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NEW SENSITIVITY-BASED METHODS FOR STRUCTURAL DAMAGE DIAGNOSIS BY LEAST SQUARE MINIMAL RESIDUAL TECHNIQUES*

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Abstract– This paper presents new sensitivity-based methods for detection of structural damage using incomplete noisy modal data. These methods are based on the first-order derivative of modal parameters. Changes of natural frequency do not usually provide spatial information on the structural damage. They are also not sensitive to the local damage. In this paper, a new sensitivity function is proposed using method of Lagrange multipliers in order to deal with these weaknesses when applying natural frequency in the sensitivity-based damage diagnosis. Mode shape is the other vibrational data which leads to better results in comparison with natural frequency. However, usually some mode shape's sensitivities require all modes to obtain exact sensitivity functions. Thus, an improved sensitivity of mode shape is presented to constitute an applicable formulation based on using incomplete modes. To determine the damage quantity, a powerful iterative method named Least-Square Minimal Residual (LSMR) technique is proposed in the condition of incomplete modes. Subsequently, Regularized Least-Square Minimal Residual (RLSMR) method is presented to detect structural damage when the incomplete modal parameters are contaminated by noise. Applicability and effectiveness of the proposed methods are numerically verified using two practical examples consisting of a six-story shear building and a planner truss. Eventually, numerical results indicate that the LSMR and RLSMR are influential algorithms for precisely determining the damage severity. Furthermore, obtained results of damage diagnosis process in the free-noise data show that the proposed sensitivities of natural frequency and mode shape can provide reliable and accurate results for structural damage diagnosis.

Keywords– Structural damage diagnosis, sensitivity-based method, Least Square Minimal Residual, incomplete noisy modal data, regularization method

1. INTRODUCTION

A structure may be prone to damage when being subjected to severe loadings like strong earthquakes or blasts or when its inherent properties like cross-section, modulus of elasticity and dynamic properties (mass, stiffness) are adversely changed. Recently vibration-based techniques have been widely applied in the structural health monitoring and damage detection process due to advances in dynamic testing. In these techniques, it is assumed that the physical-dynamic properties of the structure are changed when the damage occurs. Vibrational responses of the structure depend directly on these properties; therefore the occurrence of damage leads to adverse alteration in the structure's vibrational responses. Throughout recent decades, development of experimental modal analysis techniques has facilitated the accurate measurement of modal data in different types of structures. Hence, the structure's modal data including natural frequencies and mode shapes are the most practical dynamic information for structural damage diagnosis. Consequently, changes in structure's physical properties (mass and stiffness) will cause the

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main alteration in its modal data. This alteration can be considered as a criterion for identifying and detecting the damage in the structures.

Several methods have been developed to detect damage using changes of modal data or methods, which are dependent on them, such as mode shape curvature, modal strain energy (MSE), modal assurance criterion (MAC) and modal flexibility. Cawley and Adams [1] detected structural damage (i.e. a crack) by decreasing natural frequency and increasing damping ratio. Messina et al. [2] utilized the difference between a few number of structure's natural frequencies in the undamaged and damaged states to identify the damage location and then find its extent. Lee and Chung [3] presented a method for locating the damage (e.g. find the position of crack) and then determining the damage extent using a few natural frequencies. Moreover, Kim et al. [4] proposed a method for non-destructively locating and estimating the extent of damage in the structures in which a few natural frequencies or mode shapes are available. Su et al. [5] located the damage in a shear building in which properties of stories (stiffness and mass) change during the life cycle. They utilized the sub-structural natural frequencies of the shear building and identified the damage location by comparing the natural frequencies of sub-structures in different stages. Esfandiari et al. [6] proposed a new frequency-based technique in the form of an improved sensitivity equation of natural frequency to detect any number of localized damages which induce reduction in a structure's stiffness. Moreover, Ranjbaran [7] derived a new and general governing differential equation for eigenvalue analysis of cracked members and also determined the exact analytical solutions for eigenvalues and mode shapes of these members.

Using natural frequency for detecting the structural damage is advantageous owing to its convenient measurement and high accuracy. However, this method cannot provide spatial information regarding the structural damages and also is not sensitive enough to the local damage. Therefore, the mode shape's changes or its sensitivity are used to evaluate damages in the structural systems. Shi et al. [8] proposed a statistical sensitivity-based approach namely multiple damage location assurance criterion (MDLAC) method which is a development of Messina's method [2]. In this method, incomplete mode shape is used instead of modal frequency. Rahai et al, [9] presented a global algorithm for damage detection based on parameter estimation method using finite element analysis and incomplete measured mode shapes. Zhu et al, [10] developed an efficient damage detection algorithm for shear buildings by using changes of the first mode shape slopes (CFMSS). Moreover, they determined the damage extent using the modified CFMSS values. Baghiee et al, [11] detected the damage location and damage extent as crack in a reinforced concrete beam equipped with CFRP by means of several efficient methods such as frequency changes, modal assurance criterion, coordinate modal assurance criterion (COMAC) and modal curvatures. Yan et al, [12] proposed a new sensitivity equation of modal strain energy for structural damage diagnosis. They presented a statistic damage detection algorithm based on the closed-form equation of element's modal strain energy sensitivity with ambient vibration measurements where only the operational mode shapes are available.

In this paper, new sensitivity-based methods, which are based on the first-derivative of natural frequency and mode shape, are presented to detect the structural damage with incomplete noisy modal data. These methods are simple, effective and accurate as well as able to obviate weaknesses of modal data's sensitivity equations. In this regard, a new sensitivity equation of natural frequency is proposed by the method of Lagrange multipliers. The main parameters in this sensitivity equation are Lagrange multipliers and dynamic properties of structure. Hence, these multipliers and inherent properties of structure may overcome the main weaknesses and ambiguities of natural frequency's sensitivity in the structural damage diagnosis. To establish the sensitivity of mode shape, the modal method is improved with respect to existence of incomplete modal data and using flexibility matrix. In the improvement approach, some inefficiency and computational errors of the mode shape's sensitivity based on modal method are entirely obviated and a powerful algorithm is presented for determining the damage extent. Subsequently, a novel technique called Least Square Minimal Residual (LSMR) is proposed to solve the damage equation (penalty function) when incomplete modal data are available or the sensitivity matrices have some singular values close to zero. The LSMR is an iterative method that applies bidiagonalization process for solving the linear mathematical equations. In general, the problem of dynamic systems is ill-conditioned and requires some regularizations. Therefore, Regularized Least Square Minimal Residual (RLSMR) technique is presented which solves the Tikhonov regularization objective function in an iterative manner. The RLSMR develops LSMR method by adding a regularization parameter to the linear mathematical equations. Therefore, by adding this regularized parameter and also applying the property of RLSMR, the effects of perturbations (noisy data) on the damage diagnosis process are reduced significantly. Two practical examples including a six-story shear building and a planner truss are numerically modelled to verify the proposed methods. It can be concluded that the LSMR and RLSMR are potentially able to detect damage extent in an iterative manner by comparing the amounts of predicted and induced damages. Moreover, it can be observed that the amounts of determined relative error for the improved sensitivity of mode shape (ISMS) and proposed sensitivity of natural frequency (PSNF) are negligible. Moreover, the results confirm that ISMS is more accurate than the modal method and remains valid when this method fails. Furthermore, the correctness of PSNF is dependent on Lagrange multipliers and dynamic properties of structure. Therefore, this method can be applied in LSMR and RLSMR for detecting the local damages with a high accuracy even when incomplete noisy modal data are available.

