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Synchronization in oscillator networks with time delay and limited non-homogeneous coupling strength

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Abstract Motivated by the needs of multi-agent systems in the presence of sensing and communication which is delayed, intermittent, and asynchronous, we present a Kuramoto-type model as the delay inherent in such systems is taken into account. First, we have investigated a condition on maximum delay for the frequency entrainment of non-identical Kuramoto oscillators with heterogeneous delays and a constant coupling gain. Our next mission is to investigate the model of delayed coupled Kuramoto oscillators, which are characterized by non-identical natural frequencies and non-homogeneous coupling strength. We assume that the difference between the coupling gains is less than a certain limited value M, and on the basis of Lyapunov stability theorem, we present a strictly positive lower bound for M to achieve a consensus on the derivatives of the phases. This consensus property is even more surprising because the phases themselves do not necessarily reach a consensus. We apply our results to these oscillators and show that synchronization is guaranteed for appropriate initial conditions.

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1 Introduction

Synchronization phenomena in large populations of interacting elements are the subject of intense research efforts in physical, biological, chemical, and social systems. Over the past two decades, myriad researchers have spent a great deal of time studying synchronization, which combines ideas from nonlinear dynamics and network theory. Examples in biology and physics include groups of synchronously flashing fireflies [1] and Josephson junctions [2] and the natural circadian rhythms of the human brain [3]. A successful approach to the problem of synchronization consists of modeling each member of the population as a phase oscillator. Synchronizations the process by which interacting, oscillating objects affect each other such that they spontaneously lock to a certain frequency or phase [4].

Collective synchronization was first studied by Wiener [5], who conjectured its involvement in the generation of alpha rhythms in the brain. It was then taken up by Winfree [6] who used it to study circadian rhythms in living organisms. Winfree's model was significantly extended by Kuramoto [7] who developed results for what is now popularly known as the Kuramoto model. Similar to "all-to-all" case, a problem of frequency synchronization for a system of Kuramoto oscillators in the case where interconnection graph is not necessary to be symmetric is considered [8]. It was also observed that the system achieves synchronization for arbitrary topologies when the oscillators are identical. Recent works in the control community investigate the close relationship between Kuramoto oscillators and consensus networks [9].

The pinning synchronization on complex networks with weak coupling was investigated in [10]. Also some selected links were strengthened to reduce the energy cost in the process of achieving and keeping synchronization. Kuramoto oscillator networks are what this paper deals with, but we included a feature that was usually ignored in the previous dynamical network models and is focused on synchronization in general oscillator networks with delays. Time delay is a fundamental reality of physical systems due to the finite speeds of transmission and spreading as well as traffic congestion. Since the finite speed of signal transmission over a distance gives rise to a finite time delay, it is well known that the information flow in complex network is not instantaneous in general. In view of the time-delay phenomenon, which is frequently encountered in practical situations, the complex network is further extended to include coupling delays among its nodes, and synchronization conditions of these networks have been investigated analytically. For example, the problem of sampled-data exponential synchronization of complex dynamical networks with time-varying coupling delay and variable sampling is investigated in [11]. By the usage of the time-dependent Lyapunov functional approach and convex combination technique, sufficient conditions were proposed to ensure the stability of error dynamics.

