Double Field Domain Walls with Explicit Symmetry Breaking

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We study the dynamics of domain walls in a double-field model in which the U(1) symmetry is broken both spontaneously and explicitly. The global U(1) symmetry of the system is restored when the symmetry breaking parameter λ is set to zero. Two pairs of degenerate kinks exist in the model, and they are related to each other by a Z_2 transformation. We first calculate the single domain wall solutions and then investigate the collision processes. These include simple scattering, pair annihilation, pair capture, and other interesting processes. The possibility of the domain wall being punctured by a string is also investigated.

I. INTRODUCTION

Domain walls are topological solitons which appear in certain nonlinear field equations having disconnected degenerate vacuua [1]. These interesting objects occur in different systems, including phase transitions in the early universe [2], magnetism [3], optics [4], and brane world scenarios [5]. Once formed, domain walls can bend, collide, and annihilate each other [2]. When viewed as one-dimensional, localized objects, domain walls are usually called kinks. This is why we use the "domain wall" and "kink" terms interchangeably throughout this paper.

Apart from gravitational interactions, domain walls and kinks interact with each other via short range forces and collide without losing their identities [6–11]. Like other topological solitons, domain walls are stable, due to the boundary conditions at spatial infinity from the wall. Their existence, therefore, is essentially dependent on the presence of degenerate vacua [7, 12]. In the case of domain walls in three spatial dimensions, the curvature of the wall also leads to acceleration. This acceleration can lead to the emission of scalar and gravitational radiation [2].

Topology provides an elegant way of classifying domain walls in various sectors according to the mappings between the degenerate vacua of the field and the points at spatial infinity [1]. For the sine-Gordon (SG) system in $1 + 1$ dimensions, these mappings are between $\phi = 2n\pi$, $n \in \mathbb{Z}$ and $x = \pm\infty$, which correspond to kinks and antikinks of the SG

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system. More complicated mappings occur for solitons in higher dimensions [8].

Coupled systems of scalar fields with soliton solutions have found interesting applications in double-strand, long molecules, like DNA molecules [13–16], bi-dimensional QCD [17], and hybrid (double-field) inflationary models in cosmology [18]. Analytical and numerical properties of such models have been investigated by many authors, including Bazeia *et al.* [19] and Riazi *et al.* [7, 20]. Inspired by the coupled systems introduced in [7, 21], we investigate a new coupled system of two real scalar fields. The present model may be used as a tentative model of a double-field inflation. In the present paper, we are interested in the domain wall interactions within this model.

The field potential we start with reads

$$V(\phi,\psi) = (\phi^2 + \psi^2 - 1)^2 + \frac{1}{2}\lambda\psi^2, \tag{1}$$

in which ϕ and ψ are real scalar fields and λ is a constant controlling the explicit symmetry breaking. This potential is similar to, but not the same as, that of the hybrid inflationary model [18]. Note that the potential along the ϕ axis has always two degenerate vacua at $\phi = \pm 1$, while the potential along the ψ axis has minima at $\psi = \pm 1$ only if $\lambda < 4$. For $\lambda \geq 4$, the potential has only one minimum at $\psi = 0$ along this axis (besides the two absolute minima). The two minima at $\psi = \pm 1$ are in fact saddle points for $\lambda < 4$.

In the hybrid inflationary model there are two scalar fields, one playing the role of a rapidly decaying (water-fall) field, triggered by another (inflationary) scalar field [18]. Depending on the choice of the Lagrangian density, the model may lead to the formation of domain walls. In what follows, we show that the potential (1) leads to the formation of domain walls, and in subsequent sections we investigate how they interact with each other.

