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On Open Subgroups of Topologized Fundamental Group

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Abstract

In this talk we are interested to focus on open subgroups of topologized fundamental groups and give some equivalent conditions on a topological space to make sure a subgroup of its topologized fundamental group is open, when the fundamental group equipped with the compact-open topology or the Whisker topology. Moreover, we present some conditions under which generalized coverings, semicoverings and coverings are equal.

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1 Introduction and Motivation

We recall that a continuous map $p : \tilde{X} \to X$ is a covering map if every point of X has an open neighborhood which is evenly covered by p. It is well-known that the induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is a monomorphism and so $\pi_1(\tilde{X}, \tilde{x}) \cong p_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$.

Some people extended the notion of covering maps and introduced semicoverings [3] and generalized coverings [1,2,5]. These generalizations focus on keeping some properties of covering maps and eliminating the evenly covered property. Brazas [3] introduced semicoverings by removing evenly covered property and keeping local homeomorphism and unique path lifting properties. For generalized coverings, the local homeomorphism is replaced with unique lifting property [2,5]. A subgroup H of the fundamental group $\pi_1(X, x)$ is called covering, semicovering, generalized covering subgroup if there is a covering, semicovering, generalized covering map $p : \left(\tilde{X}, \tilde{x}\right) \to (X, x)$ such that $H = p_*\pi_1(\tilde{X}, \tilde{x})$, respectively.

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It is easy to show that the class of all coverings, semicoverings and generalized coverings on X forms a category denoted by $COV(X)$, $SCOV(X)$, $GCOV(X)$, respectively. By definition $COV(X)$ is a subcategory of $SCOV(X)$. Brazas [3] showed that $COV\,(X) = SCOV\,(X) = GCOV(X)$ for a locally path connected, semilocally simply connected space X. Torabi et al.[9] showed that the first equality extended for locally path connected, semi locally small generated spaces, i.e. $COV\,(X) = SCOV\,(X)$. The authors [1, Corollary 2.13] showed that $GCOV\,(X) = SCOV\,(X)$ for locally path connected, semi locally small generated spaces.

In order to extend equal the above results, we introduce some topological properties under which the above categorical equalities hold. We show that $GCOV\,(X) = SCOV\,(X)$ if and only if X is semilocally path H-connected, and $GCOV\,(X) = COV\,(X)$ if and only if X is semilocally H-connected, when $H = \pi_1^{gc}\,(X, x_0)$.

2 Notations and Preliminaries

The definition of generalized covering maps based on unique lifting property, is as follows.

Definition 2.1. A pointed continuous map $p : \tilde{X} \to X$ has **UL (unique lifting)** property if for every connected, locally path connected space (Y, y_0) and every continuous map $f : (Y, y_0) \longrightarrow (X, x_0)$ with $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ for chosen $\tilde{x}_0 \in p^{-1}\,(x_0)$, there exists a unique continuous lifting $\tilde{f}$ with $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$. If $\tilde{X}$ be a connected, locally path connected space and $p : \tilde{X} \longrightarrow X$ is a surjection with UL property, then $\tilde{X}$ is called a **generalized covering space** for X. A subgroup $H \leq \pi_1(X, x_0)$ is called generalized covering subgroup of $\pi_1(X, x_0)$ if there is a generalized covering map $p : \left(\tilde{X}, \tilde{x}_0\right) \longrightarrow (X, x_0)$ such that $H = p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

Definition 2.2. For a pointed space (X, x_0), we define $\pi_{\mathbf{1}}^{\mathbf{gc}}\,(\mathbf{X}, \mathbf{x_0}) = \cap\,\{H \leq \pi_1\,(X, x_0) \mid H$ *is a generalized covering subgroup*$\}$.

Definition 2.3. Let $\Omega(X, x_0)$ be the space of all closed path $\alpha : \left(I, \dot{I}\right) \to (X, x_0)$ at $x_o \in X$ equipped with the compact-open topology. The fundamental group $\pi_1\,(X, x_0)$ inherits this topology by quotient map $q : \Omega(X, x_0) \to \pi_1\,(X, x_0)$ with $q : \alpha \mapsto [\alpha]$. The fundamental group with this topology is a quasitopological group and denoted by $\pi_{\mathbf{1}}^{\mathbf{q}top}\,(\mathbf{X}, \mathbf{x_0})$ [4]. On the other hand, the collection $\{[\alpha]\, i_*\pi_1\,(U, x_0) \mid [\alpha] \in \pi_1(X, x_0)$ *and* U *open subset of* $x_0\}$ form a basis for a different topology on the fundamental group which is called the Whisker topology and denoted by $\pi_{\mathbf{1}}^{\mathbf{w}h}\,(\mathbf{X}, \mathbf{x_0})$. The fundamental group with this topology is a left topological group [1, Proposition 3.2].