Coupled dynamic networks with time-varying delay have been considered in [12]. A new closed-loop coupled dynamic error system with Markovian jump parameters and interval time-varying delays has been constructed. In [13], a projective synchronization of chaotic complex system was investigated. First, the cross-projective synchronization in coupled partially linear complex nonlinear dynamic system was realized without adding any control term. Then, the substantial conditions of projective synchronization of chaotic complex systems on whole state variable with time delay and adaptive controllers are investigated. The problem of adaptive synchronization for a complex dynamical network with coupling time-varying delay is investigated in [14,15]. In the latter problem, conditions ensuring stability of congestion control schemes for arbitrary topologies with delays at the linearized level have been proposed [16]; recently, these have been extended to cover the nonlinear cases [17,18]. However, these techniques would bring some fixed coupled gains, while non-uniform Kuramoto oscillators inevitably exist in natural and man-made networks. So it should be pointed out that they restrict applications of the synchronization criteria. For this reason, in this paper, we have decided to present a synchronization phenomenon for Kuramoto oscillators with delayed, non-identical coupled strength. Non-identical coupled strength means coefficients with small bounded variations from nominal values. We consider all values of $\varepsilon_{\mathrm{ki}}$ are smaller than limited value. The methodology we used is based on an invariance principle for Lyapunov–Razumikhin functions. The main ideas of the proof are based on recent results in Münz et al. [19]. The remainder of this paper is organized as follows: Sect. 2 is devoted to some preliminary results on functional differential equations (FDE) as well as algebraic graph theory that will be used in the rest of this paper. In Sect. 3, we describe our delay coupled non-identical Kuramoto oscillator model. Our main result, synchronization of delayed Kuramoto oscillators with non-uniform coupled coefficients, is proved in Sect. 4. The conclusions and discussion are given in Sect. 5.

2 Notations and definitions

Definition 1 [*Functional differential equations (FDE)*] We denote the functional differential equations (FDE) on Ω as:

$$\dot{x}(t) = f(x_t) \tag{1}$$

It is clear that a unique solution for Eq. (1) exists if $f(\phi)$ is Lipschitz in each compact set in Ω.

Definition 2 (*Equilibrium point*) An element $\Phi \in C([-T, 0], R^n)$ is called a steady state or equilibrium of (1) if $x_t(\Phi) = \Phi$ for all $t \geq 0$. In the sequel without loss of generality, we consider $\Phi = 0$ to be the equilibrium point.

Proposition 1 *Consider $f: \Omega \to R^n$ be a locally Lipschitz function that maps bounded subsets of Ω into bounded subsets of R^n and consider* (1). *Suppose $v, \omega : R^+ \to R^+$ are continuous, non-decreasing functions, $v(s)$ be positive for $s > 0$, $v(0) = 0$, for* *such systems if there exists a Lyapunov–Razumikhin function $V : D \to R$ such that:*

$$\begin{cases} V(x) & \geq v(|x|) \text{ for } x \in D, \text{ and} \\ \dot{V}(\varphi(0)) & \leq -\omega(\varphi(0)) \text{ if } V(\varphi(0)) \\ & = \max_{-T \leq s \leq 0} V(\varphi(s)) \end{cases}$$

Then, the equilibrium $x = 0$ of (1) *is stable* [20]. *The idea of the Razumikhin-type theorem is to treat the stability problem by utilizing a Lyapunov function instead of a Lyapunov functional* [21]. *In the beginning of his research, Razumikhin considered the single system* (1) *and investigated the stability problem on the basis of first approximations. He demonstrated that the zero solution of this system is asymptotically stable if a positive-definite function $V(x)$ has a negative-definite derivative along the solution of* (1) [22].

Theorem 1 *Suppose there are a Lyapunov–Razumikhin function $V = V(x)$ and a positively invariant closed set Ω with respect to (1) such that:*

$$\dot{V}(\varphi) \leq 0 \quad \forall \varphi \in \Omega \, s.t. \; V(\varphi(0)) = \max s\, [-T, 0]\, V(\varphi(s)) \tag{2}$$

Then, for any $\varphi \in \Omega$, $x(\varphi)$ is defined and bounded on $[-T, \infty)$, $\omega(\varphi) \subseteq M_V \subseteq E_V$ and $x_t(\phi)$ converges to M_V, that is $\lim_{t\to\infty} x_t(\phi) = M_V$.

