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Non-coprime graph of a finite group

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Abstract. In this paper we introduce the non-coprime graph associated to the group G with vertex set $G\setminus\{e\}$ such that two distinct vertices are adjacent whenever their orders are relatively non-coprime. Some numerical invariants like diameter, girth, dominating number, independence and chromatic numbers are determined and it has been proved that the non-coprime graph associated to a group G is planar if and only if G is isomorphic to one of the groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_5$, $\mathbb{Z}_6$ or S_3. Moreover, we prove that non-coprime graph of a nilpotent group G is regular if and only if G is a p-group, where p is prime number. Furthermore, a connection between the non-coprime graph and known prime graph has been stated here.

Keywords: Nilpotent group, abelian group, non-coprime graph.
2010 Mathematics Subject Classification: Primary 05C25, Secondary 20B05

INTRODUCTION

Mathematicians define specific graphs on groups or rings and use advantage of the graph properties for the groups or rings and vice versa. We try to find the interplay between graph theory and group theory.

For a positive integer n, let $\pi(n)$ be the set of all prime divisors of n. If G is a finite group, we set $\pi(G)=\pi(|G|)$. The Gruenberg-Kegel graph of G, or the prime graph of G, is denoted by Γ_G and is defined as follows. The vertex set of Γ_G is the set $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq (see [1]).

We assign a simple graph to the group G which its vertex set contains non-identity elements of the group G. Moreover, two vertices are adjacent whenever their orders are relatively non-coprime. We call it non-coprime graph and denote by Π_G. In this paper, we discuss about general properties of the non-coprime graph. For instance we find the diameter of the graph associated to the abelian groups, non-abelian nilpotent groups and non-nilpotent groups with non-trivial center. Furthermore, we prove that $\mathrm{girth}(\Pi_G)=3$. The domination number, independence number and planarity of the graph are the subjects which are verified in this paper. We also observe that for a finite group G, non-coprime graph Π_G is a planar graph if and only if G is isomorphic to one of the groups

$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_5$, $\mathbb{Z}_6$ or S_3. Let G be a nilpotent group, then Π_G is a regular graph if and only if G is a p-group, where p is a prime number. We conclude that if the order of G is an odd number, then Π_G has a perfect matching. The number of edges for the non-coprime graph associated to the group $Z_{pq}\times Z_{pq}$ is obtained. The structure of the adjacency matrix of $\Pi_{Z_{pq}\times Z_{pq}}$ is presented. Furthermore, we obtain the eigenvalues of this matrix for some small prime numbers p and q using the software MATLAB. The number of the spanning trees in the non-coprime graph $\Pi_{Z_{2p^s}}$ is achieved. We also clarify to imagine $\Pi_{D_{2n}}$, where D_{2n} is dihedral group of order $2n$. $\Pi_{Z_{p^rq^s}}$ and $\Pi_{D_{2n}}$ are examples of perfect graphs, where n is an odd number. The bound for clique number of the graph are given. Finally, we attempt to find a link between non-coprime graph and prime graph.

Throughout the paper, graphs are simple and all the notations and terminologies about the graphs are standard (for instance see [2, 3, 4]).

MAIN RESULTS

Let us start with the definition of the non-coprime graph associated to a group G. We should remindthat in the following definition the group G is not necessary to be finite, but all elements have to be of finite order. Of course, when G is finite then every element has finite order. In this paper, we always assume that G is finite.

Definition 1. *Let G be a group. The non-coprime graph of G is a graph with vertex set $G\setminus\{e\}$ and two distinct vertices x and y join by an edge whenever $\gcd(|x|,|y|)\neq 1$. We denote it by Π_G.*

It is clear Π_G is not an empty graph for the group G of order greater and equal than 3.

Theorem 1. *Let G be a finite group of order greater or equal that 4. Then $\mathrm{girth}(\Pi_G)=3$.*

Proof. We may consider the following two cases:

Case 1. G has a Sylow p-subgroup P of order at least 4. In this case, non-identity elements of P induce a complete graph K_m, where $m=|P|-1\geq 3$. Thus we have a triangle and the result holds.

