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ON THE CATEGORY OF LOCAL HOMEOMORPHISMS WITH UNIQUE PATH LIFTING PROPERTY

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ABSTRACT. In this talk, we discuss on the category of local homeomorphisms of topological spaces with unique path lifting property. We intend to find a classification of these local homeomorphisms similar to that of covering maps.

This is a joint work with Hamid Torabi‡ and Behrooz Mashayekhy††.

1. Introduction

Biss [2, Theorem 5.5] showed that for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of open subgroups of its fundamental group $\pi_1(X, x)$. There is a misstep in the proof of the above theorem. In fact, Biss assumed that every fibration with discrete fiber is a covering map which is not true in general.

Torabi et al.[7] pointed out the above misstep and gave the true classification of connected covering spaces of X according to open subgroups of the fundamental group $\pi_1(X, x)$. In fact, for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of subgroups of its fundamental group $\pi_1(X, x)$, with an open normal subgroup in $\pi_1^{qtop}(X, x)$. We know every covering map is a local homeomorphism. In this talk, we intend to study the category of local homeomorphisms $p : \tilde{X} \longrightarrow X$ for a fixed topological space $X$with unique path lifting property. Our main contribution is to find a classification for these local homeomorphisms.

2. Notations and preliminaries

For a topological space X, by a path in X we mean a continuous map $\alpha : [0, 1] \to X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α, respectively. A loop α is a path with $\alpha(0) = \alpha(1)$. For a path $\alpha : [0, 1] \to X$, α^{-1} denotes a path such that $\alpha^{-1}(t) = \alpha(1 - t)$, for all $t \in [0, 1]$. Denote $[0, 1]$ by I, two paths $\alpha, \beta : I \to X$ with the same initial and terminal points are called

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homotopic relative to end points if there exists a continuous map $F : I \times I \to X$ such that

$$F(t,s) = \begin{cases} \alpha(t) & s = 0 \\ \beta(t) & s = 1 \\ \alpha(0) = \beta(0) & t = 0 \\ \alpha(1) = \beta(1) & t = 1. \end{cases}$$

Homotopy relative to end points is an equivalent relation and the homotopy class containing a path α is denoted by $[\alpha]$. For paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$, $\alpha * \beta$ denotes the concatenation of α and β which is a path from I to X such that

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The set of all homotopy classes of loops relative to the end point x in X under the binary operation $[\alpha][\beta] = [\alpha * \beta]$ forms a group which is called the fundamental group of X and is denoted by $\pi_1(X, x)$ (see[6]). The set of all loops with initial point x in X is called the loop space of X denoted by $\Omega(X, x)$ (see [5]).

The quasitopological fundamental group $\pi_1^{qtop}(X, x)$ is the quotient space of the loop space $\Omega(X, x)$ equipped with the compact-open topology with respect to the function $\Omega(X, x) \to \pi_1(X, x)$ identifying path components (see [2]). It should be mentioned that$\pi_1^{qtop}(X, x)$ is a quasitopological group in the sense of [1] and it is not always a topological group (see [3],[4]).

Definition 2.1. [5] *Assume that X and $\tilde{X}$ are topological spaces. The continuous map $p : \tilde{X} \longrightarrow X$ is called a* **local homeomorphism** *if for every point $\tilde{x} \in \tilde{X}$ there exists an open set $\tilde{W}$ such that $\tilde{x} \in \tilde{W}$ and $p(\tilde{W}) \subset X$ is open and the restriction map $p|_{\tilde{W}} : \tilde{W} \longrightarrow p(\tilde{W})$ is a homeomorphism.*

Definition 2.2. *Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map with $f(y_0) = x_0$. Let $\tilde{x_0}$ be in the fiber over x_0. If there exist a continuous function $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a lifting for f.*

Definition 2.3. *Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $\tilde{x}_0$ be in the fiber over x_0. The map p has "unique path lifting property" if for every path f in X, there exists a unique continuous function $\tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x_0})$ with $p \circ \tilde{f} = f$.*

Let X be a fixed topological space. The set of all local homeomorphisms of X with unique path lifting property forms a category. In this category a morphism from $p : \tilde{X} \to X$ to $q : \tilde{Y} \to X$ is a continuous function $h : \tilde{X} \to \tilde{Y}$ such that $p = q \circ h$.

