8^{th} **Seminar on Geometry and Topology**

Amirkabir University of Technology, December 15-17, 2015

On Semicovering Maps

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Abstract

In this talk, after reviewing the concept of covering and semicovering maps, first we give a modified definition for semicovering maps which seems simpler than the original one given by J. Brazas. Second we present some conditions under which a local homeomorphism becomes a semicovering map.

AMS subject Classification 2010: 57M10, 57M12, 57M05.

Keywords: local homeomorphism, fundamental group, semicovering map.

1 Introduction

It is well-known that every covering map is a local homeomorphism. J. Brazas [2, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \to X$ has **continuous lifting of paths** if $\rho_p : (\rho Y)_y \to (\rho X)_{p(y)}$ defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0,1] \to Y | \alpha(0) = y\}$. Also A map $p : Y \to X$ **has continuous lifting of homotopies** if $\Phi_p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. He also gave a simpler definition in [3, Remark 2.5] for semicovering maps and showed that the condition *continuous lifting of paths* is enough for defining semicovering map, which is equivalent to the original definition of semicovering map given in [2, Definition 3.1].

Fischer and Zastrow showed that a local homeomorphism with Hausdorff domain is a semicovering if and only if all lifts of paths and their homotopies exist [4, Remark 5.1].

Now in this talk, we give a modified definition for semicovering maps which seems simpler than the original one given by Brazas [2, 3]. Moreover, we obtained some conditions under which

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a local homeomorphism is a semicovering map. In fact, we prove that a local homeomorphism with unique path lifting property (UPLP) and path lifting property (PLP) is a semicovering map. Also, we show that if $p : \tilde{X} \to X$ is a local homeomorphism, $\tilde{X}$ is Hausdorff and sequential compact, then p is a semicovering map. By an example we show that a local homeomorphism is not necessary a semicovering map (see Example 3.3).

2 Notations and Preliminaries

Definition 2.1. ([6]). Assume that X and $\tilde{X}$ are topological spaces. A continuous map $p : \tilde{X} \longrightarrow X$ is called a **local homeomorphism** if for every point $\tilde{x} \in \tilde{X}$ there exists an open set $\tilde{W}$ such that $\tilde{x} \in \tilde{W}$ and $p(\tilde{W}) \subset X$ is open and the restriction map $p|_{\tilde{W}} : \tilde{W} \longrightarrow p(\tilde{W})$ is a homeomorphism.

In this talk, a local homeomorphism $p : \tilde{X} \longrightarrow X$ is denoted by $(\tilde{X}, p)$, and we always consider $\tilde{X}$ a **path connected space** and p a **surjective map.**

Definition 2.2. ([1]) Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map with $f(y_0) = x_0$. Given $\tilde{x}_0$ in the fiber over x_0. If there exists a continuous map $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a **lifting** of f.

Definition 2.3. ([7]) Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Given $\tilde{x}_0$ in the fiber over x_0. The map p has **" path lifting property"** if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x_0})$ of f.

For abbrevition we write PLP instead of Path Lifting Property.

Definition 2.4. ([7]) Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Given $\tilde{x}_0$ in the fiber over x_0. The map p has **"unique path lifting property"** if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x_0})$ of f.

For abbrevition we write UPLP instead of Unique Path Lifting Property.

Let X be a fixed topological space. The set of all local homeomorphisms onto X with unique path lifting property forms a category. In this category a morphism from $p : \tilde{X} \to X$ to $q : \tilde{Y} \to X$ is a continuous function $h : \tilde{X} \to \tilde{Y}$ such that $p = q \circ h$.