Case 2. All Sylow p-subgroups of G have order at most 3. So G must be a group of order at least 6. We can easily check that non-coprime graphs Π_{Z_6} and Π_{S_3} have a triangle and the proof is complete.

The following corollary is a direct result of the Theorem 1.

Corollary 1. Π_G *is a cyclic graph if and only if $G\cong Z_2\times Z_2$ or Z_4.*

Proof. Suppose Π_G is a cyclic graph. If $|G|\geq 5$, then we have a triangle by Theorem 1 and at least one more vertex which is a contradiction. Thus $|G|\leq 4$ and the only cyclic non-coprime graphs are $Z_2\times Z_2$ and Z_4 as required. The converse is trivial.

We are going to find the diameter of the non-coprime graph under different circumstances for the group associated to the graph. Among those we also discuss about some other properties of the graph such as being Hamiltonian and star graph.

Proposition 1. *Let G be an abelian group. Then*

(i) $\mathrm{diam}(\Pi_G)\leq 2$ and Π_G is connected.

(ii) If $|G|\geq 4$, then Π_G is Hamiltonian.

Proof.

(i) Suppose x and y are two non-adjacent vertices. Since $|xy| = |x||y|$, the vertex xy is a vertex which make the path $x : xy : y$.

(ii) If all the vertices are adjacent, then the proof is clear. Assume x does not join y. Therefore $\{x, xy\}$ and $\{y, xy\}$ are edges. Now choose another vertex t. If t join x or y, then continue the path otherwise $\{x, xt\}$ or $\{y, yt\}$ are edges. Similar to this method we can make a cycle which pass all the vertices and use each edges once.

Theorem 2. *If G is a non-abelian nilpotent group, then* $\operatorname{diam}(\Pi_G) \leq 2$.

Proof. Since G is a nilpotent group, it is direct product of its Sylow subgroups, S_i. Let $x = s_1 \cdots s_l$ and $y = s'_1 \cdots s'_k$ be two distinct vertices such that at least one of the s_i $(1 \leq i \leq l)$ and s'_j $(1 \leq j \leq k)$ are in the same Sylow subgroup so they are adjacent. Assume x and y belong to two distinct product of Sylow subgroups, thus xy is a vertex which joins both x and y and $d(x, y) = 2$. Hence $diam(\Pi_G) \leq 2$.

Theorem 3. *Let G be a non-nilpotent group, with $Z(G) \neq 1$. Then* $\operatorname{diam}(\Pi_G) \leq 3$.

Proof. Suppose x and y are two non-adjacent vertices. If there is $z \in Z(G)$ such that it is adjacent to both x and y, then $d(x, y) = 2$. Assume, this does not happen. Since x and y commute with z, order of tz divides the least common multiple of $|t|$ and $|z|$, where $t = x$ or y. Since xz and yz join by an edge, then $d(x, y) = 3$.

By Theorems 2 and 3, we deduced the non-coprime graph associated to nilpotent or non-nilpotent groups with non-trivial center is connected.

Theorem 4. *Let G be a nilpotent group such that it does not have any Sylow 2-groups of order 2 then Π_G is Hamiltonian.*

Proof. Assume G is direct product of its Sylow subgroups, S_{p_k} of order $p_k^{\alpha_k}$, $1 \leq k \leq n$. Suppose $x \in S_{p_i}$ so $|x| = p_i^{\beta}$, $0 < \beta \leq \alpha_i$. Thus $deg(x) = |G| - 1 - |\prod_{k \neq i} S_{p_k}| \geq |V(\Pi_G)|/2$. Moreover, $deg(y) \geq deg(x)$ for all other vertices. Hence the assertion is clear.