II. PRELIMINARIES

The Lagrangian density of the system is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\partial^\mu\psi\partial_\mu\psi - [(\phi^2 + \psi^2 - 1)^2 + \frac{1}{2}\lambda\psi^2], \tag{2}$$

in which λ is the $U(1)$ explicit symmetry breaking parameter. One can write this Lagrangian in terms of the complex scalar field Φ, where

$$\Phi = \phi + i\psi, \tag{3}$$

in terms of which the Lagrangian density of the system reads

$$\mathcal{L} = \frac{1}{2}\partial^\mu\Phi^\dagger\partial_\mu\Phi - (\Phi^\dagger\Phi - 1)^2 + \frac{1}{2}\lambda(\text{Im}\Phi)^2. \tag{4}$$

From either of these two forms of the Lagrangian density, the following field equations for ϕ and ψ are obtained:

$$\Box\phi = -4\phi(\phi^2 + \psi^2 - 1), \tag{5}$$

and

$$\Box\psi = -4\psi(\phi^2 + \psi^2 - 1) + \lambda\psi. \tag{6}$$

It is obvious that if $\lambda = 0$, the Lagrangian density (4) is both Lorentz invariant and also invariant under a global $U(1)$ symmetry

$$\Phi \to \Phi' = e^{i\theta}\Phi, \tag{7}$$

or

$$\phi' = \phi\cos\theta - \psi\sin\theta,$$
$$\psi' = \phi\sin\theta + \psi\cos\theta. \tag{8}$$

The corresponding energy-momentum tensor [22, 23] of the system is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \partial_\mu\psi\partial_\nu\psi - g_{\mu\nu}\mathcal{L}, \tag{9}$$

which satisfies the conservation law

$$\partial_\mu T^{\mu\nu} = 0. \tag{10}$$

In Equation (9), $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric of the (3+1)-dimensional spacetime. For ϕ and ψ functions of x and t the Hamiltonian (energy) density is obtained from Eq. (9) according to

$$\mathcal{H} = T^{00} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 + V(\phi, \psi). \tag{11}$$

In order to derive the domain wall solutions, one has to reduce the system to an effectively 1+1 dimensional spacetime by assuming that the fields depend only on one space and one time coordinate. In fact, a domain wall is nothing but a kink placed inside a 3D space [2]. Like other systems bearing topological solitons, the present system also has the following topological current:

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu}\partial_\nu\phi, \tag{12}$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor in $1+1$ dimensions and $\epsilon^{01} = +1$ which is locally conserved:

$$\partial_\mu J^\mu = 0. \tag{13}$$

The corresponding topological charge is given by

$$Q = \int_{-\infty}^{+\infty} J^0 dx = \frac{1}{2}[\phi(+\infty) - \phi(-\infty)]. \tag{14}$$

Note that since the vacua of the system reside at $(\phi, \psi) = (\pm 1, 0)$, only the ϕ-field is responsible for the topological charge.

According to the Goldstone theorem, if a continuous global symmetry is broken spontaneously, there appears a massless (Goldstone) particle for each broken group parameter [24, 25]. However, in the case of the Lagrangian density (4), in addition to the spontaneous breaking of the U(1) symmetry for $\lambda = 0$, the symmetry is broken explicitly by the λ-term. If we expand the potential around either of the vacua ($\phi = \pm 1, \psi = 0$), there appears the following mass terms:

$$V(\chi, \psi) \simeq \frac{1}{2} m_\chi \chi^2 + \frac{1}{2} m_\psi \psi^2, \tag{15}$$

where

$$m_\chi \equiv \frac{\partial^2 V}{\partial \phi^2}\Big|_{(\phi = \pm 1, \psi = 0)} = 8 \tag{16}$$

and

$$m_\psi \equiv \frac{\partial^2 V}{\partial \psi^2}\Big|_{(\phi = \pm 1, \psi = 0)} = \lambda, \tag{17}$$

where $\chi \equiv \phi - 1$. It is seen that, due to the explicit symmetry breaking term, the massless Goldstone boson (ψ) which is normally massless, has acquired a mass (λ).

Let us consider the $U(1)$-symmetric case $\lambda = 0$. The potential reduces to the well-known complex φ^4 model, and we have the global $U(1)$ symmetry:

$$\begin{aligned}\Phi \longrightarrow \Phi' &= e^{i\theta}\Phi, \\ \mathcal{L}' &= \mathcal{L}.\end{aligned} \tag{18}$$

This symmetry leads to the following conserved current and charge, as deduced from the celebrated Noether's theorem [23]:

$$J_N^\mu = i(\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*). \tag{19}$$

$$Q_N = \int J_N^0 dx. \tag{20}$$

Writing the complex scalar field Φ in the form $\Phi = Re^{i\xi}$, the current (13) and the charge (14) take the following simple forms:

$$J_N^\mu = 2R^2 \partial^\mu \xi, \tag{21}$$

and

$$Q_N = 2 \int_{-\infty}^{+\infty} R^2 \partial^0 \xi dx. \tag{22}$$

It is obvious that the Noether charge vanishes for all static solutions, including the static kinks and antikinks to be introduced shortly. For a time-varying field like $\Phi = R(x)e^{i\omega t}$, however, we have the non-vanishing Noether charge $Q_N = 2\omega \int R^2(x)dx$.