Here, $\omega(\varphi)$ is the ω-limit set of φ and non-empty, compact, connected, and invariant, If $x(\varphi)$ is a solution of (1) that is defined and bounded on $[-T, \infty)$, then the orbit through φ which is the set $\{x_t(\varphi) : t \geq 0\}$ is precompact, and $\lim_{t\to\infty} x_t(\varphi) = \omega(\varphi)$. E_V is the set of functions $\varphi \in \Omega$ which can serve as initial conditions for (1) and is defined as:

$$E_V = \left\{\varphi \in \Omega : \max_{s\in[-T,0]} V(x_t(\varphi)(s)) = \max_{s\in[-T,0]} V(\varphi(s)), \forall t \geq 0\right\} \tag{3}$$

Note that the above condition guarantees that for a Lyapunov–Razumikhin function V and for any $\varphi \in E_V$, we have $\dot{V}(x_t(\varphi)) = 0$ for any $t > 0$ such that:

$$\max_{s\in[-T,0]} V(x_t(\varphi)(s)) = V(x_t(\varphi)(0))$$

And M_V is the largest invariant set in E_V [20]. Theorem 1 will be used extensively in our work. It proves the attractivity of invariant subsets M_V for the solutions of multi-agent systems.

2.1 Graph theory

Let $G(V, E)$ be a graph with vertex set $V\{v_1, v_2, \ldots, v_n\}$ and edge set $E\{e_1, e_2, \ldots, e_m)$. For each edge $e_j = \{v_i, v_k\}$, choose one of the v_i, v_k to be the positive end of e_j and the other to be the negative end.

Definition 3 The vertex-edge incidence matrix afforded by an orientation of n-by-m matrix $Q = Q(G) = (q_{ij})$, where:

$$hq_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the positive end of } e_j \\ -1, & \text{if it is the negative end,} \\ 0, & \text{otherwise.} \end{cases}$$

The number of parents of system i, also called thein-degree d_i of vertex v_i, is defined by $d_i = \sum_{j=1}^{N} q_{ij}$ [23].

2.2 Consensus in multi-agent systems

Before illustrating the synchronization of delayed Kuramoto oscillators, we consider to a simple car following model in multi-agent systems (MAS) as follows and apply the proposed theorem 2 for Kuramoto oscillator in the next section.

$$\dot{x}_i(t) = -K_i \sum_{k=1}^{N} \frac{\alpha_{ki}}{d_i} g_{ki}(x_i(t) - x_k(t - \tau_{ki})) \tag{4}$$

where $x_i(t) \in R$ is the position of driver i at time t, $K_i \neq 0$ models the car-agility, α_{ki} are the elements of the adjacency matrix that models the sight of the drivers, d_i is the in-degree of the node i in the graph, $g_{ki} : R \to R$ are continuously differentiable, possibly nonlinear coupling functions. The main result is to provide conditions for MAS (4) to reach consensus asymptotically; i.e., all agents eventually converge to the same state $x_i = x_k$ for all i, k.

Assumption 1 There exists a set $B \subseteq R$ such that g_{ki} and K_i satisfies $K_i \frac{dg_{ki}(y)}{dy} > 0$ for all k, i, all $y \in B$. Furthermore, $x_i(t) - x_k(t - \tau_{ki}) \in B$ for all i, k and all $t \geq 0$.

Theorem 2 *Assume the communication network of MAS* (4) *contains N agents with dynamics, where g_{ki} satisfies Assumption 1, and with initial condition φ, as well as an underlying network topology of a directed graph with a spanning tree, then the consensus of the state derivatives* (4) *is asymptotically achieved for any fixed delay τ_{ki} if all initial φ_i conditions fulfill*

$\varphi_i(0) - \varphi_k(s) \in B$ *for all* $i, k \in \{1, \ldots, N\}$ *and for all* $s \in [-T, 0]$.

Proof The main ideas of the consensus proof are based on the invariance principle of Lyapunov–Razumikhin functions and results in Theorem 1. See Theorem 7 of [19].