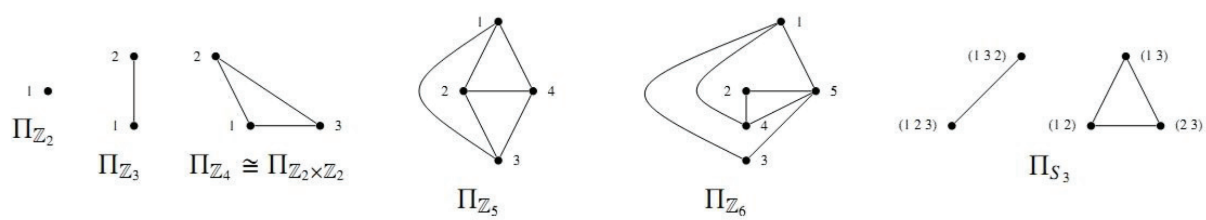

FIGURE 1. Non-coprime graphs of groups of order less than or equal 6

Proposition 2. *Let G be a group.*

(i) If G is an abelian group of order greater than 3, then Π_G is not a star graph.

(ii) If G is a non-abelian group and Π_G is connected, then Π_G is not a star graph.

Proof.

(i) It is obvious.

(ii) By Theorem 1, groups of order greater or equal than 6 has always a triangle. Therefore it is impossible for such a graph to be a star.

A dominating set for a graph Γ is a subset D of $V(\Gamma)$ such that every vertex outside D is adjacent to at least one member of D. The domination number $\gamma(\Gamma)$ is the size of the smallest dominating set of Γ. It is clear that $\gamma(\Pi_G) \leq |\pi(G)|$. Finding the smallest dominating set is an important problem in graph theory.

Theorem 5. *Let G be a group.*

(i) If G is a nilpotent group, then $\gamma(\Pi_G) = 1$

(ii) If G is a soluble group of class l, then $\gamma(\Pi_G) \leq l$.

Proof.

(i) Since G is a nilpotent group, it can be written as direct product of its Sylow subgroups $G \cong S_{p_1} \times \cdots \times S_{p_n}$, where $|S_{p_i}| = p_i^{\alpha_i}$. There exists $x_i \in S_{p_i}$, $1 \leq i \leq n$. Put $x = x_1 x_2 \cdots x_n$. It is clear that $|x| = p_1 p_2 \cdots p_n$ and every non-identity element of the group is adjacent to x. Hence the singleton $\{x\}$ is a dominating set for the graph and the assertion is clear.

(ii) As G is a soluble group, we have the following abelian series for G,

$$1 = G_0 \leq G_1 \leq G_2 \cdots \leq G_{l-1} \leq G_l = G.$$

It is clear that G_i/G_{i-1} is a nilpotent group. Therefore by the first part $x_i \in G_i \setminus G_{i-1}$ exist such that $x_i G_{i-1}$ is adjacent to all non-identity elements of G_i/G_{i-1}. Consequently x_i join to all elements of $G_i \setminus G_{i-1}$. Moreover $\cup_{i=1}^{l} G_i \setminus G_{i-1} = G \setminus \{e\}$. Hence $A = \{x_1, \cdots, x_l\}$ is a dominating set for Π_G.

Theorem 6. *Let G be a group. Then $\alpha(\Pi_G) = |\pi(G)|$, where $\alpha(\Pi_G)$ is the independence number of Π_G.*

Proof. Suppose $\pi(G) = \{p_1, p_2, \cdots, p_n\}$. There is x_i of order p_i, $1 \leq i \leq n$. Thus $\{x_1, \cdots, x_n\}$ is an independent set for Π_G and $\alpha(\Pi_G) \geq n = |\pi(G)|$. On the other hand, if $\alpha(\Pi_G) = |\{y_1, \cdots, y_k\}| = k$, then $\pi(|y_i|) \cap \pi(|y_j|) = \varnothing$ for $i \neq j$ and $n = |\pi(G)| = \pi(|y_1|) \cup \pi(|y_2|) \cup \cdots \cup \pi(|y_n|) \geq k$ this means $|\pi(G)| \geq \alpha(\Pi_G)$. Hence the assertion is clear.

Theorem 7. *Let G be a group. Then Π_G is a planar graph if and only if G is isomorphic to one of the groups* $\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_5, \mathbb{Z}_6$ or S_3.