By using the dynamical equations, it can be easily shown that the $U(1)$ current is partially conserved, and we have

$$\partial_\mu J_N^\mu = 2\lambda\phi\psi, \tag{23}$$

which is proportional to λ, like the situation arising in PCAC (partially conserved axial current). In PCAC, the explicit symmetry breaking term in the Lagrangian is usually assumed to be linear in the pseudo-Goldstone boson, the pion field π [23].

As we shall see in the next section, the symmetry of the system under $\phi \leftrightarrow \phi$ and $\psi \leftrightarrow -\psi$ leads to the appearance of two similar domain walls with the same energy per unit surface. The existence of a Z_2 symmetry breaking term in the potential (e.g., $\kappa\psi^n$, n=odd) lifts this degeneracy and makes punctured domain walls possible (see Section V).

III. DOMAIN WALL SOLUTIONS

Kink solutions in more than one spatial dimension form sheet-like structures called domain walls. When placed in 3D Euclidean space, the domain wall may be represented by a (xy) planar concentration of energy with the energy density along the z-axis highly peaked at a certain z.

For some systems (like the sine-Gordon or the ϕ^4 systems) the kink (domain wall) solution can be found analytically. For many others, including the system under consideration, analytical solutions cannot be found and one must use numerical methods. In order to find the static kink and antikink solutions which play the role of domain walls with opposite topological charges, we have employed the following numerical procedure. The algorithm starts with an approximate solution, which is the exact solution of the $U(1)$-symmetric system ($\lambda = 0$). The solution is then varied via small changes in the field values, and the total energy per unit area of the domain wall, as given by

$$E = \int H dx, \tag{24}$$

is calculated at each step. The small change in the field values is accepted if the total energy is reduced in each step, otherwise it is rejected. This procedure is iterated repeatedly, until the program reaches a minimum energy configuration. Domain wall (kink and anti-kink)) solutions obtained in this way are shown in Figure 1.

It is well known that in many nonlinear equations bearing topological solitons, static solutions satisfy the so-called Bogomolny condition. This condition puts a lower bound on the total energy of the system which is proportional to the topological charge [1, 26]. Multiplying Equation (5) by ϕ' and adding it to Equation (6) multiplied by ψ', we obtain

$$\phi'\phi'' + \psi'\psi'' = \left(\frac{1}{2}(\phi')^2 + \frac{1}{2}(\psi')^2\right)' = \frac{\partial V}{\partial\phi}\phi' + \frac{\partial V}{\partial\psi}\psi' = \frac{dV}{dx}. \tag{25}$$
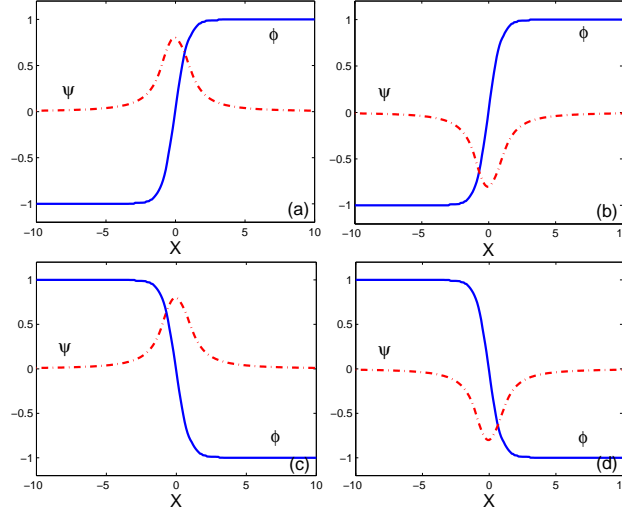
FIG. 1: Minimum energy, static solutions. The solid curves represents ϕ and the dash-dotted curves are for ψ. As one can see, under the transformation $(\phi, \psi) \rightarrow (-\phi, -\psi)$, the soliton k_1 (Fig1.a) changes to $\tilde{k}_2$ (Fig1.b) and $\tilde{k}_1$ (Fig1.c) changes to k_2 (Fig1.d).