Definition 2.4. [6] *Let $\tilde{X}$ and X be topological spaces and let $p : \tilde{X} \longrightarrow X$ be continuous. An open set U in X is* **evenly covered** *by p if $p^{-1}(U)$ is a disjoint union of open sets S_i in $\tilde{X}$, called* **sheets**, *such that $p|_{S_i} : S_i \to U$ is a homeomorphism for every i.*

Definition 2.5. [6] *If X is a topological space, then an ordered pair $(\tilde{X}, p)$ is a* **covering space** *of X if:*

(1) $\tilde{X}$ *is a path connected topological space;*
(2) $p : \tilde{X} \to X$ *is continuous;*
(3) *each* $x \in X$ *has an open neighborhood* $U = U_x$ *that is evenly covered by* p.

3. Main Results

Theorem 3.1. (Local Homeomorphism Homotopy Theorem for Paths) *Let* $(\tilde{X}, p)$ *be a local homeomorphism of* X *with unique path lifting property. Consider the following diagram of continuous maps*

$$\begin{tikzcd} I \arrow[r, "\tilde{f}"] \arrow[d, "j"'] & (\tilde{X}, \tilde{x_0}) \arrow[d, "p"] \\ I \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & (X, x_0) \end{tikzcd}$$

where $j(t) = (t, 0)$ *for all* $t \in I$. *Then there exists a unique continuous map* $\tilde{F} : I \times I \to \tilde{X}$ *which makes the diagram commutative.*

Theorem 3.2. (Lifting Criterion) *If* Y *is connected and locally path connected,* $f : (Y, y_0) \to (X, x_0)$ *is continuous and* $p : \tilde{X} \to X$ *is a local homeomorphism with unique path lifting property, where* $\tilde{X}$ *is path connected, then there exists a unique* $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ *such that* $p \circ \tilde{f} = f$ *if and only if* $f_*(\pi_1(Y, y_0) \subset p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

Corollary 3.3. *If* Y *is simply connected, locally path connected and* $p : \tilde{X} \to X$ *is a local homeomorphism with unique path lifting property, where* $\tilde{X}$ *is path connected, then any continuous map* $f : (Y, y_0) \to (X, x_0)$ *has a lifting to* $\tilde{X}$.

Corollary 3.4. *Suppose* X *is connected, locally path connected and* $p : \tilde{X} \to X$, $q : \tilde{Y} \to X$ *are local homeomorphisms with unique path lifting property where* $\tilde{X}$, $\tilde{Y}$ *are path connected. If* $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = q_*(\pi_1(\tilde{Y}, \tilde{y_0}))$, *then there exists a homeomorphism* $h : (\tilde{Y}, \tilde{y_0}) \to (\tilde{X}, \tilde{x_0})$ *such that* $p \circ h = q$.

Theorem 3.5. *Let* $p : \tilde{X} \to X$ *be a local homeomorphism with unique path lifting property and let* $x_0, x_1 \in X$ *and* $f, g : I \to X$ *be paths with* $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$ *and* $\tilde{x_0} \in p^{-1}(x_0)$. *If* $F : f \simeq g$ *rel* $\dot{I}$ *and* $\tilde{f}, \tilde{g}$ *are the lifting of* f *and* g *respectively with* $\tilde{f}(0) = \tilde{x_0} = \tilde{g}(0)$, *then* $\tilde{F} : \tilde{f} \simeq \tilde{g}$ *rel* $\dot{I}$.

Theorem 3.6. *Let* $p : \tilde{X} \to X$ *be a local homeomorphism with unique path lifting property where* $\tilde{X}$ *is path connected. If* $x_0, x_1 \in X$, $Y_0 = p^{-1}(x_0)$ *and* $Y_1 = p^{-1}(x_1)$, *then* $|Y_0| = |Y_1|$.

Theorem 3.7. *If* X *is connected, locally path connected and* H *is a subgroup of* $\pi_1(X, x)$, *then there exists a local homeomorphism* $p : \tilde{X} \to X$ *with unique path lifting property such that* $p_*(\pi_1(\tilde{X}, \tilde{x})) = H$ *if and only if* H *is an open subgroup of* $\pi_1^{qtop}(X, x)$. *Moreover there is a 1-1 correspondence between equivalent classes of local homeomorphisms of* X *(in category of local homeomorphism with unique path lifting property) and the conjugacy classes of open subgroups of the quasitopological fundamental group* $\pi_1^{qtop}(X, x)$.

Definition 3.8. *$p : \tilde{X} \to X$ is called a* **regular local homeomorphism** *if $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is a normal subgroup of $\pi_1(X, x_0)$.*

Theorem 3.9. *Every regular local homeomorphism is a cover map.*

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