Proof. Suppose Π_G is a planar graph. It is clear that the order of each Sylow p-subgroup of G is at most 5. This means $|G|$ is divisor of 60. Moreover G has at most four Sylow 2-subgroup, two Sylow 3-subgroup and one Sylow 5-subgroup. Thus, as Π_G is planar, we must have $|G| \leq 13$. The only non-coprime graphs associated to groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5,\ \mathbb{Z}_6$ and S_3 are planar (see Figure 1.). Clearly all the non-coprime graphs of p-groups of order greater than 6 and less or equal than 13 are not planar. It is enough to check all groups of order 10 and 12. It is easy to see that the non-coprime graphs of these groups are not planar. The converse can be observe by Figure 1.

By [4], we know a graph that contains a perfect matching has an even number of vertices. Therefore the perfect matching of a non-coprime graph is meaningful whenever the order of the group is an odd number. Hence we deduce the following proposition.

Proposition 3. *Π_G has a perfect matching, where $|G|$ is an odd number.*
Proof. Since every element of the group has an odd order, then each vertex is adjacent to its inverse. Hence the assertion is clear.

Proposition 4. *Π_G is a complete graph if and only if G is a p-group.*

Π_{Z_p} is a complete connected graph. Furthermore, $\Pi_{Z_{p^\infty}}$ is infinite complete non-coprime graph, where p is a prime number.

Proposition 5. *Let p and q be two distinct prime numbers. Then $\chi(\Pi_{Z_{p^r q^s}}) = \omega(\Pi_{Z_{p^r q^s}}) = p^r(q^s - 1)$, where p and q are distinct prime numbers*
Proof. Suppose $p^r < q^s$. Clearly the number of elements of order t^k in the group $Z_{p^r q^s}$ are $t^k - t^{k-1}$, where $t = p$ or q and $1 \le k \le r$ or $1 \le k \le s$, respectively. Since $p^r < q^s$, the colors which are used for the coloring of all the vertices of order q^i, $1 \le i \le s$ can be used for the vertices of order p^j, $1 \le j \le r$. Therefore $\chi(\Pi_{Z_{p^r p^s}}) = p^r q^s - 1 - \sum_{l=1}^{s} p^l - p^{l-1}$.

Proposition 6. *We have the following properties for $\Pi_{Z_{2p^s}}$, $p \ne 2$.*

(i) The edge number of $\Pi_{Z_{2p^s}}$ is obtained by the following formula,

$$|E(\Pi_{Z_{2p^s}})| = \binom{2p^s - 1}{2} - (p^s - 1).$$

(ii) The number of the spanning trees in the non-coprime graph $\Pi_{Z_{2p^s}}$ is

$$(2p^s - 3)^{p^s - 1}(2p^s - 1)^{p^s - 2}.$$

Proof.
(i) Since there is only one element of order 2 in Z_{2p^s}, then the first part is clear.
(ii) As $\Pi_{Z_{2p^s}}$ is obtained by omitting of $p^s - 1$ distinct edges of the complete graph of order $2p^s - 1$, then by [2, Lemma 4.6] the eigenvalues of Laplacian matrix are zero, $2p^s - 3$ and $2p^s - 1$ with multiplicity one, $p^s - 1$ and $p^s - 1$. Thus [2, Theorem 4.11] implies the result.

Proposition 7. *Let $G = Z_{pq} \times Z_{pq}$, where p and q are prime numbers. Then*

$$|E(\Pi_G)| = \binom{(pq)^2 - 1}{2} - (p^2 - 1)(q^2 - 1). \tag{1}$$

Moreover, there are $(p^2 - 1)$ vertices of degree $(pq)^2 - q^2 - 1$, $(q^2 - 1)$ vertices of degree $(pq)^2 - p^2 - 1$ and the rest vertices are of degree $(pq)^2 - 2$.
Proof. It is clear that two distinct vertices in Π_G are not adjacent whenever their orders are p or q. The number of vertices of order p and q are $p^2 - 1$ and $q^2 - 1$ respectively. Hence we conclude (1) easily and the rest of the assertion is clear.