3 Synchronization problem in delayed Kuramoto oscillators

In this section, we provide the model and conclude the results and contributions. Consider a set of N coupled oscillators with phases θ_i and natural frequencies ω_i. The phase of each oscillator θ_i is associated with a vertex $v_i \in V$ of an underlying undirected graph G. In particular, we consider the following delayed Kuramoto dynamics at the i-th oscillator for a complete graph the "all-to-all" case:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k(t-\tau_{ki}) - \theta_i(t)), \tag{5}$$

where K is the coupling strength between the oscillators and $\tau_{ki} i = 1, \ldots, N$ is the delay between the oscillator k and i. In order to synchronize the Kuramoto oscillators, the following equation is considered:

$$\forall i, j \in \{1, \ldots, N\} : \lim_{t\to\infty} \dot{\theta}_i(t) - \dot{\theta}_j(t) = 0 \tag{6}$$

Lemma 1 *The phase difference of the delayed Kuramoto oscillators* (5) *satisfies* $\theta_i(t) - \theta_j(t) \in D$ *for all* $i, j \in I$ *and for all* $t \geq 0$ *if the following conditions satisfied:*

- *Coupling gain K is sufficiently large,*
- *The maximal delay* $T = \max_{i\in I} \tau_{ki}$ *satisfies*

$$T < \min\left\{\frac{\arcsin(\cos(\delta)) - \frac{N(\omega_{\max}-\omega_{\min})}{K(N-2)}}{2(\max\limits_{i\in I}|\omega_i| + k)}, \frac{\delta}{2(\max\limits_{i\in I}|\omega_i| + k)}\right\} \tag{7}$$

  The initial condition of the oscillators satisfies $\theta_i(0) - \theta_j(0) \in D = \left\{x \in R \,|\, \|x\|_\infty < \frac{\pi}{2} - \delta\right\}$
- *The rate bound* $\left|\dot{\theta}_i(s)\right| \leq (K + \max_{i\in I}|\omega_i|)$ *for all* $i, j \in I$ *and all* $s \in [-T, 0]$. *[19]*

Proof $\left|\theta_i(t) - \theta_j(t)\right| \leq \frac{\pi}{2} - \delta$ is satisfied as long as the difference of the initial conditions for θ_i and θ_j at time $t = 0$ is in D [19]. Thus, we consider the case where at least two oscillators are at the critical distance. Therefore, V is defined by:

$$V = \max_{i\in I}\theta_i(t) - \min_{j\in I}\theta_j(t) \tag{8}$$

Consider the set of all points with $V = \frac{\pi}{2} - \delta$. If $\dot{V} < 0$ holds at these points, we are sure that $\left|\theta_i(t) - \theta_j(t)\right|$ cannot grow beyond $\frac{\pi}{2} - \delta$.

Choosing $\theta_U = \max_{i\in I}\theta_i(t)$ *and* $\theta_L = \min_{j\in I}\theta_j(t)$ leads to:

$$\dot{V} = \dot{\theta}_U - \dot{\theta}_L = \omega_U - \omega_L - \frac{K}{N}\sum_{k=1}^{N} (\sin(\theta_U - \theta_k(t-\tau_{kU})) + \sin(\theta_k(t-\tau_{kL}) - \theta_L)) \tag{9}$$

The delay in (9) can be rewritten as:

$$\theta_k(t - \tau_{ki}) = \theta_k(t) - \underbrace{\int_{-\tau}^{0}\dot{\theta}(t+s)\mathrm{d}s}_{=\varepsilon_{ki}} \tag{10}$$

The rate bound is $\left|\dot{\theta}_i(s)\right| \leq (K + \max_{i\in I}|\omega_i|)$, so from (10) we have:

$$\begin{aligned}|\varepsilon_{ki}| &\leq \int_{-\tau_{ki}}^{0}\left|\dot{\theta}_i(t+s)\right|\mathrm{d}s \\ &\leq \underbrace{\max_{i,k\in I}\tau_{ki}}_{=T}(\max_{i\in I}|\omega_i| + k) = \check{\delta}\end{aligned} \tag{11}$$

The second term in (7) implies that $\check{\delta} < \frac{\delta}{2}$. Therefore,

$$|\varepsilon_{kU}| \leq \check{\delta} < \frac{\delta}{2}, |\varepsilon_{kL}| \leq \check{\delta} < \frac{\delta}{2} \tag{12}$$