2. THEORY

a) The proposed method for determining the sensitivity of natural frequency

The first-order derivative of natural frequency with respect to the damage parameter b can be described by differentiation of undamped generalized eigenvalue problem:

$$\frac{d\lambda_i}{db} = \varphi_i^T \left(\frac{\partial K}{\partial b} - \lambda_i \frac{\partial M}{\partial b} \right) \varphi_i \tag{1}$$

where M and K are the mass and stiffness matrices of structure respectively; $\lambda_i = \omega_i^2$ is the i^{th} eigenvalue and ω_i is i^{th} natural frequency. Moreover, φ_i is the i^{th} mode shape (eigenvector) of the structure. In general, Eq. (1) is not a suitable equation for being used in the damage detection process. The variation of natural frequency gives no spatial information and also is not sensitive to local damages. Several researchers have demonstrated the poor performance of this equation in complex structures under damage patterns of low damage extent. Therefore, a new approach based on the method of Lagrange multipliers is proposed to constitute a new efficient technique that can obviate the existing ambiguities and weaknesses while applying Eq. (1). In mathematical optimizations, the Lagrange multipliers approach is a methodology for finding the local maxima and minima of a main function, which is subject to some equality constraints. In this method, the partial first-order derivatives of Lagrange function with respect to its unknown variables should equal zero [13, 14]. The algorithm of Lagrange's derivatives is a precise and practical method for being applied in the sensitivity analysis. Hence, in this paper, the algorithm of Lagrange multipliers method is applied to generate a new sensitivity equation for natural frequency. The main function in the Lagrange method is defined as natural frequency of the i^{th} mode and equality conditions are characterized as mass and stiffness orthogonality conditions as well as the generalized eigenvalue problem. Thus, the final equation of Lagrange function can be expressed as follows:

$$L\left(b,\lambda,\varphi,\alpha,\beta,\theta\right)=\lambda_i+\alpha_i^T\left(K-\lambda_i M\right)\varphi_i+\beta_i\left(\varphi_i^T K\varphi_i-\lambda_i\right)+\theta_i\left(\varphi_i^T M\varphi_i-1\right) \tag{2}$$

where α, β and θ are Lagrange multipliers that can be determined by the partial differentiation of Lagrange function with respect to its main variables including eigenfrequency ($\partial L/\partial\lambda_i$), mode shape ($\partial L/\partial\varphi_i$) and the multipliers of Lagrange function ($\partial L/\partial\alpha$, $\partial L/\partial\beta$ and $\partial L/\partial\theta$). Thus:

$$\frac{\partial L}{\partial\varphi_i}=\frac{\partial\lambda_i}{\partial\varphi_i}+\alpha_i^T\left(K-\lambda_i M\right)+\beta_i\left(K\varphi_i+\varphi_i^T K\right)+\theta_i\left(M\varphi_i+\varphi_i^T M\right)=0 \tag{3}$$

In general, the natural frequency is independent of mode shape and $\partial\lambda_i/\partial\varphi_i$ always equals zero. Furthermore, the mass and stiffness matrices are positive and symmetric. Thus, it can be deduced that $K\varphi_i=\varphi_i^T K$ and $M\varphi_i=\varphi_i^T M$. Eq. (3) can be rewritten as follows:

$$\frac{\partial L}{\partial\varphi_i}=\left(K-\lambda_i M\right)\alpha_i+2\beta_i\left(K\varphi_i\right)+2\theta_i\left(M\varphi_i\right)=0 \tag{4}$$

The partial derivative of Lagrange function with respect to the natural frequency is determined as:

$$\frac{\partial L}{\partial\lambda_i}=1-\alpha_i^T M\varphi_i-\beta_i=0 \tag{5}$$

On the other hand, Lagrange's partial differentiations for multipliers α, β and θ can be formulated as follows:

$$\frac{\partial L}{\partial\alpha}=\left(K-\lambda_i M\right)\varphi_i=0 \tag{6}$$

$$\frac{\partial L}{\partial\beta}=\varphi_i^T K\varphi_i-\lambda_i=0 \tag{7}$$

$$\frac{\partial L}{\partial\theta}=\varphi_i^T M\varphi_i-1=0 \tag{8}$$

To calculate Lagrange multipliers, Eq. (4) is multiplied by φ_i^T and thus:

$$\varphi_i^T\left(K-\lambda_i M\right)\alpha_i+2\beta_i\left(\varphi_i^T K\varphi_i\right)+2\theta_i\left(\varphi_i^T M\varphi_i\right)=0 \tag{9}$$

It is obvious from Eq. (6) that $\varphi_i^T\left(K-\lambda_i M\right)$ or $\left(K-\lambda_i M\right)\varphi_i$ equal zero. Based on Eqs. (7)-(8), it can be observed that $\varphi_i^T K\varphi_i=\lambda_i$ and $\varphi_i^T M\varphi_i=1$. By inserting these results into Eq. (9), it can be seen that:

$$\theta_i=-\beta_i\lambda_i \tag{10}$$

By replacing Eq. (10) in Eq. (4), it can be written:

$$\left(K-\lambda_i M\right)\alpha_i+2\beta_i\left(K\varphi_i\right)-2\beta_i\lambda_i\left(M\varphi_i\right)=0 \tag{11}$$

This equation can then be simplified as:

$$\left(K-\lambda_i M\right)\alpha_i+2\beta_i\left(K-\lambda_i M\right)\varphi_i=0 \tag{12}$$

