that the traveling wave solutions of a nonlinear evolution equation can be expressed by a polynomial in $(\frac{F}{G})$, where $G = G(\xi)$ and $F = F(\xi)$ satisfy the first order linear ordinary differential system $(FLODS)$ as follows: $F^{'}(\xi) = \lambda G(\xi)$, $G^{'}(\xi) = \mu F(\xi)$, where μ, λ are constants. This new method will play an important role in expressing the traveling wave solutions for nonlinear evolution equations via the Zoomeron equation and convection-diffusion reaction (CDR) scalar transport equation. CDR equation is practically important because the working equations of many cases fall into this filed. Typical examples are the Helmholtz equation for modeling exterior acoustics [22], constitutive equations for modeling the turbulent quantities k and ε [23], and viscoelastic constitutive equations for modeling the extra stresses in non-Newtonian fluid flows [24]. The rest of this paper is organized as follows: In Section 2, we give the description of the $(\frac{F}{G})$-expansion method. From Section 3 to Section 4, we apply this method to solve Zoomeron equation and CDR equation. In Section 5, some conclusions are given.

2. Description of the $(\frac{F}{G})$-expansion method

Suppose that a nonlinear equation is given by

$$p(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function and p is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{F}{G})$-expansion method.

Step 1. Combining the independent variables x and t into one variable $\xi = x - wt$, we suppose that

$$u(x,t) = u(\xi),\ \xi = x - wt, \tag{2}$$

where w is a nonzero constant. The traveling wave variable ξ permits us to reducing (1) to an ODE for $u = u(\xi)$,

$$p(u, u^{'}, u^{''}, u^{'''}, \ldots) = 0. \tag{3}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $(\frac{F}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (\frac{F}{G})^i, \tag{4}$$

where $G = G(\xi)$ and $F = F(\xi)$ satisfy the FLODS in the form

$$F^{'}(\xi) = \lambda G(\xi),\ G^{'}(\xi) = \mu F(\xi). \tag{5}$$

$a_0,\ a_1,\ \ldots,\ a_m,\ \lambda$ and μ are constants to be determined later, $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).
with the aid of (5), we can find the following solutions $F(\xi)$ and $G(\xi)$, which are

listed as follows:

Case 1. If $\lambda > 0$ and $\mu > 0$, then (5) has the following hyperbolic function solutions:

$$\begin{cases} F(\xi) = C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \dfrac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi), \\ G(\xi) = C_1 \dfrac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi). \end{cases} \tag{6}$$

Case 2. If $\lambda < 0$ and $\mu < 0$, then (5) has the following hyperbolic function solutions:

$$\begin{cases} F(\xi) = C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \dfrac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi), \\ G(\xi) = -C_1 \dfrac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi). \end{cases} \tag{7}$$

Case 3. If $\lambda > 0$ and $\mu < 0$, then (5) has the following trigonometric function solutions:

$$\begin{cases} F(\xi) = C_1 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \dfrac{\sqrt{\lambda}}{\sqrt{-\mu}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi) \\ G(\xi) = -C_1 \dfrac{\sqrt{-\mu}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi). \end{cases} \tag{8}$$

Case 4. If $\lambda < 0$ and $\mu > 0$, then (5) has the following trigonometric function solutions:

$$\begin{cases} F(\xi) = C_1 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2 \dfrac{\sqrt{-\lambda}}{\sqrt{\mu}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi), \\ G(\xi) = C_1 \dfrac{\sqrt{\mu}}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi). \end{cases} \tag{9}$$

Step 3. Substituting (4) and (5) into equation (3) separately yields a set of algebraic equations for $(\frac{F}{G})^i (i = 1, 2, \ldots, m)$. Setting the coefficients of $(\frac{F}{G})^i$ to zero yields a set of nonlinear algebraic equations in $a_i(i = 0, 1, 2, \ldots, m)$ and w. Solving the nonlinear algebraic equations by Maple and Mathematica, we obtain many exact solutions of (1) according to (2), (4),(6), (7),(8) and (9).