Here, prime means derivative with respect to x. We thus obtain the following first integral of the static field equations:

$$\frac{1}{2}(\phi')^2 + \frac{1}{2}(\psi')^2 - V(\phi, \psi) = C, \tag{26}$$

where C is a constant of integration. For localized solutions this constant should be zero, since the fields rest on their vacuum values at $x \rightarrow \pm\infty$. It is seen that there are two types of kinks and antikinks which are related to each other by the field transformations $\phi \leftrightarrow -\phi$ and $\psi \leftrightarrow -\psi$, or a simple parity operation. As one can see, under this operation, the soliton k_1 (Fig1.a) changes to $\tilde{k}_2$ (Fig1.b) and $\tilde{k}_1$ (Fig1.c) changes to k_2 (Fig1.d).

For low energy density walls, one can use a Newtonian formulation to find the gravitational field via the Poisson equation:

$$\nabla^2 \Phi = 4\pi G\rho, \tag{27}$$

where Φ is the gravitational potential and ρ is the domain wall equivalent mass density, given by $\rho = H/c^2$, H being the Hamiltonian density $H = T^{00}$. Using the Gauss law for a cylindrical volume bisected by the domain wall, one finds

$$\mathbf{g} = -2\pi G\sigma\mathbf{k} \tag{28}$$

for large distances from the wall (compared to the thickness of the wall). In this equation, $\mathbf{g}$ is the gravitational field strength vector, $\mathbf{k}$ is the outward unit vector perpendicular to the domain wall, and σ is the domain wall mass density.

When g is comparable to c^2/z where c is the velocity of light and z is the distance from the wall, Newtonian theory breaks down and one has to use the general theory of relativity.

The gravitational effects of a planar domain wall in the thin-wall limit can be found in [27], [28]. In the thin-wall limit one assumes that the domain wall is infinitely thin, so only the vacuum Einstein equations need to be solved on either side of the wall. By matching the vacuum solutions on the two sides of the wall (i.e., implementing the junction conditions) which is facilitated by using the Gauss-Codazzi formalism, one can obtain the appropriate metric [29].

In this case, we have to use the following metric:

$$ds^2 = -f(z)dt^2 + h(z)(dx^2 + dy^2) + dz^2, \tag{29}$$

where $f(z)$ and $h(z)$ are unknown functions. The Einstein equations with the ϕ and ψ fields as sources read

$$G^\mu_\nu = 8\pi G T^\mu_\nu = 8\pi G \left[\partial^\mu \phi \partial_\nu \phi + \partial^\mu \psi \partial_\nu \psi - \delta^\mu_\nu (\frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi + \frac{1}{2} \partial^\alpha \psi \partial_\alpha \psi - V(\phi, \psi)) \right]. \tag{30}$$

We also have the following field equations for ϕ and ψ:

$$\nabla^\mu \nabla_\mu \phi + \frac{\partial V}{\partial \phi} = 0, \tag{31}$$

and

$$\nabla^\mu \nabla_\mu \psi + \frac{\partial V}{\partial \psi} = 0, \tag{32}$$

where ∇_μ is the covariant derivative. Some single field gravitational domain wall solutions have been discussed in [30] for the case when $16\pi G \langle \phi \rangle^2 \ll 1$, where $\langle \phi \rangle$ is the vacuum expectation value of the field ϕ. According to [30], no essentially different general relativistic effect is reported. However, when $16\pi G \langle \phi \rangle^2 > 1$, new effects are observed [31–33]. It should be noted that at large inter-wall distances scalar field interactions can be neglected, since they are short-range, while at short distances the reverse is true and the nonlinear scalar field interactions take over. In the next section, we have done our numerical calculations in the limit where the gravitational effects can be ignored.