TABLE 1. Eigenvalues of the adjacency matrix of $\Pi_{\mathbb{Z}_{pq}\times\mathbb{Z}_{pq}}$

	p=2, q=2			
Eigenvalues	0	8	-1	2
Eigenvalues Multiplicity	3	1	10	1
	p=2, q=3			
Eigenvalues	0	7	-1	23
Eigenvalues Multiplicity	3	1	30	1
	p=2, q=5			
Eigenvalues	0	21	-1	71
Eigenvalues Multiplicity	3	1	92	1
	p=3, q=3			
Eigenvalues	0	63	-1	7
Eigenvalues Multiplicity	8	1	90	1

By the above proposition, we conclude that the adjacency matrix of $\Pi_{\mathbb{Z}_{pq}\times\mathbb{Z}_{pq}}$ is a square matrix of rank $(pq)^2-1$ which has four blocks that organized as follows,

The first block has $(pq)^2-p^2-q^2+1$ rows and columns with zero in diagonal entries and 1 in all the other entries.

The second block and the third block are $((pq)^2-p^2-q^2+1)\times(p^2+q^2-2)$ and $(p^2+q^2-2)\times((pq)^2-p^2-q^2+1)$ matrices, respectively. All their entries are 1.

The forth block is formed by four submatrix A_i, $i=1,2,3,4$. A_1 is $(p^2-1)\times(p^2-1)$ matrix for which its main diagonal is zero and all the other entries is 1. All entries of A_2 and A_3 are zero, where A_2 and A_3 are $(p^2-1)\times(q^2-1)$ and $(q^2-1)\times(p^2-1)$ submatrixes. Moreover, A_4 is $(q^2-1)\times(q^2-1)$ matrix such that all entries are zero and the other entries are one.

We use MATLAB to compute the eigenvalues of this adjacency matrix for some primes p and q (see TABLE 1.). It is clear that $\Pi_{\mathbb{Z}_{pq}\times\mathbb{Z}_{pq}}$ is an integral graph for p and q which are mentioned in the above tables. For the bigger prime numbers p and q we obtain decimal eigenvalues.

Theorem 8. *The non-coprime graph associated to $\mathbb{Z}_{pq}\times\mathbb{Z}_{pq}$ is not 1-planar, where p and q are prime numbers.*
Proof. Lemma 2.2 in [5] and Proposition 7 imply the result.

Proposition 8. *Let $D_{2n}=\langle a,b : a^n=b^2=1, a^b=a^{-1}\rangle$ be a dihedral group of order $2n$, where $n\geq 4$. Then*

(i) If n is a prime power and odd, then $\Pi_{D_{2n}}$ contains two complete components K_n and K_{n-1}. Moreover, $\mathrm{E}(\Pi_{D_{2n}})=4n-6$, where $\mathrm{E}(\Pi_{D_{2n}})$ denotes the energy of the graph.

(ii) If $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$, then $\Pi_{D_{2n}}$ is connected, where $p_1=2$ and p_i are odd prime numbers $2\leq i\leq t$. (iii) $\omega(\Pi_{D_{2n}})=\chi(\Pi_{D_{2n}})=n$, where n is a prime power and odd.

(iv) $\Pi_{D_{2n}}$ is not a planar graph.

Proof.
(i) We label the adjacency matrix of $\Pi_{D_{2n}}$ in the form of block-diagonal matrix which contains two blocks. By [2, Theorem 3.4] the assertion is clear.

(ii) and (iii) follows by use of the element orders of D_{2n}.

(iv) Π_{D_8} is not a planar graph by Theorem 7. If $n \geq 5$, then $a^i b$ are the vertices which are adjacent, where $0 \leq i \leq n-1$ and we have K_5 as the subgraph of $\Pi_{D_{2n}}$. The rest parts are clear.

It is clear that $\Pi_{D_{2n}}$ is not a regular graph, where n is an integer satisfies (i) and (ii) in Proposition 8.