Theorem 2.5. *Let $(\tilde{X}, p)$ be a local homeomorphism of X, Y be a connected space, $\tilde{X}$ be Hausdorff and $f : (Y, y_0) \to (X, x_0)$ be continuous. Given $\tilde{x}$ in the fiber over x_0 there is at most one lifting $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ of f.*

Proof. Let $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ and $f' : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ be two continuous map with $p \circ \tilde{f} = f$ and $p \circ f' = f$. Put $A = \{y \in Y | \tilde{f}(y) = f'(y)\}$, $B = \{y \in Y | \tilde{f}(y) \neq f'(y)\}$. Clearly $A \cup B = Y, A \cap B = \phi$ if $B = \phi$, then $\tilde{f} = f'$, hence we can assume that $B \neq \phi$. We show that A is an open subset of Y. Let $a \in A$, then $\tilde{f}(a) = f'(a) = b$. Since $(\tilde{X}, p)$ is a local homeomorphism, there exists $v \subseteq \tilde{X}$ such that $b \in v$, $p : |_v : v \to p(v)$ is homeomorphism. Put

$W = \tilde{f}^{-1}(v) \cap f'^{-1}(v)$, then $w \in W$, and W is an open subset of Y. So $p \circ \tilde{f}(w) = p \circ f'(w)$ and $\tilde{f}(w), f'(w) \in v$ and p is monomorphism on v thus $\tilde{f}(w) = f'(w)$ so $W \subseteq A$.
Now we show that A is closed. consider an arbitrary sequence $y_n \to y$ in A .So $\tilde{f}(y_n) \to \tilde{y}, f'(y_n) \to f'(y)$ since $y_n \in A$, $\tilde{f}(y_n) = f'(y_n)$ and since $\tilde{X}$ is hausdorff, $\tilde{f}(y) = f'(y)$ and $A = Y$. □

Theorem 2.6. *[5, Theorem 3.1] (local homeomorphism Homotopy theorem for paths)Let $(\tilde{X}, p)$ be a local homeomorphism of X with PLP and $\tilde{X}$ be T_2. Consider the diagram of continuous maps*

$$\begin{array}{ccc} I & \xrightarrow{\tilde{f}} & (\tilde{X}, \tilde{x}_0) \\ {\scriptstyle j}\downarrow & \overset{\tilde{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ I \times I & \xrightarrow{F} & (X, x_0), \end{array}$$

where $j(t) = (t, 0)$ for all $t \in I$. Then there exists a unique continuous map $\tilde{F} : I \times I \to \tilde{X}$ making the diagram commute.

Theorem 2.7. *[5, Theorem 3.5] Let $p : \tilde{X} \to X$ be a local homeomorphism with PLP and $\tilde{X}$ be T_2. Let $x_0, x_1 \in X$ and $f, g : I \to X$ be paths such that $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$ and $\tilde{x_0} \in p^{-1}(x_0)$. If $F : f \simeq g$ rel $\dot{I}$ and $\tilde{f}, \tilde{g}$ are the lifting of f and g respectively with $\tilde{f}(0) = \tilde{x_0} = \tilde{g}(0)$, then $\tilde{F} : \tilde{f} \simeq \tilde{g}$ rel $\dot{I}$.*

Theorem 2.8. *[5, Theorem 3.2] (Lifting criterion) If Y is connected and locally path connected, $f : (Y, y_0) \to (X, x_0)$ is continuous and $p : \tilde{X} \to X$ is a local homeomorphism with UPLP and PLP where $\tilde{X}$ is path connected, then there exists a unique $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ such that $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. Moreover, if f is a local homeomorphism, then $\tilde{f}$ is a local homeomorphism.*

Corollary 2.9. *[5, Corollary 3.3] If Y is simply connected and locally path connected and $p : \tilde{X} \to X$ is a local homeomorphism with UPLP and PLP and $\tilde{X}$ is path connected, then every map $f : (Y, y_0) \to (X, x_0)$ has a lifting.*

Corollary 2.10. *[5, Corollary 3.4] If X is connected and locally path connected, $(\tilde{X}, p)$, $(\tilde{Y}, q)$ are local homeomorphisms with UPLP and PLP, $\tilde{Y}$ is T_2 and path connected, $\tilde{X}$ is path connected, and $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = q_*(\pi_1(\tilde{Y}, \tilde{y_0}))$, then there exists a homeomorphism $h : (\tilde{Y}, \tilde{y_0}) \to (\tilde{X}, \tilde{x}_0)$ such that $p \circ h = q$.*

Definition 2.11. [5, Definition 3.8] $p : \tilde{X} \to X$ is called a **regular local homeomorphism** if $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is a normal subgroup of $\pi_1(X, x_0)$.