Inserting (10) into (9) results in

$$\begin{aligned}\dot{V} &= \dot{\theta}_U - \dot{\theta}_L \\ &= \omega_U - \omega_L - \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k + \varepsilon_{kU}) \\ &\quad + \sin(\theta_k - \theta_L - \varepsilon_{kL})\end{aligned} \tag{13}$$

As stated before, $\dot{V}$ has to be negative. Since the coupling gain K is the tunable parameter in (5), we have to find a strictly positive lower bound for

$$\begin{aligned}&\sin(\theta_U - \theta_k + \varepsilon_{kU}) + \sin(\theta_k - \theta_L - \varepsilon_{kL}) \\ &\quad = 2\sin\left(\frac{\theta_U - \theta_k + \varepsilon_{kU} - \varepsilon_{kL}}{2}\right) \\ &\quad\quad \times \cos\left(\frac{\theta_U - \theta_k + \varepsilon_{kU} - (\theta_k - \theta_L - \varepsilon_{kL}}{2}\right)\end{aligned} \tag{14}$$

We want to find a lower bound for equation in (14), so we use the relations

$$0 \le \theta_U - \theta_k \le \frac{\pi}{2} - \delta, 0 \le \theta_k - \theta_L \le \delta - \frac{\pi}{2} \quad (15)$$

By summing up the results from (12) and (15), we obtain:

$$-2\check{\delta} + \delta - \frac{\pi}{2} \le \theta_U - \theta_k + \varepsilon_{kU} - (\theta_k - \theta_L - \varepsilon_{kL}) \le \frac{\pi}{2} - \delta + 2\check{\delta} \quad (16)$$

Inequality (16) directly yields to a lower bound for the cosine term in (14); i.e.,

$$\cos\left(\frac{\theta_U - \theta_k + \varepsilon_{kU} - (\theta_k - \theta_L - \varepsilon_{kL}}{2}\right) \ge \cos\left(\frac{\frac{\pi}{2} - \delta + 2\check{\delta}}{2}\right) \quad (17)$$

Similarly, we attain a lower bound for the sine term in (14), i.e.,

$$\sin\left(\frac{\theta_U - \theta_k + \varepsilon_{kU} - \varepsilon_{kL}}{2}\right) \ge \sin\left(\frac{\frac{\pi}{2} - \delta - 2\check{\delta}}{2}\right) \quad (18)$$

The inequalities (17) and (18) give a bound for (14); i.e.,

$$\sin(\theta_U - \theta_k + \varepsilon_{KU}) + \sin(\theta_k - \theta_L - \varepsilon_{KL}) \ge 2\sin\left(\frac{\frac{\pi}{2} - \delta - 2\check{\delta}}{2}\right)\cos\left(\frac{\frac{\pi}{2} - \delta + 2\check{\delta}}{2}\right) = \cos(\delta) - \sin\left(2\check{\delta}\right) \quad (19)$$

We abbreviate $\eta = \cos(\delta) - \sin\left(2\check{\delta}\right)$, since $\check{\delta} < \frac{\delta}{2} < \frac{\pi}{8}$. So we have $\eta > 0$. Combining (13) and (19) results in an upper bound for $\dot{V}$, i.e.,

$$\dot{V} \le \omega_{\max} - \omega_{\min} - \frac{K}{N}(N-2)\eta \quad (20)$$

The factor $N - 2$ in (20) comes from the fact that we consider no delay in the self-coupling. Therefore, one of the sine summands in (9) vanishes for $k \in \{U, L\}$; because the other summand is positive, we bound them by zero.

