Unique Path Lifting from Homotopy Point of View

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Abstract

The aim of this paper is to introduce the concepts of path homotopically lifting and its role in the category of fibrations. At first, we have some various notions, closely related to path lifting and unique path lifting; and their relations are supplemented by examples. Then, we study some results in the category of fibration with these notions instead of unique path lifting.

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1 Introduction

A map $p : E \to B$ is called a fibration if it has homotopy lifting property with respect to an arbitrary space X, namely, given maps $\tilde{f} : X \to E$ and $F : X \times I \to B$ such that $F \circ j = p \circ \tilde{f}$ for $j : X \to X \times I$ by $j(x) = (x, 0)$, there is a map $\tilde{F} : X \times I \to E$ such that $\tilde{F} \circ j = \tilde{f}$ and $p \circ \tilde{F} = F$.

$$\begin{tikzcd}
X \arrow[r, "\widetilde{f}"] \arrow[d, "j"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"] & B
\end{tikzcd}$$

Also, a map $p : E \to B$ is said to have unique path lifting property (upl) if, given paths w and w' in E such that $p \circ w = p \circ w'$ and $w(0) = w'(0)$, then $w = w'$. Fibrations with upl, as a generalization of covering spaces are important. It is well known that every fiber

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(inverse image of a singleton) of a fibration with unique path lifting has no nonconstant path [4, Theorem 2.2.5].

In fact, unique path lifting causes a lot of results about a fibration $p : E \to B$, like injectivity of p_*, uniqueness of lifting of a given map and being homeomorphic of any two fibers [4]. Unique path lifting has an important role in the various topological concepts such as covering theory and new generalizations of covering theory, for example [1, 2, 3]. At first, we consider path lifting in the homotopy category and also will discuss about the uniqueness of this type of path lifting and classical path lifting. In fact, their relations will be introduced by some examples. Then, in the last section we would supplement the relations between these new notions in the presence of fibrations. For example, we call a map $p : E \to B$ has weakly unique path homotopically lifting property (wuphl) if, given paths w and w' in E such that $w(0) = w'(0)$, $w(1) = w'(1)$, $p \circ w \simeq p \circ w'$ rel $\{0, 1\}$, we have, $w \simeq w'$ $rel\{0, 1\}$. We will show that every loop in each fiber of a fibration with wuphl is nullhomotopic, which is a homotopy analogue of the same result when we have unique path lifting. Throughout this paper, a *map* $f : X \to Y$ means a continuous function and $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ will denote the homomorphism induced by f.

2 Main results

Path lifting is the lifting of paths in the category Top. We can consider path lifting problem in the htop and get a new feature of lifting problem.

Definition 2.1. Let $p : E \to B$ be a map. A path $\widetilde{\alpha} : I \to E$ is called a homotopically lifting of a path α if $po\widetilde{\alpha} \simeq \alpha$ rel $\{0, 1\}$.

Definition 2.2. Let $p : E \to B$ be a map and $\widetilde{\alpha}$ *and* $\widetilde{\beta}$ be paths in E, then we say that
(i) p has **unique path lifting (upl)** if

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0),\ po\widetilde{\alpha} = po\widetilde{\beta} \Rightarrow \widetilde{\alpha} = \widetilde{\beta}.$$

(ii) p has **homotopically unique path lifting (hupl)** if

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0),\ po\widetilde{\alpha} = po\widetilde{\beta} \Rightarrow \widetilde{\alpha} \simeq \widetilde{\beta}\ rel\ \{0, 1\}.$$

(iii) p has **weekly homotopically unique path lifting (whupl)** if

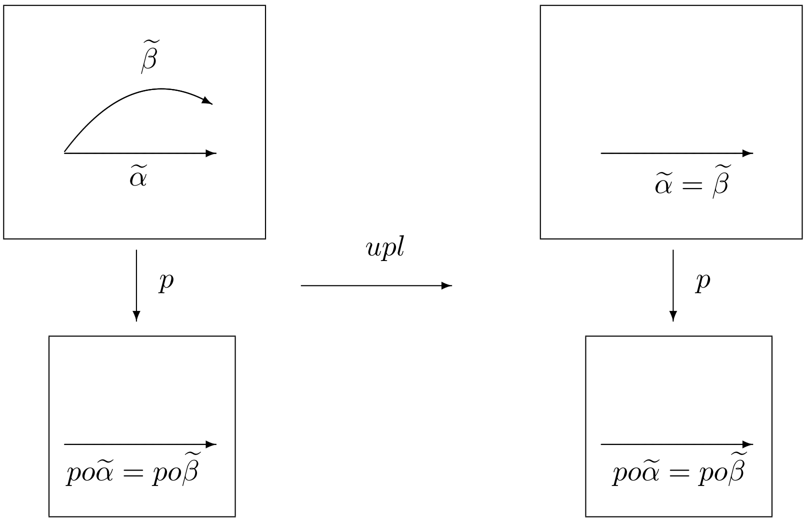
$$\widetilde{\alpha}(0) = \widetilde{\beta}(0),\ \widetilde{\alpha}(1) = \widetilde{\beta}(1),\ po\widetilde{\alpha} = po\widetilde{\beta} \Rightarrow \widetilde{\alpha} \simeq \widetilde{\beta}\ rel\ \{0, 1\}.$$

