On Generalized Covering Subgroups of a Fundamental Group

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Abstract

In this talk, after reviewing concepts of covering, semicovering and generalized covering subgroups introduced by J. Brazas, we give a new criterion for a subgroup $H \leq \pi_1(X, x_0)$ to be a generalized covering subgroup.

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1 Introduction

Recently, the notion of covering space has been extended using eliminating some of its properties and keeping some others [1,2,3,5]. For instance, semicoverings are introduced by eliminating the evenly covered property and keeping local homeomorphismness and unique path lifting property [2]. In the case of generalized coverings, local homeomorphismness has been replaced with unique lifting property [1,3,5]. It is well-khown that for connected and locally path connected spaces every covering is a semicovering and every semicovering is a generalized covering. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a map and $H = p_*\pi_1(\tilde{X}, \tilde{x}_0) \leq \pi_1(X, x_0)$. Then H is called a covering, a semicovering or a generalized covering subgroup if p is covering, semicovering or generalized covering map, respectively. It is shown that H is a covering subgroup if and only if it contains an open normal subgroup of $\pi_1^{qtop}(X, x_0)$ [2,6]. Brazas showed that H is a semicovering subgroup if and only if it is an open subgroup of $\pi_1^{qtop}(X, x_0)$. He also proved that H is a generalized covering subgroup if and only if $p_H : \tilde{X}_H \to X$ has the uniqe path lifting property, where $p_H : \tilde{X}_H \to X$ is the well-known endpoint projection [3]. Now in this talk, we show that for a connected and locally path connected space X, a subgroup H of $\pi_1(X, x_0)$ is a generalized covering subgroup if and only if $(p_H)_*\pi_1\left(\tilde{X}_H, e_H\right) = H$.

2 Notations and Preliminaries

Definition 2.1. A pointed continuous map $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ has **UL (unique lifting)** property if for every connected, locally path connected space (Y, y_0) and every continuous map $f : (Y, y_0) \longrightarrow (X, x_0)$ with $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$, there exists a

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unique continuous lifting $\tilde{f}$ with $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$. If $\tilde{X}$ is a connected, locally path connected space and $p : \tilde{X} \longrightarrow X$ is surjective with UL property, then $\tilde{X}$ is called a **generalized covering space** for X. A subgroup $H \leq \pi_1(X, x_0)$ is called a **generalized covering subgroup** of $\pi_1(X, x_0)$ if there is a generalized covering map $p : \left(\tilde{X}, \tilde{x}_0\right) \longrightarrow (X, x_0)$ such that $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

Definition 2.2. A map $f : Y \longrightarrow X$ has **UPL (unique path lifting)** property if it has UL property for the closed interval $I = [0, 1]$. A map $f : Y \longrightarrow X$ has $\mathbf{U}PL'$ **(only unique path lifting)** property if any two paths $\alpha, \beta : [0, 1] \to Y$ are equal whenever $f \circ \alpha = f \circ \beta$ and $\alpha(0) = \beta(0)$.

Definition 2.3. Let H be a subgroup of $\pi_1(X, x_0)$ and $P(X, x_0) = \{\alpha : (I, 0) \to (X, x_0) | \, \alpha \textit{ is a path}\}$ be a path space. Then $\alpha_1 \sim \alpha_2 \; mod \, H$ if both $\alpha_1(1) = \alpha_2(1)$ and $\left[\alpha_1 * {\alpha_2}^{-1}\right] \in H$. It is easy to check that this is an equivalence relation on $P(X, x_0)$. The equivalence class of α is denoted by $\langle \alpha \rangle_H$. Now one can define the quotient space $\tilde{X}_H = \frac{P(X, x_0)}{\sim}$ and the map $p_H : (\tilde{X}_H, e_H) \to (X, x_0)$ by $p_H(\langle \alpha \rangle_H) = \alpha(1)$, where e_H is the class of constant path at x_0.

For $\alpha \in P(X, x_0)$ and an open neighborhood U of $\alpha(1)$, a continuation of α in U is a path $\beta \in P(X, x_0)$ of the form $\beta = \alpha * \gamma$, where $\gamma(0) = \alpha(1)$ and $\gamma(I) \subseteq U$. Thus we can define a set $\langle U, \langle \alpha \rangle_H \rangle = \{\langle \beta \rangle_H \in X_H | \; \beta \textit{ is a continuation of } \alpha \textit{ in } U\}$. It is shown that the subsets $\langle U, \langle \alpha \rangle_H \rangle$ as defined above form a basis for a topology on $\tilde{X}_H$ for which the function $p_H : (\tilde{X}_H) \to X$ is continuous [7, Theorem 10.31]. Moreover, if X is path connected, then p_H is surjective. This topology on $\tilde{X}_H$ is called the Whisker topology [4].