Theorem 9. *Let G be a nilpotent group. Then Π_G is a regular graph if and only if G is a p-group.*

Proof. If G is an abelian group such that Π_G is a regular graph then every vertex is adjacent to each generators. Thus degree of each vertex is complete and so by Proposition 2.13 the result is clear. Now if G is non-abelian nilpotent group, then $G \cong S_{p_1} \times \cdots \times S_{p_n}$, where S_{p_i} are distinct Sylow p_i-subgroups of G. We conclude that the vertices which belong to the different Sylow p_i-subgroups have distinct degrees so the result is clear.

Let X and Y be two graphs with two disjoint vertex sets. The product of X and Y, denoted by $X \times Y$ is the graph with vertex set $V(X) \times V(Y)$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $\{x_1, x_2\} \in E(X)$ and $\{y_1, y_2\} \in E(Y)$. The sum of X and Y, denoted by $X+Y$, is the graph with vertex set $V(X) \times V(Y)$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $\{x_1, x_2\} \in E(X)$ and $y_1 = y_2$ or $\{y_1, y_2\} \in E(Y)$ and $x_1 = x_2$.

Theorem 10. *Let (a,b) be a vertex of $\Pi_{G_1} + \Pi_{G_2}$, where G_1 and G_2 are two groups. Then* $\deg((a,b)) = \deg(a) + \deg(b)$.

Proof. The proof is clear by definition of the sum of two graphs.

Proposition 9. *Let Z_n be a cyclic group of order n.*

(i) $\Pi_{\mathsf{Z}_p} + \Pi_{\mathsf{Z}_q}$ is $(p+q-4)$-regular, where p and q are prime numbers.

(ii) $\chi(\Pi_{\mathsf{Z}_p} + \Pi_{\mathsf{Z}_q}) = p+q-3$.

(iii) $2\sqrt{pq-p-q} \leq \mathsf{E}(\Pi_{\mathsf{Z}_p} + \Pi_{\mathsf{Z}_q}) \leq \dfrac{(p-1)(q-1)(1+\sqrt{(p-1)(q-1)})}{2}$.

(iv) $\Pi_{\mathsf{Z}_2} + \Pi_{\mathsf{Z}_p}$ is a complete graph.

Proof.

(i) Follows by Theorem 10.

(ii) Clearly $\chi(\Pi_{\mathsf{Z}_p} + \Pi_{\mathsf{Z}_q}) \geq p+q-3$. Now [2, Theorem 6.8] implies that the greatest eigenvalue of $\Pi_{\mathsf{Z}_p} + \Pi_{\mathsf{Z}_q}$ is $p+q-4$ with multiplicity one. Hence the result follows by [2, Theorem 3.18].

(iii) By [6, Theorems 10 and 12] is clear. (iv) It is obvious.

In the following we show that for all p-groups of certain order, there is only one non-coprime graph.

Proposition 10. *Let G be a finite p-group. Then G and H have isomorphic non-coprime graphs if and only if* $|G| = |H|$.

As an example for the Proposition 10, if $\Pi_{D_8} \cong \Pi_H$, then $H \cong D_8$, Q_8 or an abelian group of order 8. Moreover, it shows that being abelian or non-abelian does not inherit via graph isomorphism.

Proposition 11. *If* $\Pi_G \cong \Pi_{S_3}$, *then* $G \cong S_3$.

Proposition 11 is not valid for all groups for example non-coprime graphs associated to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ are isomorphic but $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ are not.

NON-COPRIME GRAPH AND PRIME GRAPH

Finally we try to find a connection between non-coprime graph and prime graph.

Theorem 11.*Suppose* Γ_G *and* Π_G *are prime graph and non-coprime graph associated to the group* G. *Then* $\gamma(\Gamma_G) \le \gamma(\Pi_G)$.