Once again, the result of Eq. (6) is used and the vector of Lagrange multiplier α can be calculated by solving the equation below:

$$\left(K-\lambda_i M\right)\alpha_i=0 \tag{13}$$

Subsequently, Lagrange multiplier β can be determined through Eq. (5) as follows:

$$\beta_i=1-\alpha_i^T M\varphi_i \tag{14}$$

As a consequence, the last multiplier of Lagrange function can be computed using Eq. (10) after calculating the multiplier β. By determining the Lagrange multipliers, the total first-order derivative of Eq. (2) with respect to damage parameter b should be formulated to constitute the sensitivity of natural frequency.

$$\frac{dL}{db}=\frac{d\lambda_i}{db}+\frac{d\left[B\left(b\right)\vartheta_L\right]}{db} \tag{15}$$

where

$$B\left(b\right)=\left[\left(K-\lambda_i M\right)\varphi_i \quad \varphi_i^T K\varphi_i-\lambda_i \quad \varphi_i^T M\varphi_i-1\right] \tag{16}$$

$$\vartheta_L=\left[\alpha_i^T \quad \beta_i \quad \theta_i\right]^T \tag{17}$$

Derivative of the second expression of Eq. (15) can be expressed as:

$$\frac{d\left[B\left(b\right)\vartheta_L\right]}{db}=\frac{dB\left(b\right)}{db}\vartheta_L+B\left(b\right)\frac{d\vartheta_L}{db} \tag{18}$$

It should be noted that the ϑ_L (Lagrange multipliers' vector) will be computed based on the following conditions:

$$\left\langle\vartheta_L,B\left(b\right)\right\rangle=\left(\vartheta_L\right)^T\cdot B\left(b\right)=0 \tag{19}$$

$$\left\langle\vartheta_L,B\left(b\right)+\frac{dB\left(b\right)}{db}\right\rangle=\left(\vartheta_L\right)^T\cdot\left(B\left(b\right)+\frac{dB\left(b\right)}{db}\right)=0 \tag{20}$$

According to these equations, the first expression of the right-hand side of Eq. (18) should equal zero. On the other hand, $d\vartheta_L/db$ always equals zero. Therefore, $d\left[B\left(b\right)\vartheta_L\right]/db=0$ and this expression is eliminated from the total derivative of Lagrange function. As a result:

$$\frac{dL}{db}=\frac{d\lambda_i}{db} \tag{21}$$

The total derivative of Lagrange function can be extended by partial derivative of its variables as follows:

$$\frac{dL}{db}=\frac{\partial L}{\partial b}+\left(\frac{\partial L}{\partial\varphi_i}\frac{d\varphi_i}{db}\right)+\left(\frac{\partial L}{\partial\lambda_i}\frac{d\lambda_i}{db}\right)+\left(\frac{\partial L}{\partial\alpha}\frac{d\alpha}{db}\right)+\left(\frac{\partial L}{\partial\beta}\frac{d\beta}{db}\right)+\left(\frac{\partial L}{\partial\theta}\frac{d\theta}{db}\right) \tag{22}$$

As expressed before, the expressions $\partial L/\partial\varphi_i$, $\partial L/\partial\lambda_i$ and differentiation of the Lagrange's multipliers ($\partial L/\partial\alpha$, $\partial L/\partial\beta$ and $\partial L/\partial\theta$) should equal zero for computing the Lagrange multipliers. Thus:

$$\frac{dL}{db}=\frac{\partial L}{\partial b}=\frac{\partial\lambda_i}{\partial b}+\left[\left(\frac{\partial K}{\partial b}-\lambda_i\frac{\partial M}{\partial b}\right)\varphi_i \quad \left(\varphi_i^T\frac{\partial K}{\partial b}\varphi_i\right) \quad \left(\varphi_i^T\frac{\partial M}{\partial b}\varphi_i\right)\right]\vartheta_L \tag{23}$$

In this equation, the partial derivative of natural frequency with respect to *b* always equals zero. According to Eqs. (21)- (23), it is clear that the total derivative of natural frequency is equivalent to the partial derivative of Lagrange function. Thus, the proposed sensitivity of natural frequency (PSNF) can be formulated as follows:

$$\frac{d\lambda_i}{db} = \alpha_i^T \left(\frac{\partial K}{\partial b} - \lambda_i \frac{\partial M}{\partial b} \right) \varphi_i + \beta_i \left(\varphi_i^T \frac{\partial K}{\partial b} \varphi_i \right) + \theta_i \left(\varphi_i^T \frac{\partial M}{\partial b} \varphi_i \right) \tag{24}$$

In this study, it is assumed that the structural damage is not accompanied by a change in the mass matrix. In addition, it is assumed that stiffness reduction leads to occurrence of damage in the structure. Hence, the first-order derivative of mass matrix is omitted from Eq. (24). Accordingly, the sensitivity function of natural frequency for detecting damage extent can be written as:

$$\frac{d\lambda_i}{db} = \left(\alpha_i^T + \beta_i \varphi_i^T \right) \left(\frac{\partial K}{\partial b} \right) \varphi_i \tag{25}$$

As can be seen in Eq. (25), the proposed sensitivity of natural frequency is formulated by using Lagrange's multipliers and the variation of stiffness matrix that can be easily computed by linear algebra and simple mathematical formulations. Moreover, another advantage of the proposed method is that PSNF does not require mode shapes of the damaged structure and therefore, does not increase the volume of measurement process and equipment's cost.

b) The improved method for determining the sensitivity of mode shape

Based on vibration theory, the mode shapes of a structure are independent; thus, it is possible to constitute the sensitivity (first-order derivative) of each mode shape separately. In this regard, several researchers have introduced different methods for calculating the sensitivity of mode shape. For instance, many methods including the model method [15, 16], Nelson's method [17-19], the algebraic method [20, 21] and the modal method [22-24] have been proposed or developed to create a simple and reliable way for calculating the sensitivity of mode shape. However, it is required to improve the structure of some mentioned approaches to better their performance. Therefore, these methods should be modified for the case of incomplete measured modes and even complicated structural systems. One of these approaches is the modal method that was introduced by Fox and Kapoor [22]. The sensitivity of mode shape can be described by expressing its derivative as a linear combination of all modes:

$$\frac{d\varphi_i}{db} = \sum_{j=1}^{n} c_j \varphi_j \tag{26}$$

in which:

$$c_j = \begin{cases} \dfrac{\varphi_j^T}{\lambda_i - \lambda_j} \left(\dfrac{\partial K}{\partial b} - \lambda_i \dfrac{\partial M}{\partial b} \right) \varphi_i & if \quad i \neq j \\ -0.5 \varphi_j^T \dfrac{\partial M}{\partial b} \varphi_j & if \quad i = j \end{cases} \tag{27}$$