Remark: From the above cases, it is concluded that the proposed method can produce more traveling solutions compare with the $(\frac{G'}{G})$-expansion method. This can be easily seen from the characteristic equation of (5).

3. Application to the Zoomeron equation

In this section, we will apply the $(\frac{F}{G})$-expansion method to construct the traveling solutions for Zoomeron equation [25]

$$(\frac{u_{xy}}{u})_{tt} - (\frac{u_{xy}}{u})_{xx} + 2(u^2)_{xt} = 0, \tag{10}$$

where $u(x, y, t)$ is the amplitude of the relevant wave mode. We know that this equation was introduced by Calogero and Degasperis [26]. The travelig wave variable below,

$$u(x, y, t) = u(\xi), \ \xi = x - cy - wt, \tag{11}$$

Permits us to convert (10) into an ODE for $u(x, y, t) = u(\xi)$ in the form

$$c(1 - w^2)u^{''} - 2wu^3 - Ru = 0, \tag{12}$$

where R is a constant of integration and $w \neq \{0, 1\}$.
By balancing between $u^{''}$ with u^3 in (12) we get $m = 1$. Consequently, we get

$$u(\xi) = a_0 + a_1(\frac{F}{G}) \qquad a_1 \neq 0, \tag{13}$$

where $a_0,\ a_1$ are constants to be determined later

By substituting (13) into (12) and collecting all terms with the same power of $(\frac{F}{G})$ together, the left- hand side of (12) is converted into another polynomial in $(\frac{F}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_0,\ a_1,\ \lambda,\ R,\ \mu$ and w as follows:

$$\begin{aligned}
&(\frac{F}{G})^1 : -2wa_0^3 - ra_0 = 0,\\
&(\frac{F}{G})^2 : -6wa_0^2a_1 + 2cw^2a_1\mu\lambda - Ra_1 - 2ca_1\lambda\mu = 0,\\
&(\frac{F}{G})^3 : -6wa_0a_1^2 = 0,\\
&(\frac{F}{G})^4 : -2cw^2a_1\mu^2 - 2wa_1^3 + 2ca_1\mu^2 = 0.
\end{aligned}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

$$a_0 = 0,\ a_1 = \sqrt{\frac{c(1 - w^2)}{w}},\ R = 2cw^2\mu\lambda - 2c\lambda\mu. \tag{14}$$

Substituting (14) into (13) we can obtain four types of traveling wave solutions of the Zoomeron (10) as follows:

When $\lambda > 0$ and $\mu > 0$, we obtain the hyperbolic function solutions

$$u_1(x,y,t) = \sqrt{\frac{c(1-w^2)}{w}} \left(\frac{C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi)} \right), \tag{15}$$

where $\xi = x - cy - wt$, C_1 and C_2 are arbitrary constants.
In particular, if $C_1 = 0,\ C_2 \neq 0$, we have the solitary wave solution

$$u_1(x,y,t) = \sqrt{\frac{\lambda c(1-w^2)}{\mu w}} \left(\tanh(\sqrt{\lambda}\sqrt{\mu}\xi) \right). \tag{16}$$

When $\lambda < 0$ and $\mu < 0$, we obtain the hyperbolic function solutions

$$u_2(x,y,t) = \sqrt{\frac{c(1-w^2)}{w}} \left(\frac{C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)} \right), \tag{17}$$

where $\xi = x - cy - wt$, C_1 and C_2 are arbitrary constants.
If $C_1 = 0,\ C_2 \neq 0$, we have the solitary wave solution

$$u_2(x,y,t) = -\sqrt{\frac{\lambda c(1-w^2)}{\mu w}} \left(\tanh(\sqrt{-\lambda}\sqrt{-\mu}\xi) \right). \tag{18}$$

When $\lambda > 0$ and $\mu < 0$, we obtain the trigonometric function solutions

$$u_3(x,y,t) = \sqrt{\frac{c(1-w^2)}{w}} \left(\frac{C_1 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{-\mu}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi)} \right), \tag{19}$$

where $\xi = x - cy - wt$, C_1 and C_2 are arbitrary constants.
When $\lambda < 0$ and $\mu > 0$, we obtain the trigonometric function solutions

$$u_4(x,y,t) = \sqrt{\frac{c(1-w^2)}{w}} \left(\frac{C_1 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{\mu}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi)} \right), \tag{20}$$

where $\xi = x - cy - wt$, C_1 and C_2 are arbitrary constants.