(iv) p has **unique path homotopically lifting (uphl)** if

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0),\ po\widetilde{\alpha} \simeq po\widetilde{\beta}\ rel\ \{0, 1\} \Rightarrow \widetilde{\alpha} \simeq \widetilde{\beta}\ rel\ \{0, 1\}.$$

(v) p has **weekly unique path homotopically lifting (wuphl)** if

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0),\ \widetilde{\alpha}(1) = \widetilde{\beta}(1),\ po\widetilde{\alpha} \simeq po\widetilde{\beta}\ rel\ \{0, 1\} \Rightarrow \widetilde{\alpha} \simeq \widetilde{\beta}\ rel\ \{0, 1\}.$$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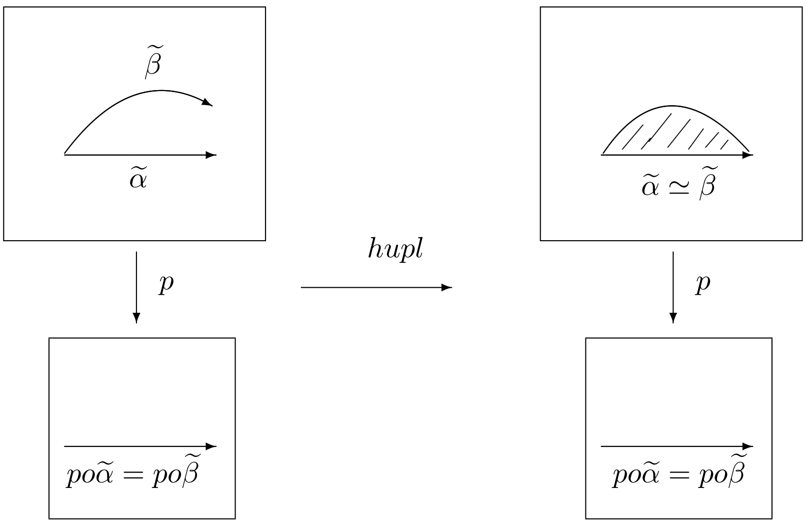
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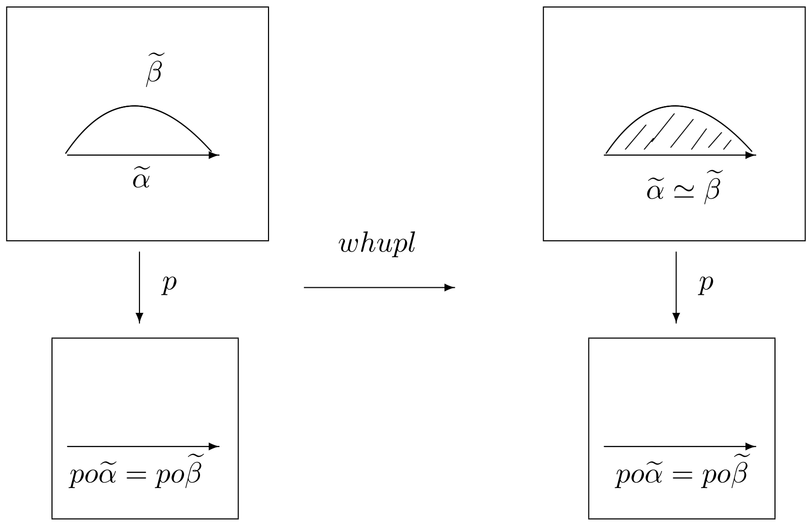
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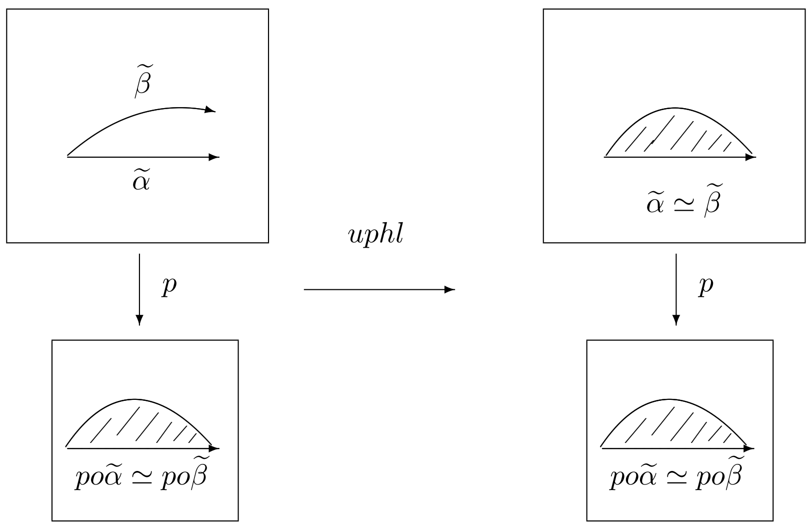
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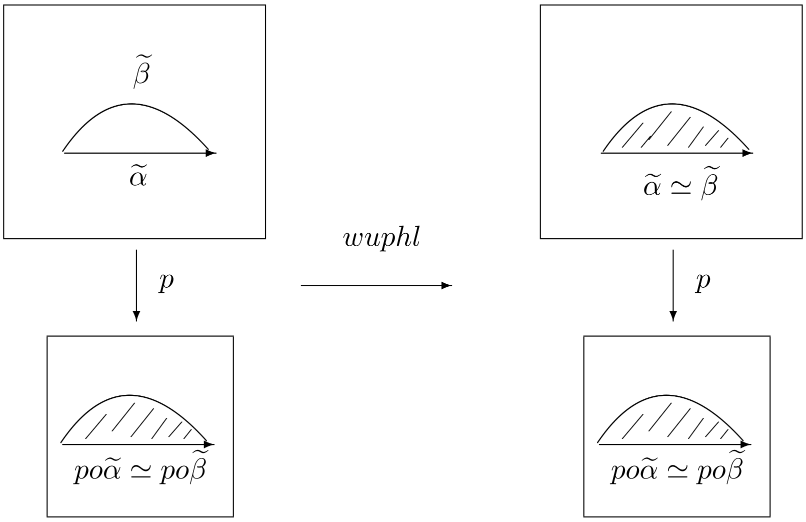