The inequality (20) directly yields a condition for K such that $\dot{V} < 0$ if $V = \frac{\pi}{2} - \delta$, namely

$$K > \frac{N(\omega_{\max} - \omega_{\min})}{(N-2)\eta} \quad (21)$$

And solving the inequality for T gives:

$$T < \frac{\arcsin(\cos(\delta) - \frac{N(\omega_{\max} - \omega_{\min})}{K(N-2)}}{2(\max_{i\in I}|\omega_i| + k)} \quad (22)$$

Since (22) holds by (7), we get $\dot{V} < 0$, therefore $\left|\theta_i(t) - \theta_j(t)\right| \le \frac{\pi}{2} - \delta$ for all $t \ge 0$. [19]

Theorem 3 *The Kuramoto oscillators (5) achieve the synchronization under the conditions of Lemma 1.*

Proof First, we observe that (5) is a special case of (4) with $g_{ki}(y) = \sin(y)$. Also we consider all-to-all coupled oscillators; i.e., $\alpha_{ki} = 1$ for all i, k and $d_i = N$ for all i. According to assumption 1, B is given by $B = \left\{x \in R \,|\, \|x\|_\infty < \frac{\pi}{2}\right\}$. Lemma 1 yields to $D = \left\{x \in R \,|\, \|x\|_\infty < \frac{\pi}{2} - \delta\right\}$ such that $\theta_i(t) - \theta_j(t) \in D$ for all $i, j \in \{1, \ldots, N\}$ and $t \ge 0$. Lemma 1 results in following inequality:

$$\left|\theta_i(t) - \theta_j\left(t - \tau_{ij}\right)\right| \le \max_{i,j\in I}\left|\theta_i(t) - \theta_j\left(t - \tau_{ij}\right)\right| \le \max_{i,j\in I}(\left|\theta_i(t) - (\theta_j(t)\right| + \int_{-\tau_{ij}}^{0}\left|\dot{\theta}_i(t+s)\right| \mathrm{d}s) \le \frac{\pi}{2} - \delta + T\left|\max_{i\in I}|\omega_i| + k\right|$$

Since (7) holds, we have:

$$\frac{\pi}{2} - \delta + T\left|\max_{i\in I}|\omega_i| + k\right| < \frac{\pi}{2}$$

Consequently, $\theta_i(t) - \theta_j\left(t - \tau_{ij}\right) \in B$ forall $i, j \in I$ and $t \ge 0$. According to Theorem 2, in such condition the synchronization Eq. (6) is satisfied [19].

4 Synchronization in delayed Kuramoto oscillators with limited heterogeneity coupled strength

In this section, we propose a Kuramoto model with limited heterogeneity coefficients. In the previous part, we assumed that coupling gain K is a tunable and fixed parameter, whereas in real models of classical Kuramoto, this value is not constant and there is much heterogeneity. For example, in the network-reduced model of a power system with non-trivial transfer conductance, there is some equivalence between the classic power network model equations and a non-uniform Kuramoto model which is described by non-homogeneous coupling and non-uniform phase shifts. The establishment of sufficient conditions for synchronization of non-uniform Kuramoto oscillators can be derived from concise and purely algebraic conditions which relate synchronization and transient stability of a power network to the underlying system parameters and initial conditions. Since we consider a non-uniform Kuramoto model which states that coupling gains are heterogeneous, the differences between them are less than a certain level similar to known M. In other words, the constant coefficients K in Eq. (5) are replaced with heterogeneous coefficients $(K+\in_{ki})$. Therefore, Eq. (5) will be changed as follows:

$$\dot{\theta}_i(t) = \omega_i + \frac{1}{N}\sum_{k=1}^{N}(K+\in_{ki}) \times \sin(\theta_k(t-\tau_{ki}) - \theta_i(t)) \tag{23}$$

We consider all values of $\in_{ki}$ to be smaller than limited value M by using results in a synchronous delayed system and the maximum delay Lemma 1. In this part, we obtain the maximum amount available extensions of M. Similar to Eq. (13), the derivative of Lyapunov function in case coupling values are heterogeneously obtained.