4. Application to the CDR equation

We consider the one dimensional convection-diffusion reaction equation by using $(\frac{F}{G})$-expansion[27, 28]

$$\phi_t + u\phi_x - k\phi_{xx} + F = 0, \tag{21}$$

Where the reaction term F is a source. We remark that geologists, civil engineers, mathematicians, and so on, frequently use different terminology in describing the phenomena embodied in (21). As found in [23], if the tracer is radioactive with decay rate c, then $F = c\phi$ and we obtain the linear CDR

$$\phi_t + u\phi_x - k\phi_{xx} + c\phi = 0. \tag{22}$$

If the tracer is a biological species with logistic growth rate $F = r\phi(1 - \dfrac{\phi^p}{R})$, where r is the growth constant, R is the carrying capacity, and $p \neq 1$ is a positive quantity. Then we start with the CDR equation (convection-diffusion equation with growth) in the form

$$\phi_t + u\phi_x - k\phi_{xx} + \alpha\phi - \beta\phi^p = 0, \tag{23}$$

where k, α and β are nonzero constants. To solve equation (23), consider the wave transformation

$$\phi(x,t) = \phi(\xi), \qquad \xi = x - wt, \tag{24}$$

where w is constants that to be determined later.
By using the transformation ξ, (24), equation (23) can be coverted to following ODE

$$\alpha\phi + (u-w)\phi^{'} - k\phi^{''} - \beta\phi^p = 0. \tag{25}$$

By balancing $\phi^{''}$ and ϕ^p we get

$$m + 2 = mp \Rightarrow m = \frac{2}{p-1} \tag{26}$$

To get a colsed form analytic solution, the parameter m should be integer. A transformation formula

$$\phi = v^{\frac{2}{p-1}} \tag{27}$$

Should be use to achieve our goal. This in turn transforms (25) to

$$\alpha v^2 + \frac{2}{p-1}(u-w)v^{'}v - \frac{2k}{p-1}v^{''}v - \frac{2k(3-p)}{(p-1)^2}(v^{'})^2 - \beta v^{(4)} = 0. \tag{28}$$

Balancing $v^{''}v$ and $v^{(4)}$, we find

$$m + 2 + m = 4m \Rightarrow m = 1.$$

Consequently from (4) we get

$$v(\xi) = a_0 + a_1(\frac{F}{G}). \tag{29}$$

Substituting (29) into equation (28), collecting the coefficients of each power of $(\dfrac{F}{G})$, and solve the system of algebraic equations using Maple, we obtain the set of solution:

$$a_0 = a_0,\ a_1 = a_1,\ \mu = \mu\ \lambda = \frac{a_0^2\mu}{a_1^2} \tag{30}$$
$$w = -\frac{2k\mu a_0 p - u a_1 p + 6a_0\mu k + u a_1}{a_1(p-1)}.$$

Substituting these results into (29) and using (6)-(9) we can obtain four types of traveling wave solutions of the CDR (23) as follows:
When $\lambda > 0$ and $\mu > 0$, we obtain the hyperbolic function solutions

$$\phi_1(x,t) = \left[a_0 + a_1\left(\frac{C_1\cosh(\frac{a_0\mu}{a_1}\xi) + C_2\frac{a_0}{a_1}\sinh(\frac{a_0\mu}{a_1}\xi)}{C_1\frac{a_1}{a_0}\sinh(\frac{a_0\mu}{a_1}\xi) + C_2\cosh(\frac{a_0\mu}{a_1}\xi)}\right)\right]^{\frac{2}{p-1}}, \tag{31}$$