IV. DOMAIN WALL COLLISIONS

Flat domain walls are essentially kink solitons. As in other kink-bearing systems, an important question is the form of the inter-kink potential and the behavior of the solitons in collisions with each other. Kinks and antikinks of different nonlinear systems behave differently in collisions. In most cases, the following situations arise: 1) A pair of kinks or antikinks which have the same topological charge repel each other. They retain their

original shape after the collision. In the sine-Gordon system, which is integrable, the pair retain their original speeds after the collision. In non-integrable systems like the ϕ^4 system, part of the energy is converted into small amplitude waves which are radiated away, and the final speed of the solitons is less than their initial speed [34]. 2) In the sine-Gordon system, the collision between a kink and an antikink does not lead to their destruction and the pair retain their initial speeds after the collision. The force between the pair is velocity-dependent [35]. It is attractive at relatively large distances and repulsive at short distances. In non-integrable systems like ϕ^4, ϕ^6, or the double-sine-Gordon system, the collision process between a kink and an antikink is more complicated and interesting phenomena happen [36, 37]. For example, the pair annihilate each other when their relative velocity is smaller than a first threshold v_1. For velocities larger than v_1 and smaller than a second threshold v_2, there appear scattering windows in which the pair leave the interaction region with a speed smaller than their initial speed. Some small amplitude waves are radiated away in this process. Velocities larger than v_2 lead to the scattering of the pair and emission of radiation. 3) In the sine-Gordon system, there is a bound state (breather) solution in which a kink and an antikink oscillate around the center of mass of the system indefinitely. Breather solutions in non-integrable systems like ϕ^4 are unstable and lead to the annihilation of the pair after transient oscillations.

In this section, we look for the above possibilities in the system under investigation. Since analytical calculations are not possible here, we employ the modified finite-difference method described in [7].

Figure 2 shows the simple scattering of a kink and antikink. The velocity of each soliton (in units c) is 0.6 for this process. Figures 3–5 show examples of some interesting interactions for the system considered in this paper. In Figure 3, the pair annihilate each other into a pair of neutral wave packets which leave the interaction area with larger velocities. Figure 4 shows the formation of a bound pair emitting the residual energy in the form of lower amplitude scalar waves. In Figure 5, the collision leads to the excitation of each domain wall and the excited wall relaxes into its lower energy state, emitting neutral waves within a short time. By neutral waves, we mean waves which do not carry net topological charge. In these plots, positive topological charge density is shown in red and negative charge in blue in order to better illustrate the location and the fate of charged objects. By plotting the charge densities, we will be able to track the charged lumps in the process of collisions and interactions. For example, in this way we would be able to detect whether a positively charged soliton bounces back a negatively charged one, or it tunnels through it. Even other fates are also possible (e.g., two neutral wave packets emerging (see Fig. 3)). All these possibilities are clearly distinguished when we plot the charge densities.

Quantum mechanically, the radiation emitted by the accelerating kinks is in the form of scalar particles [38, 39]. In three space dimensions, domain walls can undergo acceleration and deformations due to their own tension, except in the very special cases of static solutions. The radiation emitted from deformed domain walls has been calculated both analytically and numerically [40]. Radiation due to periodically deformed kinks has been calculated analytically in [41, 42].