In this study, the structural damage is introduced by the stiffness reduction factor and thus mass variation does not influence on the sensitivity of modal data. Hence, the coefficient c_j can be modified as:

$$c_j = \frac{\varphi_j^T}{\lambda_i - \lambda_j} \left(\frac{\partial K}{\partial b} \right) \varphi_i \tag{28}$$

The modal method is widely used in engineering applications particularly in sensitivity analysis due to its simplicity. However, this method requires all mode shapes to obtain an exact sensitivity equation of each mode. Measurement of all modes is a significant computational task in the modal analysis techniques and sometimes only a few modes dominate the dynamic behaviour of structure. Therefore, when the modal data are incomplete, the modal method does not provide appropriate results. In this study, an improved method is presented to deal with this weakness by developing the modal method. It is assumed that *m* modes are identified for a structural system with *n* degrees of freedom. Hence, the improved sensitivity of mode shape (ISMS) can be described as follows:

$$\frac{d\varphi_i}{db} = \sum_{j=1}^{m} c_j \varphi_j + \sum_{j=m+1}^{n} d_j \varphi_j \tag{29}$$

Assuming *n* and *m* modes are available, Eq. (29) can be modified as:

$$\frac{d\varphi_i}{db} = \sum_{j=1}^{m} c_j \varphi_j + \left(\sum_{j=1}^{n} d_j \varphi_j - \sum_{j=1}^{m} d_j \varphi_j \right) \tag{30}$$

Considering Wang's theory [23], it can be assumed in modal analysis that $m \ll n$ and therefore, $\lambda_i \ll \lambda_j$. Accordingly, d_j can be written as:

$$d_j = -\frac{\varphi_j^T}{\lambda_j}\left(\frac{\partial K}{\partial b}\right)\varphi_i \tag{31}$$

The second part of Eq. (30) can be expressed as:

$$q_j = -\sum_{j=1}^{n}\left(\frac{\varphi_j^T}{\lambda_j}\left(\frac{\partial K}{\partial b}\right)\varphi_i \right)\varphi_j + \sum_{j=1}^{m}\left(\frac{\varphi_j^T}{\lambda_j}\left(\frac{\partial K}{\partial b}\right)\varphi_i \right)\varphi_j \tag{32}$$

It is clear that d_j is a $(n \times 1)$ vector. Based on the fundamental concept of matrix analysis and linear algebra, Eq. (32) can be rewritten as:

$$q_j = -\sum_{j=1}^{n}\left(\frac{\varphi_j \varphi_j^T}{\lambda_j} \right)\left(\frac{\partial K}{\partial b}\right)\varphi_i + \sum_{j=1}^{m}\left(\frac{\varphi_j^T}{\lambda_j}\left(\frac{\partial K}{\partial b}\right)\varphi_i \right)\varphi_j \tag{33}$$

The first expression of Eq. (33) is similar to the main equation of flexibility matrix that can be determined by modal parameter [25] as follows:

$$F = \sum_{j=1}^{n} \frac{1}{\lambda_j} \varphi_j \varphi_j^T \tag{34}$$

On the other hand, the flexibility matrix always equals the inverse of stiffness matrix ($F=K^{-1}$). Thus, the equation of mode shape's sensitivity can be described as:

$$\frac{d\varphi_i}{db} = \sum_{j=1}^{m}\left(\frac{\varphi_j^T}{\lambda_i - \lambda_j}\left(\frac{\partial K}{\partial b}\right)\varphi_i \right)\varphi_j + \sum_{j=1}^{m}\left(\frac{\varphi_j^T}{\lambda_j}\left(\frac{\partial K}{\partial b}\right)\varphi_i \right)\varphi_j - K^{-1}\left(\frac{\partial K}{\partial b}\right)\varphi_i \tag{35}$$

Final equation of mode shape's sensitivity for damage detection is formulated as follows:

$$\frac{d\varphi_i}{db} = \sum_{j=1}^{m}\left(\left(\frac{\varphi_j^T}{\lambda_i - \lambda_j} + \frac{\varphi_j^T}{\lambda_j} \right)\left(\frac{\partial K}{\partial b}\right)\varphi_i \right)\varphi_j - K^{-1}\left(\frac{\partial K}{\partial b}\right)\varphi_i \tag{36}$$

As can be observed, the improved sensitivity of mode shape (ISMS) has been constituted by adding the flexibility matrix (inversed stiffness) and a coefficient (φ_j^T / λ_j) to the old version of modal method. It is clear that the flexibility matrix directly depends on the mode shape and the reciprocal of natural frequency. Moreover, the effect of high-frequency components on the flexibility matrix is reduced with the increase of natural frequency. Thus, this matrix has enough accuracy for using a number of low-order modes and frequencies. Therefore, presence of flexibility matrix in Eq. (36) leads to an increase in the precision of mode shape's sensitivity.

c) Damage detection by Least-Square Minimal Residual (LSMR) method with incomplete modal data

Least-square method is the most currently used mathematical approach for solving damage equation when sensitivity function is available. In general, the damage equation can be introduced by truncation of Taylor series which is defined as Penalty function:

$$S\,b = \Delta y \tag{37}$$

where S and Δy are sensitivity (coefficient matrix) and difference matrix of modal data, respectively. Furthermore, b is the damage vector that should be calculated. It should be noted that Δy depends on to the way the sensitivity function is used. In other words, when the sensitivity of natural frequency is applied, Δy is defined as the difference between natural frequency of undamaged and damaged structures.

To utilize the sensitivity of mode shape in the algorithm of damage diagnosis, Δy is described as the discrepancy between mode shapes of undamaged and damaged structures. Since modal data are incomplete, the sensitivity matrices are defined as rectangular matrices. Therefore, Eq. (37) can be solved by the direct least-square method as follows:

$$b = \left(S^T S\right)^+ S^T \Delta y \tag{38}$$

As some singular values of sensitivity matrix are close to zero, the condition number of S^TS is a large quantity and therefore its inverse cannot be accurately determined. In other words, the results of Eq. (38) do not have enough precision. Thus, in these cases iterative methods such as Least-Square Minimal Residual (LSMR) have got better performance in comparison with direct approaches. In general, this method is proposed to solve linear systems $Sb=\Delta y$ and least square problems by minimizing the norm $\|Sb-\Delta y\|$. It is analytically equivalent to the Minimal Residual method and is applied to the normal equation $S^TSb_k=S^T\Delta y$ so that the quantities $\|S^Tr_k\|$ are monotonically decreasing. It should be mentioned that $r_k=\Delta y - Sb_k$ is the residual for the current iterate b_k. In fact, b_k is an approximate solution for the equation $Sb=\Delta y$ which is obtained from LSMR algorithm in the k^{th} step by applying the condition $min\|S^Tr_k\|$. In practice, it can also be seen that $\|r_k\|$ is monotonically reduced. The LSMR algorithm uses an algorithm of Golub and Kahan, which is expressed as the bidiagonalization procedure [26, 27]. To reduce S to a lower bidiagonal form, bidiagonalization process can be performed as follows:

$$\xi_1 u_1 = \Delta y \tag{39}$$

$$\eta_1 v_1 = S^T u_1 \tag{40}$$

For $i=1,2,\ldots$ the last two equations can be expressed as:

$$\xi_{i+1} u_{i+1} = S v_i - \eta_i u_i \tag{41}$$

$$\eta_{i+1} v_{i+1} = S^T u_{i+1} - \xi_{i+1} v_i \tag{42}$$

The scalars $\xi_i \geq 0$ and $\eta_i \geq 0$ are chosen such that $\|u_i\|_2=\|v_i\|_2=1$. By using the bidiagonalization process, the LSMR method yields an approximate solution for determining the unknown parameter b_k at the k^{th} step so that $\|S^Tr_k\|$ is monotonically decreasing.

$$b_k = V_k z_k \tag{43}$$

where V_k is a matrix with k columns, each column of which is the vector v. Thus:

$$V_k = [v_1 \quad v_2 \quad \dots \quad v_k] \tag{44}$$

Furthermore, z_k is given by minimizing the norm $\|S^T r_k\|$ in which $r_k = \Delta y - Sb_k$ is the residual for the approximate solution b_k. The LSMR improves the solution of direct least-square method by using both iterative technique and bidiagonalization process. In particular, when the sensitivity matrix is approximately obtained or constituted as a rectangular matrix with singular values close to zero, the LSMR is more influential and more reliable than the direct techniques.

d) Damage detection by Regularized Least-Square Minimal Residual (RLSMR) method with incomplete noisy modal data

The least-square problems such as LSMR and LSQR and other methods related to this problem are generally ill-conditioned and require some regularization. This means that in a huge number of dynamic systems, small perturbations such as presence of noise in the modal data and inaccurate structural modeling may lead to great unrealistic errors. Moreover, the direct inverse solutions for the damage equation usually yield poor results since the errors in modal data measurements may be greatly amplified due to the nature of ill-posed problems. Thus, regularization methods are needed to filter out the influence of noise on the measured modal data. The two most currently used regularization approaches are truncated singular value decomposition (TSVD) and Tikhonov regularization method (TRM) [28-30]. These methods improve the conditions of linear problem and therefore lead to a reliable numerical solution. Although from a theoretical point of view both methods are similar, Tikhonov regularization may be more useful in the variational form. Hence, in the present study, the objective function of Tikhonov method is adopted to redefine the linear least-squares problem as minimization of Tikhonov objective function as follows:

$$J(b) = \|Sb - \Delta y\|_2^2 + \gamma \|b\|_2^2 \tag{45}$$

where, $\|\cdot\|$ indicates the Euclidean norm. The Tikhonov objective function consists of two parts including the residual norm and the solution norm. Indeed, the upper objective function is constructed by adding an additional norm to the traditional least-square minimization technique. This added norm is known as the solution norm $\|b\|$ which is adjusted by a regularization parameter called γ. Since γ depends on the size of perturbation in data, selection of the regularization parameter is a crucial step in the regularization methods. Moreover, for regularized solution of Eq. (45), the optimal regularization parameter should be estimated to obtain a meaningful amount for the unknown quantity (damage severity). There are two popular methods including L-curve method (LCM) and generalized cross validation (GCV) for determining the regularization parameter [31-33]. In the generalized cross validation, the optimal regularization parameter is calculated as the parameter that leads to minimal average prediction error for all omitted data points. In fact, GCV is utilized to maximize the predictability of regularized solution by appropriate setting of the regularization parameter. Hence, the generalized cross validation is adopted here and can be defined as:

$$GCV(\gamma) = \frac{\left(1/m\right)\left\|\left(A_\gamma - I\right)\Delta y\right\|_2^2}{\left[\left(1/m\right) trace\left(I - A_\gamma\right)\right]^2} \tag{46}$$

where m is the number of quantities obtained from measurements (frequencies and mode shapes), and

$$A_\gamma = S\left(S^T S + \gamma I\right)^{-1} S^T \tag{47}$$

The expression *trace(I-A$_\gamma$)* in the denominator is readily determined by truncated singular value decomposition [30]. It should be noted that if γ is too small, then the objective function of Tikhonov regularization will be too close to the original ill-conditioned problem. In addition, if γ is extremely large, the Eq. (45) will greatly deviate from original problem. After choosing the optimal regularization parameter, Eq. (45) is expanded to directly obtain the damage unknown parameter.

$$J(b) = \Delta y^T \Delta y - 2b^T S^T \Delta y + b^T S^T S b + \gamma b^T b \tag{48}$$

According to Eq. (48), it can be seen that the Tikhonov regularization method is a direct approach. Since the sensitivity matrices are rectangular matrices and some of their singular values are about zero, the direct method and pseudo-inverse techniques have unreliable computational error in the damage detection process. Thus, a new iterative method based on LSMR algorithm namely Regularized Least-Square Minimal Residual (RLSMR) method is introduced to expand Tikhonov's objective function for achieving more trustworthy results in comparison with the case of direct method. Accordingly, the minimization term of Eq. (48) can be rewritten as follows:

$$\begin{pmatrix} S \\ \gamma I \end{pmatrix} b = \begin{pmatrix} \Delta y \\ 0 \end{pmatrix} \tag{49}$$

To determine the unknown quantity *b*, the above equation should be minimized as:

$$min\left\| \begin{pmatrix} S \\ \gamma I \end{pmatrix} b - \begin{pmatrix} \Delta y \\ 0 \end{pmatrix} \right\|_2^2 \tag{50}$$

Assuming that:

$$\bar{S} = \begin{pmatrix} S \\ \gamma I \end{pmatrix} \tag{51}$$

and

$$\bar{y} = \begin{pmatrix} \Delta y \\ 0 \end{pmatrix} \tag{52}$$

Therefore, the damage parameter *b* can be calculated in the k^{th} iterative step by applying the condition $min\left\|\bar{S}^T \bar{r}_k\right\|$, where $\bar{r}_k = \bar{y} - \bar{S} b_k$ is the residual for the regularized problem. By using bidiagonalization process of LSMR algorithms, it can be written:

$$b_k = \bar{V}_k \bar{z}_k \tag{53}$$

where $\bar{V}_k$ and $\bar{z}_k$ are determined by Eqs. (39)-(42) in a similar way. The only difference is that expression $\bar{S}$ and $\bar{y}$ should be replaced with *S* and Δy. Although the proposed RLSMR algorithm is potentially able to solve the most ill-conditioned problems, the precision of this method is dependent on the regularization parameter. In fact, the main difference between LSMR and RLSMR methods lies in the regularization coefficient. Thus, choosing the regularization parameters plays an important role in ill-conditioned problems.

