On the Whisker Topology on Fundamental Group

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Abstract

In this talk, after reviewing concepts of compact-open topology, Whisker topology and Lasso toplogy on fundamental groups, we present some topological properties for the Whisker topology on a fundamental group.

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1 Introduction

The concept of a natural topology on the fundamental group appears to have originated with Hurewicz [8] in 1935. The topology inherited from the loop space by quotient map, where equipped with compact-open topology, on fundamental group is denoted by $\pi_1^{qtop}(X, x_0)$. Spanier [10, Theorem 13 on page 82] introduced a different topology that Dydak et al. [4] called it the Whisker topology and denoted by $\pi_1^{wh}(X, x_0)$. They also introduced a new topology on $\pi_1(X, x_0)$ and called it the Lasso topology to characterize the unique path lifting property which is denoted by $\pi_1^{l}(X, x_0)$ and showed that this topology makes the fundamental group a topological group [3]. However Biss [2] claimed that $\pi_1^{qtop}(X, x_0)$ is a topological group, but it is shown that the multiplication map is not continuous, in general, hence $\pi_1^{qtop}(X, x_0)$ is a quasitopological group (see [6]). In this talk, we show that $\pi_1^{wh}(X, x_0)$ is not a topological group, in general. In addition, it is not even a semitopological group, but it has some properties similar to toplogical groups. For instance, every open subgroup of $\pi_1^{wh}(X, x_0)$ is also a closed subgroup of $\pi_1^{wh}(X, x_0)$ and $\pi_1^{wh}(X, x_0)$ is T_0 if and only if it is T_2. Moreover, $\pi_1^{wh}(X, x_0)$ is a homogeous and regular space, and it is totally seperated if and only if is T_0.

2 Notation and Preliminaries

Definition 2.1. Let H be a subgroup of $\pi_1(X, x_0)$ and $P(X, x_0) = \{\alpha : (I, 0) \to (X, x_0) | \, \alpha$ *is a path*$\}$ be a path space. Then $\alpha_1 \sim \alpha_2 \; mod \; H$ if $\alpha_1(1) = \alpha_2(1)$ and $\left[\alpha_1 * \alpha_2{}^{-1}\right] \in H$. It is easy to check that this is an equivalence relation on $P(X, x_0)$. The equivalence class of α is denoted by $\langle\alpha\rangle_H$. Now one can define the quotient space $\tilde{X}_H = \frac{P(X,x_0)}{\sim}$ and the

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map $p_H : (\tilde{X}_H, e_H) \to (X, x_0)$ by $p_H(\langle\alpha\rangle_H) = \alpha(1)$ where e_H is the class of constant path at x_0.

For $\alpha \in P(X, x_0)$ and an open neighborhood U of $\alpha(1)$, a continuation of α in U is a path $\beta \in P(X, x_0)$ of the form $\beta = \alpha * \gamma$, where $\gamma(0) = \alpha(1)$ and $\gamma(I) \subseteq U$. Thus we can define a set $\langle U, \langle\alpha\rangle_H\rangle = \{\langle\beta\rangle_H \in X_H | \ \beta \ is \ a \ continuation \ of \ \alpha \ in \ U\}$ where U is an open neighborhood of $\alpha(1)$ in X. It is shown that the subsets $\langle U, \langle\alpha\rangle_H\rangle$ as defined above form a basis for a topology on $\tilde{X}_H$ for which the function $p_H : \tilde{X}_H \to X$ is continuous [9, Theorem 10.31]. Moreover, if X is path connected, then p_H is surjective. This topology on $\tilde{X}_H$ is called the Whisker topology [4].

Definition 2.2. Let $p_e : \tilde{X}_e \to X$ be the defined end point projection map for $\{e\} \leq \pi_1(X, x_0)$ and put $p_e^{-1}(x_0)$ as a subspace of $(\tilde{X}_e, \tilde{x}_0)$ with its default Whisker topology. One can transfer this topology by the bijection $f : \pi_1(X, x_0) \to p_e^{-1}(x_0)$ into $\pi_1(X, x_0)$ with $[\alpha] \mapsto \langle\alpha\rangle_H$. The fundamental group with Whisker topology is denoted by $\pi_1^{wh}(X, x_0)$. Fishcer and Zastrow [7, Lemma 2.1.] have shown that the Whisker topology is finer than the inherited topology from loop space with compact-open topology on $\pi_1(X, x_0)$ which is denoted by $\pi_1^{qtop}(X, x_0)$.