3 Main Results

By Theorems 2.6 and 2.8 we can prove that the following results:

Theorem 3.1. *If $p : \tilde{X} \longrightarrow X$ is a local homeomorphism with UPLP and PLP, then $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is an open subgroup of $\pi_1^{qtop}(X, x)$.*

Now, the following theorem is one of the main results of this talk which modifies the definition of semicovering maps.

Theorem 3.2. *A map $p : \tilde{X} \longrightarrow X$ is a semicovering map if and only if it is a local homeomorphism with UPLP and PLP.*

Note that there exists a local homeomorphism without UPLP and PLP.

Example 3.3. Let $\tilde{X} = ([0,1] \times \{0\}) \bigcup (\{1/2\} \times [0, 1/2)$ with coherent topology with respect to $\{[0, 1/2] \times \{0\},\ (1/2, 1] \times \{0\},\ \{1/2\} \times (0, 1/2)\}$ and let $X = [0, 1]$. Define $p : \tilde{X} \to X$ by $p(s,t) = \begin{cases} s & t = 0 \\ s + 1/2 & s = 1/2 \end{cases}$. It is routine to check that p is an onto local homeomorphism which dose not have UPLP and PLP.

Theorem 3.4. *If $\tilde{X}$ is Hausdorff and sequential compact and $p : \tilde{X} \longrightarrow X$ is a local homeomorphism, then p has Path Lifting Property.*

Corollary 3.5. *If $\tilde{X}$ is Hausdorff and sequential compact and $p : \tilde{X} \longrightarrow X$ is a local homeomorphism, then p is a semicovering map.*

Proof. $\tilde{X}$ is Hausdorff and sequential compact so by Theorem 3.4 p has PLP and by Theorem 2.5 p has UPLP. Hence Theorem 3.2 implies that p is a semicovering map.

□

Definition 3.6. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a local homeomorphism with PLP and $\tilde{X}$ be T_2 and path connected. A **local homeomorphism transformation** is a homeomorphism $h : \tilde{X} \to \tilde{X}$ such that $p \circ h = p$. We define $LH(\tilde{X}/X) = \{h : \tilde{X} \to \tilde{X} | p \circ h = p, h\ is\ homeomorohism\}$. Clearly $LH(\tilde{X}/X)$ forms a group with ordinary composition. Also $LH(\tilde{X}/X)$ acts on fiber $p^{-1}(x_0)$ by $h\tilde{x_0} = h(\tilde{x_0})$ for $h \in LH(\tilde{X}/X)$ and $\tilde{x_0} \in p^{-1}(x_0)$.

Theorem 3.7. *Let X be connected, locally path connected, and let $x_0 \in X$ and $(\tilde{X}, p)$ be a local homeomorphism of X with PLP and $\tilde{X}$ be T_2 and path connected. Then $LH(\tilde{X}/X)$ acts on $p^{-1}(x_0)$ transitively if and only if $(\tilde{X}, p)$ is a regular local homeomorphism of X.*

Theorem 3.8. *Let $(\tilde{X}, p)$ be a local homeomorphism of X with PLP and $\tilde{X}$ be T_2 and path connected.*

1. *If $h \in LH(\tilde{X}/X)$, and $h \neq 1_{\tilde{X}}$, then h has no fixed point.*

2. *If $h_1, h_2 \in LH(\tilde{X}/X)$ and there exists $\tilde{x} \in \tilde{X}$ with $h_1(\tilde{x}) = h_2(\tilde{x})$, then $h_1 = h_2$.*

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