Some properties of the space $\tilde{X}_H$ and the map p_H are as follows: The map $p_H : \tilde{X}_H \to X$ has the path lifting property. Moreover, every path α in X beginning at x_0 can be lifted to a path $\tilde{\alpha}$ in $\tilde{X}_H$ beginning at e_H and end at $\langle \alpha \rangle_H$ [7, Theorem 10.32]. For every $H \leq \pi_1(X, x_0)$ the space $\tilde{X}_H$ is path connected [7, Corollary 10.33].

Brazas [3, theorem 24] showed that a subgroup $H \leq \pi_1(X, x_0)$ is a generalized covering subgroup of $\pi_1(X, x_0)$ if and only if $p_H : \tilde{X}_H \longrightarrow X$ has $\mathbf{U}PL'$property.

3 Main results

In the trivial case $H = 1$, clearly $H \leq (p_H)_* \pi_1(\tilde{X}_H, e_H)$. Fischer and Zastrow [5] using this fact found an equivalent condition for UPL property in $p_e : \tilde{X}_e \to X$. They also showed that a space X admits a generalized universal covering if and only if $p_e : \tilde{X}_e \to X$ has UPL' property [5, Lemma 2.8]. Then Brazas extended the result for every generalized covering subgroup [3, Lemma 21] and showed that for any subgroup $H \leq \pi_1(X, x_0)$, $H \leq (p_H)_* \pi_1(\tilde{X}_H, e_H)$ [3, corollary 20]. Moreover, he showed that if $p_H : \tilde{X}_H \to X$ has UPL property, then $H = (p_H)_* \pi_1(\tilde{X}_H, e_H)$ [3, Lemma 21]. In the following theorem we investigate the convers of the above result.

Theorem 3.1. *For any $H \leq \pi_1(X, x_0)$, if $(p_H)_* \pi_1(\tilde{X}_H, e_H) \leq H$, then $p_H : \tilde{X}_H \to X$ has UPL property.*

The following corollary is the main result of this talk.

Corollary 3.2. *Let $H \leq \pi_1(X, x_0)$. Then the end point projection $p_H : \tilde{X}_H \to X$ is a generalized covering map if and only if $(p_H)_* \pi_1 \left(\tilde{X}_H, e_H\right) = H$.*

Proof. Brazas showed that $H \leq (p_H)_* \pi_1(\tilde{X}_H, e_H)$ for any subgroup H of $\pi_1(X, x_0)$ [3, Corollary 20]. Combining this fact with Theorem 3.1 implies that if $(p_H)_* \pi_1 \left(\tilde{X}_H, e_H\right) = H$, then $p_H : \tilde{X}_H \to X$ has **UPL (unique path lifting)** property. The convers holds using [3, Lemma 21].

Brazas [3, Theorem 15] showed that for any collection of generalized covering subgroups of $\pi_1(X, x_0)$, the intersection of them is also a generalized covering subgroup. But its proof is too long and need to use pullbacks. We will give a simple proof using Corollary 3.2.

Corollary 3.3. *If $\{H_j \mid j \in J\}$ is any set of generalized covering subgroups of $\pi_1(X, x_0)$, then $H = \cap_{j\in J} H_j$ is a generalized covering subgroup.*

Proof. At first, we show that $(p_H)_* \pi_1 \left(\tilde{X}_H, e_H\right) \leq \cap (p_{H_j})_* \pi_1 \left(\tilde{X}_{H_j}, e_{H_j}\right) = H$ then, use Theorem 3.1 and assume that $[\alpha] = [p_H \circ \widetilde{\alpha}] = (p_H)_* [\widetilde{\alpha}] \in (p_H)_* \pi_1 \left(\tilde{X}_H, e_H\right)$ where $\widetilde{\alpha}: I \to \tilde{X}_H$ is a loop in $\tilde{X}_H$ at e_H with $\widetilde{\alpha}(t) = \langle \beta_t \rangle_H$. We define for every $j \in J$, $\widetilde{\alpha}_j : I \to \tilde{X}_{H_j}$ by $\widetilde{\alpha}_j(t) = \langle \beta_t \rangle_{H_j}$. It is clear that $\widetilde{\alpha}_j$ is a loop at e_{H_j}, $p_H \circ \widetilde{\alpha} = p_{H_j} \circ \widetilde{\alpha}_j$ and so $[p_H o \widetilde{\alpha}] = \left[p_{H_j} o \widetilde{\alpha}_j\right] = [\alpha]$ for every $j \in J$. Therefore, $(p_H)_* \leq H$. Now using Theorem 3.1 the result holds.

For a pointed space (X, x_0) we define: $\pi_1^{gc}(X, x_0) = \bigcap \{H \leq \pi_1(X, x_0) | H \text{ is a generalized covering subgroup}\}$.

Corollary 3.4. *For a pointed space (X, x_0), $\pi_1^{gc}(X, x_0)$ is a generalized covering subgroup.*

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