e) Noisy measurement

In experimental modal tests, there may be some deviations in the results due to the existence of noise in measurements. In the numerical examples, this noise is simulated by adding a series of pseudo-random values to the theoretically calculated frequencies and mode shapes [34]. In other words, due to the complexity of the measurement process, an amount of noise may be inserted in measured data which contaminates the modal data. Thus, in order to investigate the effect of noise on the results obtained by the proposed damage detection method a random noise is considered as follows:

$$\varphi_i^* = \varphi_i \left(1+\varepsilon_r\right) \tag{54}$$

where φ* and φ are the mode shape with and without noise, respectively. Moreover, ε_r is a random number. In this study, two values equal to 1% and 5% are applied to mode shapes and natural frequencies as proportional random noises.

3. APPLICATION

a) A six-story shear building

To investigate the effectiveness of proposed damage detection algorithms, a six-story shear building is considered as shown in Fig. 1. Formulation of discrete systems is carried out in order to generate mass and stiffness matrices for this shear building [35]. It is assumed that the slabs are confined amongst beams and behave as rigid body; hence, the stiffness of each story is computed summing the stiffness of columns. Furthermore, the mass of each story is calculated summing half of the weight of top and bottom walls as well as the slab weight. After determining the structure's properties, natural frequencies and mode shapes of the shear building are calculated through the generalized eigenvalue problem. In practice, it is not possible or sometimes not necessary to identify all of the vibrational modes. Thus, only three of the first calculated mode shapes and natural frequencies are used. The initial physical properties of the shear building are presented in Table 1.

Table 1. Physical properties of the six-story shear building

Physical Properties	Story 1	Story 2	Story 3	Story 4	Story 5	Story 6
Mass (Ton)	10	10	10	8	8	6
Stiffness (Ton/m)	125	125	111	95	95	83

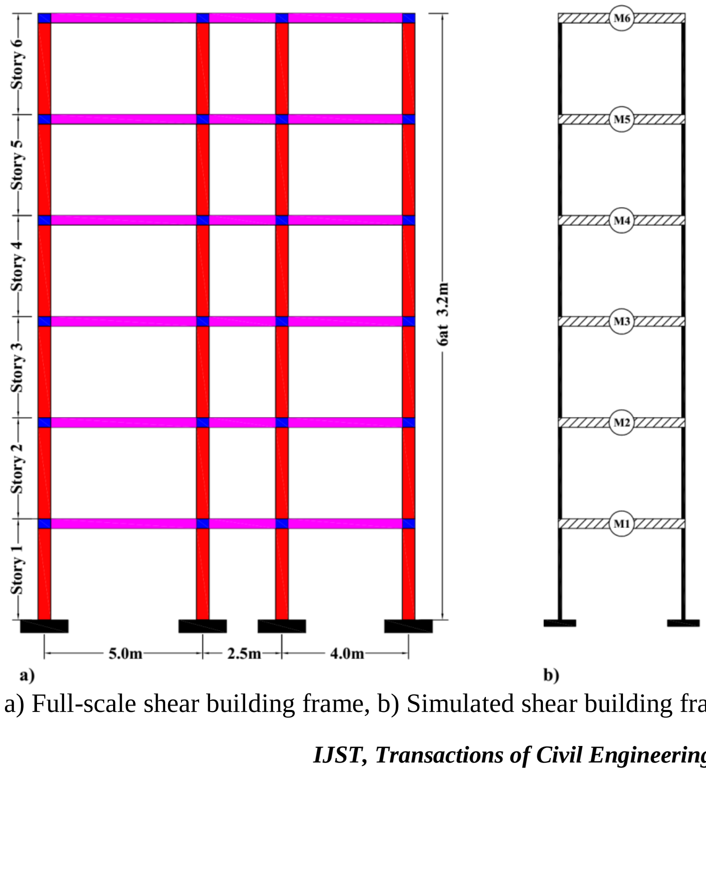


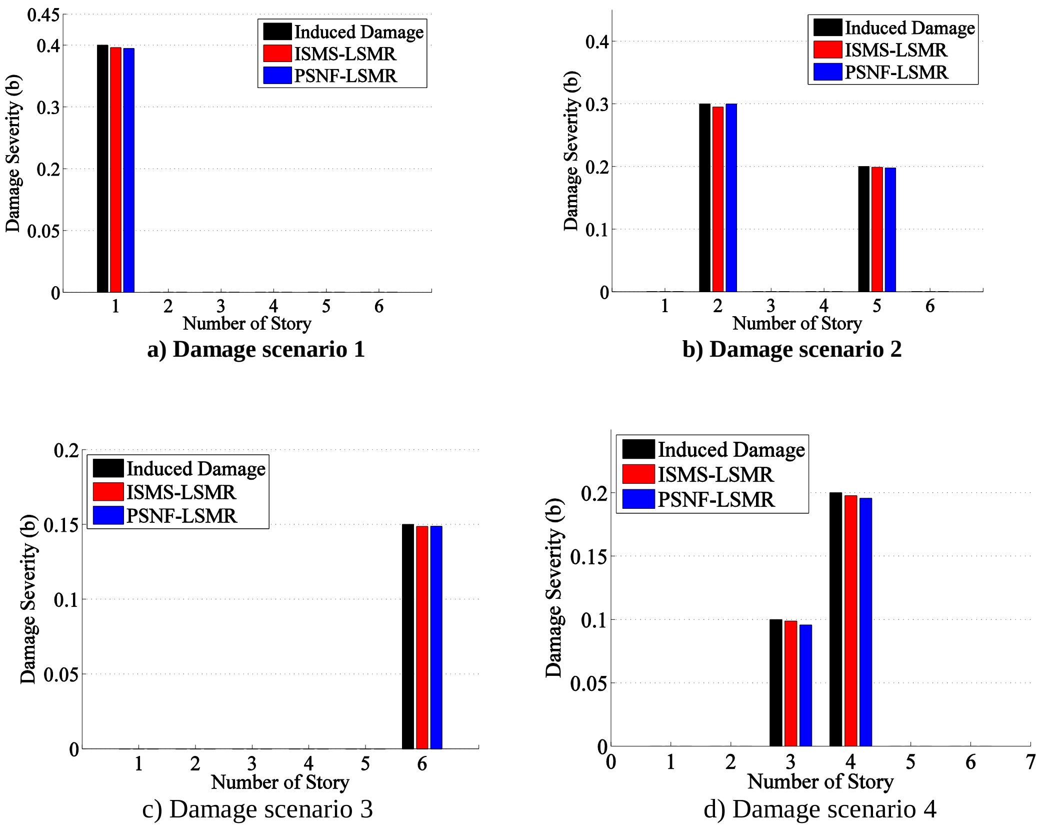
Fig. 1. a) Full-scale shear building frame, b) Simulated shear building frame

To evaluate the effectiveness of proposed structural damage diagnosis methods, several damage scenarios are considered by imposing reduction factors on the stiffness matrix of the shear building. Specifications of these scenarios are summarized in Table 2.