Let us examine the bound pair in more detail. If there is a stable kink-antikink bound
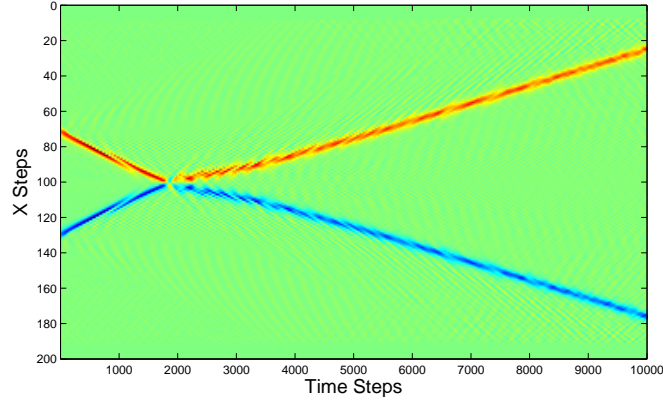
FIG. 2: Domain wall (kink-antikink) simple scattering at $v = 0.6$. The topological charge density is plotted on the (x, t) plane. The red color indicates positive and blue indicates negative topological charge. It is assumed that the collision is side-by-side and the gravitational effects are ignored.
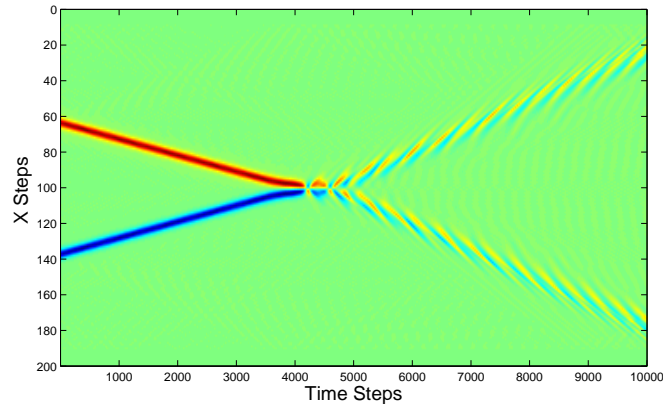

FIG. 3: Annihilation of a kink-antikink pair into a pair of neutral wave packets at $v = 0.36$. The red color indicates positive and blue indicates negative topological charge.

state, then one should be able to obtain it via an energy-minimization procedure. To this end, we have followed an energy-minimization algorithm, which produces a minimum-energy solution, starting with a trial pair of functions which satisfy the boundary conditions and the general functional form of the soliton pair. The initial guess functions read

$$\phi(x) = \frac{2}{1 + x^2} - 1, \tag{33}$$

and
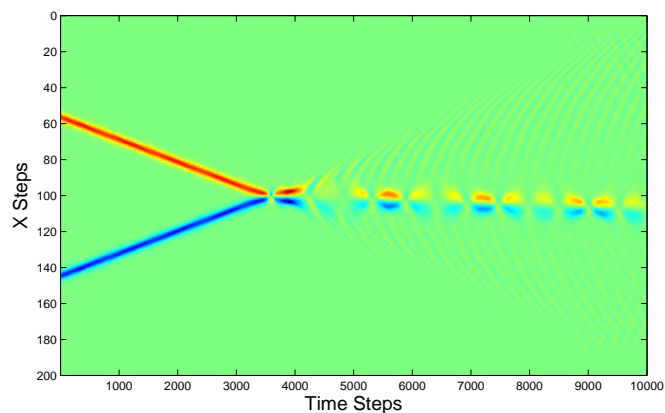
$$\psi(x) = \frac{x}{1 + x^2}. \tag{34}$$

FIG. 4: Formation of a soliton molecule via kink-antikink capture at $v = 0.5$. Note that the surplus energy is radiated away. The red color indicates positive and blue indicates negative topological charge. This system is different from a breather, since in a breather the positively and negatively charged kinks periodically interchange their position.
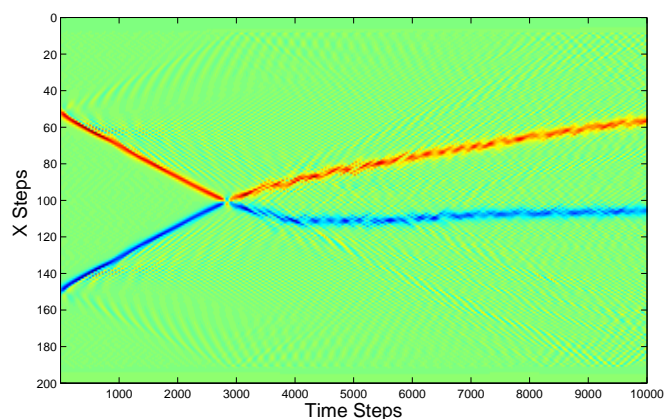

FIG. 5: Kink-antikink interaction, leading to the excitation and subsequent de-excitation and recoil of the kink and antikink ($v = 0.7$). The red color indicates positive and blue indicates negative topological charge.