$$\dot{V} = \omega_U - \omega_L - \frac{1}{N}\sum_{k=1}^{N}(K+\in_{kU})\sin(\theta_U - \theta_k + \varepsilon_{kU}) + (K+\in_{kL})\sin(\theta_k - \theta_L - \varepsilon_{kL}) \tag{24}$$

Proposition 2 *Consider the oscillators with heterogeneous coupling gain (23). The related derivative of Lyapunov function defined in (24) can be rewritten based on Kuramoto model with fixed strength as it is illustrated in (13).*

Proof Let $f(X)$ and $f(Y)$ be the derivatives of Lyapunov function, respectively, for delayed Kuramoto model with heterogeneous and fixed strength coupling. It is clear that $f(X)$ and $f(Y)$ can be rewritten as:

$$\underbrace{f(K+\in)}_{f(X)} = \omega_U - \omega_L - \frac{1}{N}\sum_{k=1}^{N}(K+\in_{kU}) \times \sin(\theta_U - \theta_k + \varepsilon_{kU}) + (K+\in_{kL})\sin(\theta_k - \theta_L - \varepsilon_{kL}) \tag{25}$$

$$\underbrace{f(K)}_{f(Y)} = \dot{\theta}_U - \dot{\theta}_L = \omega_U - \omega_L - \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k + \varepsilon_{kU}) + \sin(\theta_k - \theta_L - \varepsilon_{kL}) \tag{26}$$

where

$$X = (K+\in_{1U}, K+\in_{2U}, \ldots, K+\in_{1L}, \ldots, K+\in_{KL}) \tag{27}$$

$$X - Y = (\in_{1U}, \in_{2U}, \ldots.., \in_{1L}, \ldots., \in_{KL}) \tag{28}$$

Considering Taylor expansions for linear functions around points K and after omitting the higher-order terms (relating to the fact that values of $\in_{ki}$ are closed to each other), one can rewrite $f(X)$ as follows:

$$f(X) = f(Y) + \nabla f(Y)(X-Y) \tag{29}$$

which is equal to

$$f(K+\in) = f(K) + \nabla f(K)(\in) \tag{30}$$

Lemma 2 *For the delayed Kuramoto oscillators (24) with heterogeneous coefficients $(K+\in_{ki})$, the synchronization condition is satisfied under the following inequality $\forall i, j \in I : |\varepsilon_{ki}| \le M$*

$$M \le \frac{\omega_{\max} - \omega_{\min} - \frac{K}{N}(N-2)\left(\cos(\delta) - \sin\left(2T\left(\max_{i\in I}|\omega_i| + k\right)\right)\right)}{\frac{1}{N}(N-2)\left\{\cos(\delta) - \sin\left(2T\left(\max_{i\in I}|\omega_i| + k\right)\right)\right\}} \tag{31}$$

Proof As is proved in Lemma 1, the maximum delay is calculated for $\dot{V} < 0$, namely $f(K) < 0$. Similarly, this method is applied for proving the stability of delayed Kuramoto (13). So the following inequality should be established:

$$f(K+\in) < 0 \tag{32}$$

Inserting (29) into (32) results in

$$f(K) + \nabla f(K)(\in) < 0 \rightarrow \in < -\frac{f(K)}{\nabla f(K)} \tag{33}$$

First via Eq. (26), we get gradient around each point of K as:

$$\nabla f(K) = -\frac{1}{N}\{\sin(\theta_U - \theta_1(t-\tau_{1U})) + \sin(\theta_1(t-\tau_{1L}) - \theta_L) + \cdots + \sin(\theta_U - \theta_N(t-\tau_{NU})) + \sin(\theta_N(t-\tau_{NL}) - \theta_L) \tag{34}$$

Inserting (26) and (34) into (33) causes

$$\in < \frac{-\left\{\omega_U - \omega_L - \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k\,(t-\tau_{kU})) + \sin\,(\theta_k\,(t-\tau_{kL}) - \theta_L)\right\}}{-\frac{1}{N}\left\{\sin\,(\theta_U - \theta_1\,(t-\tau_{1U})) + \sin\,(\theta_1\,(t-\tau_{1L}) - \theta_L)\ldots + \sin\,(\theta_U - \theta_N\,(t-\tau_{NU})) + \sin\,(\theta_N\,(t-\tau_{NL}) - \theta_L)\right\}} \tag{35}$$