Example 2.3. Every continuous map from a simply connected space to any space has wuphl and whupl. Note that every injective map has upl and also, for injecive map, wuphl and uphl are equivalent.

By a direct verification we have the following proposition

Proposition 2.4. *Let $p : E \to B$ be a map and $e \in p^{-1}(b)$, for $b \in B$. Then*
i) Injectivity of $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is equivalent to wuphl.
ii) Injectivity of $p_* : \pi_1(E, e) \to \pi_1(B, b)$ implies that p has whupl.

It is notable that that converse of (ii) is not necessarily true, for seeing this, refer to Example 2.7.
In the next proposition, we show that in Top, uniqueness and homotopically uniqueness of path lifting are equivalent.

Proposition 2.5. *upl $\Leftrightarrow$ hupl*

Proof. By definitions, $upl \Longrightarrow hupl$. Now let $p : E \to B$ be a map with hupl and $\tilde{\alpha}$ *and* $\tilde{\beta}$ be paths in E such that $\tilde{\alpha}(0) = \tilde{\beta}(0)$, $p \circ \tilde{\alpha} = p \circ \tilde{\beta}$. Define, for every $t \in I$, $\tilde{\alpha}_t$, $\tilde{\beta}_t : I \to E$ such that $\tilde{\alpha}_t(s) = \tilde{\alpha}(st)$ and $\tilde{\beta}_t(s) = \tilde{\beta}(st)$. By definitions, $\tilde{\alpha}_t(0) = \tilde{\beta}_t(0)$ and $p \circ \tilde{\alpha}_t = p \circ \tilde{\beta}_t$. Then hupl imply that $\tilde{\alpha}_t \simeq \tilde{\beta}_t \; rel \; \{0, 1\}$, specially $\tilde{\alpha}_t(1) = \tilde{\beta}_t(1)$ which implies $\tilde{\alpha}(t) = \tilde{\beta}(t)$ and since t is arbitrary, $\tilde{\alpha} = \tilde{\beta}$. □