Fischer and Zastrow [5, Lemma 2.1] showed that the Whisker topology is, in general, finer than the quotient topology inherited from the compact-open topology.

3 Main Results

Recall that a space X is called semilocally simply connected at x $\in$ X if there exists an open neighborhood U *of* x such that all loops in U at x be nullhomotopic in X or equivalently the

induced homomorphism $i_* : \pi_1(U,x) \to \pi_1(X,x)$ from the inclusion map $i : U \to X$ be trivial. Furthermore, a space is called semilocally simply connected if it is semilocally simply connected at each point. In continue, we extend these concepts to any subgroups of the fundamental group and present their topological equivalents.

Definition 3.1. Let H be an arbitrary subgroup of $\pi_1(X,x)$. Then we define the following concepts.

a) The space X is called ***semi locally* H*-connected at* x ∈ X** if there exists an open neighborhood U *of* x such that $i_*\pi_1(U,x) \leq H$. Generalization of this definition to the whole of space needs to determine the equivalence subgroups in fundamental groups when the base point changes.

b) A topological space X is called ***semi locally path* H*-connected*** if for every path α beginning at x_0 there exists an open neighborhood U_α of $\alpha(1)$ such that $i_*\pi_1(U_\alpha, \alpha(1)) \leq [\alpha^{-1}H\alpha]$. This definition also is a generalization of semilocally simply connected space where H is the trivial subgroup.

c) A topological space X is called ***semi locally* H*-connected*** if for every $x \in X$ and for every path α from x_0 *to* x, the space X is semi locally $[\alpha^{-1}H\alpha]$-connected at $x \in X$.

Theorem 3.2. *For $H \leq \pi_1(X,x_0)$ the space X is semi locally H-connected at* $\mathrm{x}_0 \in \mathrm{X}$ *if and only if H is an open subgroup of ${\pi_1}^{wh}(X,x_0)$.*

Trivially, semi locally H-connectedness at a point is strongly dependent on choice of the point. For instance, the Hawaiian Earring (HE, a) is semi locally 1-connected at $x \in HE$, where x is any non-base point $(x \neq a)$ and clearly is not semi locally 1-connected at $\mathrm{a} \in HE$.

Proposition 3.3. *A connected, locally path connected space X is semi locally path H-connected for $H \leq \pi_1(X,x_0)$ if and only if H is an open subgroup of ${\pi_1}^{qtop}(X,x_0)$.*

Corollary 3.4. *For a connected, locally path connected space X, the categorical equivalence $SCOV(X) = GCOV(X)$ holds if and only if X is semi locally path ${\pi_1}^{gc}(X,x_0)$-connected.*

Corollary 3.5. *If X is homotopicaly Hausdorff relative to H and the index of H in $\pi_1(X,x_0)$ is finite, then X is semi locally path H-connected.*

Proposition 3.6. *A connected, locally path connected space X is semi locally* H*-connected for $H \leq \pi_1(X,x_0)$ if and only if H is a covering subgroup of $\pi_1(X,x_0)$.*

Corollary 3.7. *For a connected, locally path connected space X, the categorical equivalence $GCOV(X) = COV(X)$ holds if and only if X is semi locally ${\pi_1}^{gc}(X,x_0)$-connected.*

Corollary 3.8. *If X is a connected, locally path connected and semi locally ${\pi_1}^{gc}(X,x_0)$-connected space, then ${\pi_1}^{gc}(X,x_0) = {\pi_1}^{sp}(X,x_0)$ and X is coverable in the sense of [6, Definition 2.4.], where ${\pi_1}^{sp}(X,x_0)$ is the Spanier subgroup of $\pi_1(X,x_0)$.[7,8].*

Corollary 3.9. *Every semi locally* H*-connected space is semi locally path H-connected.*

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