Assuming that all values of $\in_{ki}$ are smaller than limited value M results in (36)

$$M = \frac{\left\{\omega_U - \omega_L - \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k\,(t-\tau_{kU})) + \sin\,(\theta_k\,(t-\tau_{kL}) - \theta_L)\right\}}{\frac{1}{N}\left\{\sin\,(\theta_U - \theta_1\,(t-\tau_{1U})) + \sin\,(\theta_1\,(t-\tau_{1L}) - \theta_L)\ldots + \sin\,(\theta_U - \theta_N\,(t-\tau_{NU})) + \sin\,(\theta_N\,(t-\tau_{NL}) - \theta_L)\right\}} \tag{36}$$

According to (20), we can find an upper bound for the numerator of (36) as follows:

$$\omega_U - \omega_L - \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k\,(t-\tau_{kU})) + \sin\,(\theta_k\,(t-\tau_{kL}) - \theta_L) \le \omega_{\max} - \omega_{\min} - \frac{K}{N}\,(N-2)\,\eta \tag{37}$$

To calculate M, we have to find a strictly positive lower bound for the denominator of (36). Moreover, we consider no delay in the self-coupling, so one of the sine summands in (9) vanishes for $k \in \{U, L\}$, because the other summand is positive. Based on inequality (19), we obtain:

$$\frac{1}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k\,(t-\tau_{kU})) + \sin\,(\theta_k\,(t-\tau_{kL}) - \theta_L) \ge \frac{1}{N}\,(N-2)\,\eta \tag{38}$$

Inverting (38) gives

$$\left[\frac{1}{N}\sum_{k=1}^{N}\sin(\theta_U - \theta_k\,(t-\tau_{kU})) + \sin\,(\theta_k\,(t-\tau_{kL}) - \theta_L)\right]^{-1} \le \left(\frac{1}{N}\,(N-2)\,\eta\right)^{-1} \tag{39}$$

With inserting (37) and (39) into (36), yields:

$$M \le \frac{\omega_{\max} - \omega_{\min} - \frac{K}{N}\,(N-2)\,\eta}{\frac{1}{N}\,(N-2)\,\eta} \tag{40}$$

As before, we know $0 < \eta = \cos\,(\delta) - \sin\left(2\check{\delta}\right)$. Combining (11) and (40) results in an upper bound for M, as:

$$M \le \frac{\omega_{\max} - \omega_{\min} - \frac{K}{N}\,(N-2)\left(\cos\,(\delta) - \sin\left(2T\left(\max_{i\in I}|\omega_i| + k\right)\right)\right)}{\frac{1}{N}\,(N-2)\left\{\cos\,(\delta) - \sin\left(2T\left(\max_{i\in I}|\omega_i| + k\right)\right)\right\}}$$

Consequently, we get $\dot{V} < 0$. We computed the maximum acceptable value of M, which is the largest region where $f(x) < 0$. The largest sphere around the K point of the delayed system is stable in which the system stays synchronized.

5 Conclusions

In this paper, we have studied the problem of synchronization of coupled oscillator networks in the Kuramoto framework with delay. First, we have investigated a condition on maximum delay for the synchronization of non-identical Kuramoto oscillators with heterogeneous delays and constant coupling gain. Then, we have considered that the model of delay coupled Kuramoto oscillators are characterized by non-identical natural frequencies and heterogeneous coupling gain, but the difference between the values is less than a certain limited value as M. Based on the stability theory of the Lyapunov function and by using previous conclusions, we proposed the certain criteria of M. A completely non-uniform delay coupled Kuramoto oscillators with coupling weight matrix similar to the one obtained in the case of network congestion control would be desirable, at least in the case in which there is a unique attracting set. This will be the focus of our future research.

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