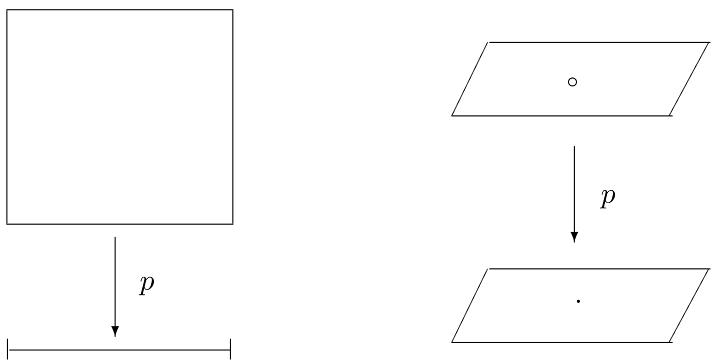
Proposition 2.6.
(i) upl $\Rightarrow$ whupl,

(ii) uphl $\Rightarrow$ whupl,
(iii) uphl $\Rightarrow$ wuphl,
(iv) uphl $\Rightarrow$ upl,
(v) wuphl $\Rightarrow$ whupl.

Proof. Use definitions. Just for (iv), a method like in the proof of the previous proposition is needed. $\square$

The following example shows that the converse of all the parts of Proposition 2.6 is not true.

Example 2.7.
For, (i) *wuphl* $\not\Rightarrow$ *uphl*, (ii) *whupl* $\not\Rightarrow$ *uphl* and (iii) *whupl* $\not\Rightarrow$ *upl*, let $E = \{0\} \times [0,1] \times [0,1]$ and $B = \{0\} \times [0,1] \times \{0\}$, and $p : E \to B$ is the vertical projection.
Also, for (iv) *upl* $\not\Rightarrow$ *uphl* and (v) *whupl* $\not\Rightarrow$ *wuphl*, let $E = \{(x, y, 2) \in R^3\} - \{(0, 0, 2)\}$, $B = \{(x, y, 0) \in R^3\}$ and $p : E \to B$ be again the vertical projection.



Remark 2.8. Moreover, there is no relation between upl and wuphl, because the part (i) of the example 2.7 imply that, *wuphl* $\not\Rightarrow$ *upl* and by (ii), we have, *upl* $\not\Rightarrow$ *wuphl*.

Since, uphl imply upl and also, a map with upl has unique lifting property for path connected space, we have

Corollary 2.9. *If a map has uphl, it has the unique lifting property for path connected spaces.*

3 Fibrations and homotopically liftings

In this section, we compare and study the notions introduced in section 2 in presence of fibrations.

Proposition 3.1. *For fibrations we have:*
(i) upl (hupl) $\Rightarrow$ uphl
(ii) upl (hupl) $\Rightarrow$ wuphl

Proof. For (i) see [4, Lemma 2.3.3], also, (ii) come from definition and (i). $\square$

Corollary 3.2. *For fibrations, upl (hupl) and uphl are equivalent.*

Remark 3.3. We already saw that even within assumption fibration, the converse of (i) of this proposition is true, moreover, the map in example 2.7 (i), is a fibration with wuphl which has not upl, then, the converse of (ii) is failed.

In the following theorem, we show that considering lifting in the homotopy category makes that paths in fibers are homotopically constant.

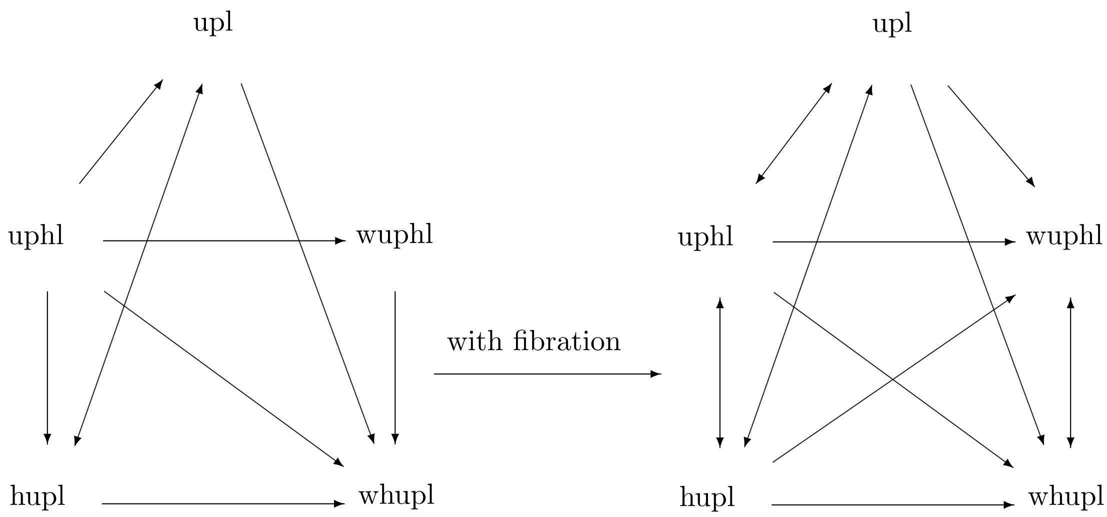
Theorem 3.4. *If $p : E \to B$ is a fibration, then p has wuphl if and only if every loop in each fiber is nullhomotopic.*