Table 2. Damages in the six-story shear building

Case Number	Story no.	Damage
		Stiffness reduction (%)
1	1	-40%
2	2	-30%
	5	-20%
3	6	-15%
4	3	-10%
	4	-20%

The induced damages change the structural properties of the shear building and therefore dynamic responses of structure will be altered. These alterations may be indicated by adverse changes of mode shapes or decrease of natural frequencies. However, these changes do not yield precise information for identifying the damage location or detecting its extent. Thus, detection of multiple damage cases is usually carried out by the sensitivity analysis derived from the analytical model of the structure. In this paper, the sensitivity matrices are composed of the first-order derivative of the modal parameters with respect to each damage variable. As a result, the iterative least square minimal residual method is used to detect the induced damages in the shear building. Initially, the quantities of induced damages are determined by LSMR when the modal data are not contaminated by noise. Next, the regularized least square minimal residual technique is used and the damage extents are calculated in an iterative manner when incomplete modal data are contaminated. Figures 2a-d, illustrate the damages predicted in all damage scenarios by LSMR.



a) Damage scenario 1 **b) Damage scenario 2**

c) Damage scenario 3 d) Damage scenario 4

Fig. 2. Damage quantification in the shear building by LSMR in all damage scenarios

As results show, the LSMR potentially enables detecting induced damages by the proposed sensitivity matrices despite incompleteness of modal data. It is obvious that the extent of imposed damages have been precisely determined and almost no computational error is observed. The main reasons for the high accuracy of this method for damage quantification lies in using iterative steps instead of direct methods and also unique property of LSMR. Therefore, the iterative LSMR method is a reliable tool for detecting the structural damage in the case of incomplete modal data. It should be noted that the LSMR has got a preferable capability in solving linear equations like $Sb=\Delta y$ for the complex structures with a large number of degrees of freedom. In other words, the LSMR can accurately solve the linear equation by the iterative algorithm even in the cases where S is a rectangular matrix with a large number of coefficients or the most measured data are incomplete. Furthermore, the number of iterative steps depends on the precision and dimensions of sensitivity matrices. Provided that the accuracy of the sensitivity matrices is higher and their condition number is smaller, a fewer number of iterative steps is required.

Because the great importance of sensitivity functions in the damage quantification process, the accuracy of these functions should be verified by investigating the relative error values. In this regard, four sensitivity equations including the proposed sensitivity of natural frequency (PSNF) by Eq. (25), the classical sensitivity of natural frequency (CSNF) presented in Eq. (1), improved sensitivity of mode shape (ISMS) by Eq. (36) and sensitivity of mode shape obtained by the modal method (SMSM) introduced through Eq. (26) are used. Relative error is evaluated through computing the ratio of amounts of induced and computed damages in all damage scenarios when the first three and the whole of measured modes are present, respectively. Accordingly, LSMR is used as the reference method and different damage extents are computed by mentioned sensitivity functions. Figures 3a-b, depict the relative error for the first three modes and all modes that is the ratio of amounts of induced and predicted damages in the shear building.

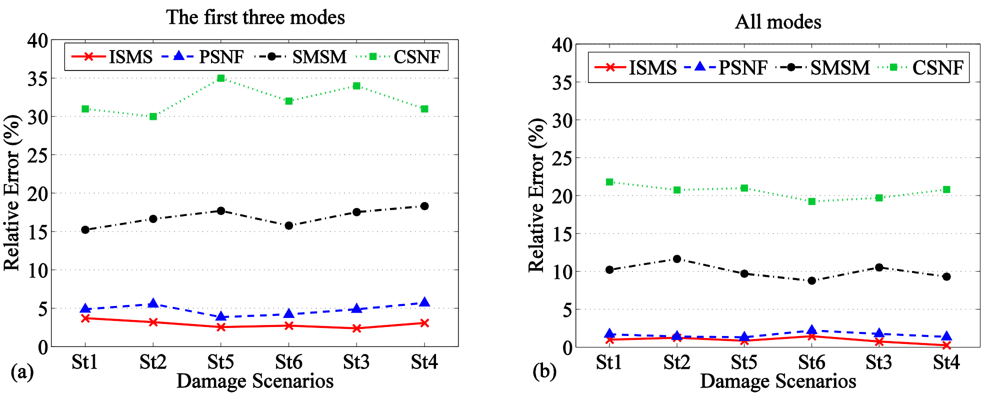
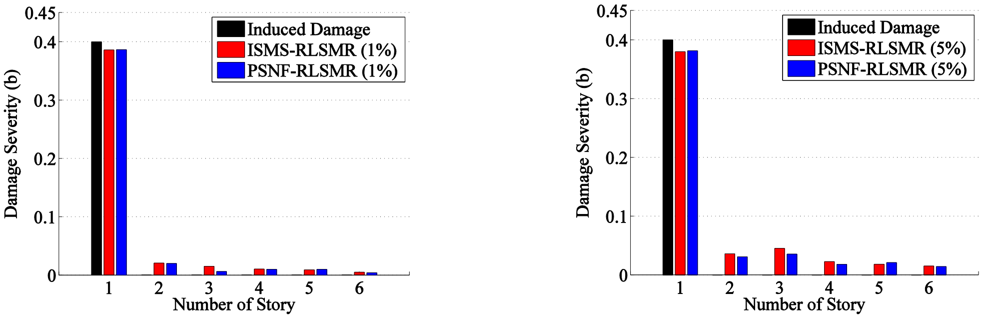


Fig. 3. The values of relative errors between sensitivity functions in the shear building, a) the first three modes, b) all modes