It is obvious that these trial functions have zero total topological charges, comprising equal negative and positive charges of the kink and antikink constituents. The initial guess, together with the minimum energy solution are shown in Figure 6. This minimum-energy bound state of the kink and antikink closely conforms with the pair formed in the numerical experiment shown in Figure 4.
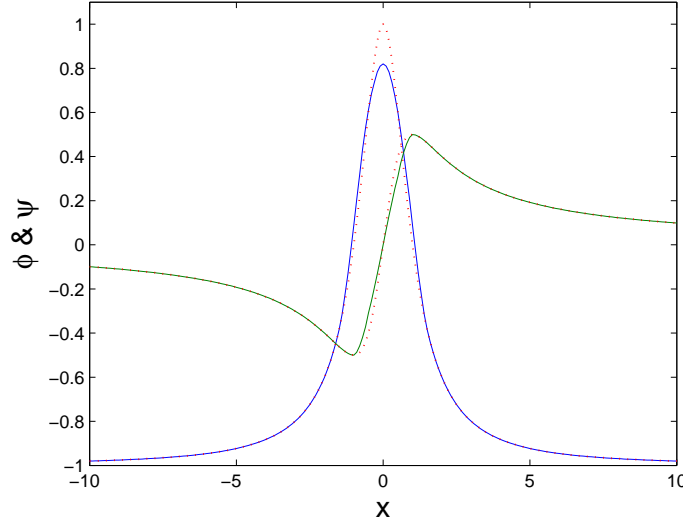
FIG. 6: Initial guess functions (33) and (34) shown as dashed curves, together with the minimum energy kink-antikink bound state solution (solid lines).

V. CAN THE DOMAIN WALLS GET PUNCTURED?

Some domain walls can get punctured. Here, we follow the criteria presented in [2] to check if our domain walls can get punctured, too. A punctured domain wall has a hole in it. The boundary of this hole is formed by a closed string. The potential considered in [2] is the following:

$$V(\Phi) = \frac{\lambda}{4}(|\Phi|^2 - \eta^2)^2 - \frac{\alpha\eta}{32}(\Phi + \Phi^*)^3, \tag{35}$$

in which $0 < \alpha \ll \lambda$. It is seen that this potential differs from (1) in their second terms. The extrema of this potential are located at $\psi = 0$ and

$$\psi = \eta \left[\frac{3\alpha + \sqrt{9\alpha^2 + 64\lambda^2}}{8\lambda}\right], \quad \chi = n\pi. \tag{36}$$

Here, ψ and χ are the modulus and phase of Φ (i.e., $\Phi = \psi \exp(i\chi)$). A domain wall exists when we have two disconnected vacua at boundaries (e.g., $\chi(-\infty) = 0$ and $\chi(+\infty) = 2\pi$). Now the path from $\chi = 0$ to $\chi = 2\pi$ can be contracted by lifting it over the top of the potential at $\psi = 0$. In this way, a patch of the domain wall can be bounded by a string and a hole can form [43] (see Figure 7).

Now we turn to the potential (1) to see if it is topologically possible to have the same situation. As mentioned in the Introduction, we have two disconnected points at ($\phi = \pm 1$, $\psi = 0$) as true vacua. There is a potential barrier located at $\phi = \psi = 0$ (false vacuum)