3 Main results

In this section we are going to present some interesting properties of $\pi_1^{wh}(X, x_0)$. At first, it seems necessary to characterize the open subsets and subgroups of $\pi_1^{wh}(X, x_0)$. Let $[\alpha] \in \pi_1(X, x_0)$. for every open subset U of x_0 there is a bijection $\varphi_\alpha : i_*\pi_1(U, x_0) \to (U, [\alpha]) \bigcap p_e^{-1}(x_0)$ defined by $\varphi_\alpha([\gamma]) = [\alpha * \gamma]$. It is easy to check that φ_α is a well defined bijection.

The collection $\{[\alpha]i_*\pi_1(U, x_0) \mid [\alpha] \in \pi_1(X, x_0) \ and \ U \ open \ subset \ of \ x_0\}$ form a basis for the Whisker topology on $\pi_1(X, x_0)$. Moreover, these basis elements are closed and hence they are clopen subsets.

The left (right) topological group is a group equipped with a topology that makes all of the left (right) translations continuous. A semitopological group is a left topological group which is also a right topological group [1, Section 1.2.]. $\pi_1^{wh}(X, x_0)$ is not a right topological group in general, hence it is not a semitopological group. For example see the Hawaiian earring is not a topological group since the inverse map in $\pi_1^{wh}(HE, *)$ is not continous [4]. Recall that a non-empty topological space X is called a $G-$ space, for a group G, if it is equipped with an action of G on X . A homogeneous space is a $G-$ space on X which G acts transitively.

Proposition 3.1. *$\pi_1^{wh}(X, x_0)$ is a homogenous space.*

Proof. Clearly $\pi_1^{wh}(X, x_0)$ acts on itself. To show that this action is transitive, it is enough to prove that left translation map in $\pi_1^{wh}(X, x_0)$ is homeomorphism. It is known that every left topological group is a homogenous space. Hence $\pi_1^{wh}(X, x_0)$ is a homogenous space.

Corollary 3.2. *Every open subgroup of $\pi_1^{wh}(X, x_0)$ is a closed subgroup.*

Recall that a topological space is called totally separated if for every pair of disjoint points there exists a clopen subset which contains one of points and does not contain another. The following proposition state some separation axioms for $\pi_1^{wh}(X, x_0)$.

Proposition 3.3. *For a connected and locally path connected space X, the following statement are equivalent:*

1. $\pi_1^{wh}(X, x_0)$ *is* T_0.
2. $\pi_1^{wh}(X, x_0)$ *is* T_1.
3. $\pi_1^{wh}(X, x_0)$ *is* T_2.
4. $\pi_1^{wh}(X, x_0)$ *is* T_3 $(T_3 = regular + T_1)$.
5. $\pi_1^{s}(X, x_0) = 1$, *where* $\pi_1^{s}(X, x_0)$ *is the collection of small loops at* x_0.
6. $\pi_1^{wh}(X, x_0)$ *is totally separated.*
   Moreover, $\pi_1^{wh}(X, x_0)$ *is regular.*

Corollary 3.4. *If the right translation in* $\pi_1^{wh}(X, x_0)$ *are continuous, then* $\pi_1^{wh}(X, x_0)$ *is a topological group.*

It seems interesting to know that when $\pi_1^{wh}(X, x_0)$ *has the countable axiom properties.*

Proposition 3.5. *If* X *is a first countable space, then* $\pi_1^{wh}(X, x_0)$ *is also first countable.*

Proof. Let β_{x_0} be a countable neighborhood basis at x_0 and let $[f] \in \pi_1^{wh}(X, x_0)$. Then the colection $\beta_f = \{[f]i_*\pi_1(V, x_0) \mid V \in \beta_{x_0}\}$ form a countable neighborhood basis at $[f]$.

Proposition 3.6. *The closure of trivial element in* $\pi_1^{wh}(X, x_0)$ *is equals to* $\pi_1^{s}(X, x_0)$.

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