Similarly, we can replace, wuphl with whupl, then

Theorem 3.5. *A fibration $p : E \to B$ has whupl if only if every loop in each fiber is nullhomotopic.*

Corollary 3.6. *If $p : E \to B$ is a fibration then whupl and wuphl are equivalent.*

So, the relation between this five kinds of the paths lifting is as the following



Remark 3.7. Note that the converse of Corollary 3.6 is not necessarily correct, because if $p : \{*\} \to I$ is the constant map $* \mapsto 0$, then p has wuphl and whupl but p is not a fibration. For if, let $\widetilde{f} : X \to \{*\}$ be $x \mapsto *$ and $F : X \times I \to I$ be $F(x,t) = t$, then $p \circ \widetilde{f} = F \circ j$. But there is not a map $\widetilde{F} : X \times I \to \{*\}$ such that $p \circ \widetilde{F} = F$ because $p \circ \widetilde{F}(x, 0.5) = p(*) = 0$ but $F(x, 0.5) = 0.5$.

We know that If $p : E \to B$ is a fibration with upl, then the induced homomorphism by p, $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism. By Proposition 2.4 and Corollary 3.6, we have:

Corollary 3.8. *If $p : (E, e_0) \to (B, b_0)$ be a fibration, then whupl is equivalent to the injectivity of $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$.*

Although for fibrations with upl, the induced homomorphism on the fundamental groups is monomorphism, but for fibrations with wuphl, we have more than this.

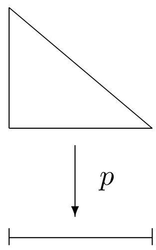
Theorem 3.9. *For a fibration $p : E \to B$ with wuphl and path connected fibers, the induced homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is an isomorphism.*

Proof. Let $[\alpha] \in \pi_1(B, b_0)$ and $\widetilde{\alpha}$ be the lifting of α started at e_0. $\widetilde{\alpha}(1) \in p^{-1}(b_0)$ and assume λ is a path in $p^{-1}(b_0)$ from $\widetilde{\alpha}(1)$ to e_0. $[\widetilde{\alpha} * \lambda] \in \pi_1(E, e_0)$ and $p_*([\widetilde{\alpha} * \lambda]) = [\alpha * c_{b_0}] = [\alpha]$ and hence p_* is onto. Injectivity of p_* comes from Proposition 2.4, (ii). $\square$

Note that path connectedness of fibers is essential in the previous theorem. For example, let p be the exponential map $R \to S^1$ which is a covering map and so is a fibration with upl. Then, by the Proposition 3.1 (ii), p has wuphl, but p_* is not an isomorphism.

It is well known that for fibrations, fibers have the same homotopy type and for fibrations with upl and path connected base space, every two fibers are homeomorphic [4, Lemma 2.3.8]. In the following example, we show this fact fails if we replace upl by wuphl (whupl).

Example 3.10. Let $E = \{(x, y) \in R^2 | x \geq 0, y \geq 0, y \leq 1 - x\}$, $B = [0, 1]$ and $p : E \to B$ be projection on the first component which is clearly continuous. For two given maps $F : X \times I \to B$ and $\widetilde{f} : X \to E$ in which $F \circ j = p \circ \widetilde{f}$, define $\widetilde{F} : X \times I \to E$ by $\widetilde{F}(x, t) = (F(x, t), pr_2 \circ \widetilde{f})$ which is continuous and show that p is fibration. But, $p^{-1}(0) = \{0\} \times I$ and $p^{-1}(1) = \{(1, 0)\}$ and this shows that p has not upl, has wuphl and fibers are not necessarily homeomorphic.



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