which separates these two points. Therefore, it is topologically possible for the field to be contracted by lifting it over the false vacuum. If the Z_2 symmetry in the direction of the ψ-field is broken by a term like $\kappa\psi^n$, n=odd, then the punctured area will have a lower energy density compared to the domain wall. The topology of the field-3D space mapping is shown in Figure 7.
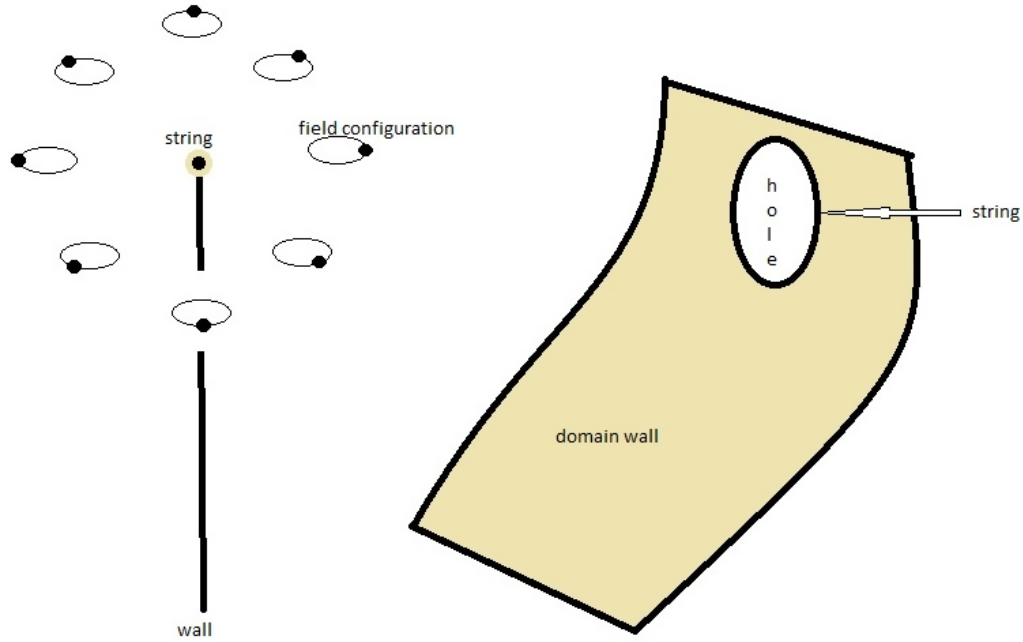

FIG. 7: Domain wall punctured by a closed string. The small ellipses indicate the field configuration on the (ϕ, ψ) plane. It is assumed that the ψ-field Z_2 symmetry is broken and $\psi > 0$ vacuum has a lower energy than the $\psi < 0$ vacuum.

Whether the puncture tends to get larger and larger or likes to pinch off, is an interesting question which needs further investigation. Some effects resulting from the wall being punctured, in the context of the symmetron model are discussed in [44].

VI. CONCLUSION

Kink-bearing systems show very interesting phenomena [6–11]. Domain walls are in fact mathematically the same structures, extended in two more spatial dimensions. A complex scalar field with $U(1)$ symmetry is well known and worked out thoroughly in field theory [23]. When the global $U(1)$ symmetry is made local, it leads to the appearance of electric charge and electromagnetic interactions. A system comprised of a complex scalar field coupled to the U(1) gauge field with spontaneously broken symmetry (the so-called

abelian Higgs model) is known to bear cosmic string solutions. Motivated by these interesting properties, we considered a double-real-field Lagrangian with a $U(1)$-breaking term. We obtained static domain wall solutions and showed that there are two degenerate pairs of kinks and antikinks in the system, related to each other by the symmetry operations $\phi \leftrightarrow \phi$ and $\psi \leftrightarrow -\psi$. Several numerical experiments were performed to explore what happens in the parallel collision of domain walls at various relative velocities. It was observed that different interesting phenomena may happen. Examples include simple scattering, pair annihilation into neutral wave packets, formation of soliton molecules (bound kink-antikink pairs), and excitation-decay process. The soliton molecule formed in some kink-antikink collisions approximately conforms with the solution obtained via minimizing the energy of a pair of guess functions adapted to the required topological charge and boundary conditions. In order to distinguish charged solitons from neutral wave packets and follow the evolution of each charged soliton, we preferred to plot charge densities rather than the fields or energy densities, which are more common in the literature.

Another observation to be pointed out is that in many examples, the resulting dynamics is not symmetrical about the pair center of mass. In other words, there is a left-right asymmetry which constitutes yet another difference with other well-known non-integrable systems. The model is also interesting in the sense that in the case of vanishing coupling constant $\lambda = 0$, it reduces to a global U(1) system. For $0 < \lambda < 4$ two pairs of degenerate kinks appear which are transformed to each other by a parity transformation. For $\lambda > 4$ the system essentially reduces to the ϕ^4 system.

Finally, we discussed the topological possibility for the domain wall being punctured by a string loop. We argued that the form of the potential allows the field to lift over the potential barrier at $\phi = \psi = 0$ and form a string at the boundary of the domain wall, forming a hole bounded by a string.

Acknowledgements

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