where $\xi = x - (-\frac{2k\mu a_0 p - u a_1 p + 6a_0\mu k + u a_1}{a_1(p-1)})t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu < 0$, we obtain the hyperbolic function solutions

$$\phi_2(x,t) = \left[a_0 + a_1\left(\frac{C_1\cosh(\frac{a_0\mu}{a_1}\xi) - C_2\frac{a_0}{a_1}\sinh(\frac{a_0\mu}{a_1}\xi)}{-C_1\frac{a_1}{a_0}\sinh(\frac{a_0\mu}{a_1}\xi) + C_2\cosh(\frac{a_0\mu}{a_1}\xi)}\right)\right]^{\frac{2}{p-1}}, \tag{32}$$

where $\xi = x - (-\frac{2k\mu a_0 p - u a_1 p + 6a_0\mu k + u a_1}{a_1(p-1)})t$, C_1 and C_2 are arbitrary constants.
When $\lambda > 0$ and $\mu < 0$, we obtain the trigonometric function solutions

$$\phi_3(x,t) = \left[a_0 + a_1\left(\frac{C_1\cos(\frac{a_0}{a_1}\sqrt{\mu}\sqrt{-\mu}\xi) + C_2\frac{a_0}{a_1}\frac{\sqrt{\mu}}{\sqrt{-\mu}}\sin(\frac{a_0}{a_1}\sqrt{\mu}\sqrt{-\mu}\xi)}{-C_1\frac{a_1}{a_0}\frac{\sqrt{-\mu}}{\sqrt{\mu}}\sin(\frac{a_0}{a_1}\sqrt{\mu}\sqrt{-\mu}\xi) + C_2\cos(\frac{a_0}{a_1}\sqrt{\mu}\sqrt{-\mu}\xi)}\right)\right]^{\frac{2}{p-1}}, \tag{33}$$

where $\xi = x - (-\frac{2k\mu a_0 p - u a_1 p + 6a_0\mu k + u a_1}{a_1(p-1)})t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu > 0$, we obtain the trigonometric function solutions

$$\phi_4(x,t) = \left[a_0 + a_1\left(\frac{C_1\cos(\sqrt{-\frac{a_0^2\mu}{a_1^2}}\sqrt{\mu}\xi) - C_2\frac{\sqrt{-\frac{a_0^2\mu}{a_1^2}}}{\sqrt{\mu}}\sin(\sqrt{-\frac{a_0^2\mu}{a_1^2}}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\frac{a_0^2}{a_1^2}\mu}}\sin(\sqrt{-\frac{a_0^2\mu}{a_1^2}}\sqrt{\mu}\xi) + C_2\cos(\sqrt{-\frac{a_0^2\mu}{a_1^2}}\sqrt{\mu}\xi)}\right)\right]^{\frac{2}{p-1}}, \tag{34}$$

where $\xi = x - (-\frac{2k\mu a_0 p - ua_1 p + 6a_0\mu k + ua_1}{a_1(p-1)})t$, C_1 and C_2 are arbitrary constants.

In particular, if $C_1 = 0,\ C_2 \neq 0,\ p = 2,\ \lambda\mu > 0$, then (31) and (32) become

$$\phi(x,t) = a_0\left(1 + \tanh(\frac{a_0\mu}{a_1}(x + \frac{2k\mu a_0 p + ua_1 p - 6a_0\mu k - ua_1}{a_1(p-1)}t))\right), \tag{35}$$

which are the solitary wave solutions of the CDR equation (23).

5. Conclusion

In this paper, $(\frac{F}{G})$ -expansion method is used to obtain more general exact solutions of the nonlinear evolution equations. The advantages of the $(\frac{F}{G})$ -expansion method is that it is possible to obtain more traveling wave solutions with distinct physical structures. Form our results, some results previous known as traveling wave solutions and soliton-like solutions can be recovered. Moreover, the proposed method is capable of greatly can be minimizing the size of computational work compared to the existing technique. Finally, it is worth to mention that the implementation of this proposed method is very simple and straightforward, and it can also be applied to other nonlinear evolution equations arising in mathematical physics.

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