Proof. Let $\gamma(\Pi_G) = k$ and $A = \{x_1, \cdots, x_k\}$ be the dominating set for Π_G. Therefore $p_i \mid |x_i|$, $1 \le i \le k$. We claim $I = \{p_1, p_2, \cdots, p_k\}$ is a dominating set for Γ_G. Assume $p \in \pi(G) \setminus I$. Since A is a dominating set of Π_G we deduce $p \in \pi(|x_i|)$ for some i. Thus $pp_i \mid |x_i|$ and a suitable power of x_i is of order pp_i. This means p and p_i are adjacent in the prime graph and I is its dominating set. Hence the result is clear.

Example 1.*In this example we present some groups which confirm the validity of the above theorem.*

(i) $\gamma(\Gamma_{S_3}) = |\{2,3\}|$ and $\gamma(\Pi_{S_3}) = |\{(1\,2),(1\,2\,3)\}|$ so $\gamma(\Gamma_{S_3}) = \gamma(\Pi_{S_3})$.

(ii) $\gamma(\Gamma_{\mathbb{Z}_5 \times S_3}) = |\{5\}|$ and $\gamma(\Pi_{\mathbb{Z}_5 \times S_3}) = |\{(a,(1\,2)),(a,(1\,2\,3))\}|$ so $\gamma(\Gamma_{\mathbb{Z}_5 \times S_3}) < \gamma(\Pi_{\mathbb{Z}_5 \times S_3})$, where a is a generator of $\mathbb{Z}_5$.

Theorem 12.*The prime graph associated to the group* G *is connected if and only if* Π_G *is connected.*

Proof. Suppose Γ_G is connected graph. Let $x, y \in V(\Pi_G)$ two non-adjacent vertices. There are distinct prime numbers p, q such that $p \mid |x|$ and $q \mid |y|$. Connectedness of Γ_G implies that $r_1, \cdots, r_t \in \pi(G)$ and $z_0, z_1, \cdots, z_t \in G$ exist such that $|z_0| = pr_1, |z_1| = r_1r_2, |z_2| = r_2r_3 \cdots, |z_t| = r_tq$. Hence we find a path between x and y. Conversely let p and q be two distinct prime numbers. Therefore x and y are p and q elements, respectively. Since Π_G is connected there is a path between x and y. Suppose $x = x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = y$ is the shortest path from x to y. Moreover $(|x_i|, |x_{i+1}|) \ne 1$ and $(|x_i|, |x_j|) = 1$ where $i - j > 1$. Let $p_i \mid |x_i|$ and $|x_{i+1}|$ and put $p = p_1$ and $q = p_n$. It is clear that p_1 and p_2 are adjacent vertices in Γ_G, because $p_1p_2 \| x_2|$ so there is an element x_2^k of order p_1p_2 for $k \in \mathbb{N}$. Hence by continuing this process the result follows.

Theorem 13.*Let* G *be a finite group. If* Γ_G *is Hamiltonian, then* Π_G *is Hamiltonian. In particular, the converse is valid whenever* G *is an abelian group.*

Proof. Suppose $|G| = \prod_{i=1}^{n} p_i^{\alpha}$, where p_i prime numbers. There is a cycle which passes all the vertices of the graph Γ_G. With out loss of generality assume that $p_1 \sim p_2 \cdots \sim p_n \sim p_1$ is the cycle. Therefore $y_1, \cdots, y_n$ are elements of orders p_ip_{i+1}, $1 \le i \le n$. We have $x_1, \cdots, x_n \in V(\Pi_G)$ such that $|x_i| = p_i$. It is clear that we can make a cycle as $x_1 \sim y_1 \sim x_2 \sim y_2 \cdots \sim x_n \sim y_n \sim x_1$. If z is a vertex of the non-coprime graph distinct from x_i and y_i, then we have the path $\cdots \sim x_s \cdots \sim z \sim \cdots \sim y_s$ by using of the orders. Hence the rest is clear.

CONCLUSION

In this research, the non coprime graph of a group is defined and the diameter, girth, connectivity, Hamiltonian, planarity, independence number and domination number are determined. Moreover, some more properties of the graph for nilpotent and abelian groups are also investigated. Furthermore, the energy of this graph for some special cases of the group are computed here.

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