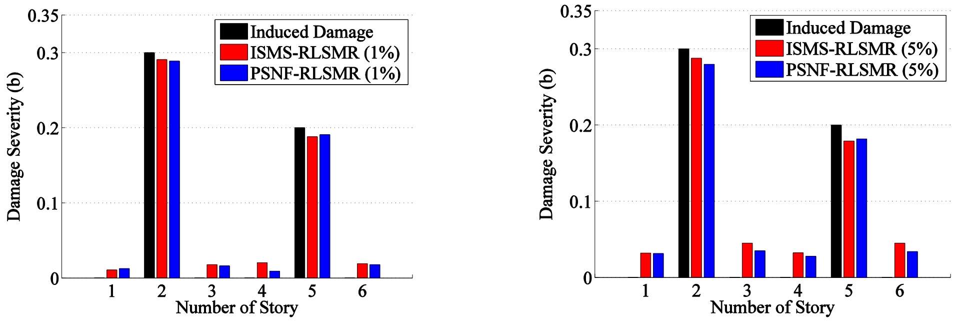
As can be seen from Figs. 3a-b, the values of relative errors of PSNF and ISMS are negligible and approximately close to zero when all modes are present. Thus, it can be concluded that these sensitivity equations have been precisely formulated. In contrast, equations of CSNF and SMSM lead to considerable amounts of error in spite of well-organized LSMR being utilized. Thus, these sensitivity functions are not appropriate criteria for damage detection process particularly when incomplete measured modes are present. It should be kept in mind that CSNF is a simple sensitivity method of poor performance particularly in the complex structures under low damage patterns conditions. Furthermore, the relative errors show that SMSM may deviate from exact results in complex structures and thus requires some modifications.

After determining the damage extents by noise-free data, the measured modal data are polluted by some random noises including 1% and 5%, respectively. Subsequently, the optimal regularization parameter γ is calculated by generalized cross validation (GCV) in each damage scenario. Eventually, the damage extents b_k will be determined by RLSMR in several iterations. The results of damage detection process in the shear building in the case of noisy data are presented in Figs. 4-7.



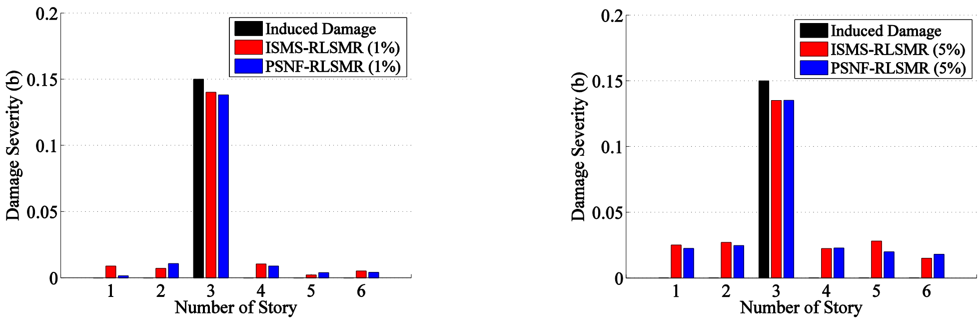
a) Damage scenario 1, shear building b) Damage scenario 1, shear building

Fig. 4. Damage quantification in the shear building by RLSMR, damage scenario 1, a) 1% noisy data, b) 5% noisy data



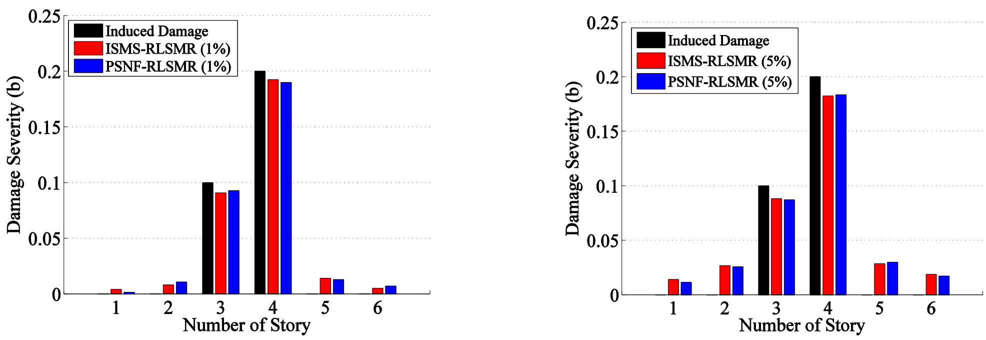
a) Damage scenario 2, shear building b) Damage scenario 2, shear building

Fig. 5. Damage quantification in the shear building by RLSMR, damage scenario 2, a) 1% noisy data, b) 5% noisy data



a) Damage scenario 3, shear building b) Damage scenario 3, shear building

Fig. 6. Damage quantification in the shear building by RLSMR, damage scenario 3, a) 1% noisy data, b) 5% noisy data



a) Damage scenario 4, shear building b) Damage scenario 4, shear building

Fig. 7. Damage quantification in the shear building by RLSMR, damage scenario 4, a) 1% noisy data, b) 5% noisy data

As shown in Figs. 4-7, the extent of induced damages is calculated by RLSMR with a good precision even when noise pollutes the modal data. In other words, obtained results show that in cases of multiple damages and despite incompleteness of measured modes, amount of computational error is negligible and can be neglected. In fact, these results obviously indicate the high capability of RLSMR in the case of ill-conditioned problems. It is clear that the proposed sensitivity functions perform well in the case of noise-free data and therefore, combining these functions with optimal regularization parameter in matrix $\bar{S}$, improves the results of damage quantification. Hence, it is obvious that the LSMR is not directly able to attain these accurate results and the optimal regularization parameter should be precisely determined. The advantage and applicability of GCV has been verified by many researchers [36, 37]. Consequently, the proposed method of RLSMR can be introduced as an influential and precise algorithm instead of using other direct regularization methods such as Tikhonov regularization method and singular value decomposition. Moreover, there are damage extents predicted in undamaged members of the shear building. This may be caused by complexity in the calculation of sensitivity matrices, existence of noise in data and also modal data being incomplete. However, as will be shown later, this error is insignificant and does not have any important effect on damage quantification algorithm.

b) A planner truss

In order to investigate the damage detection algorithms further, a two-dimensional truss is considered as shown in Fig. 8. Basic characteristics of the structure include Young modulus E=200 GPa and density ρ=7850 kg/m^3. All members of the truss are modeled as L-shaped double equal angles of 100 mm width and 5 mm thickness. Each node of the truss has two degrees of freedom (DOF). In this example, the first five vibration modes of the structure are used to simulate incompleteness condition for modal parameters. Two types of random noises including 1% and 5% are imposed on the extracted mode shapes and natural frequencies, respectively.

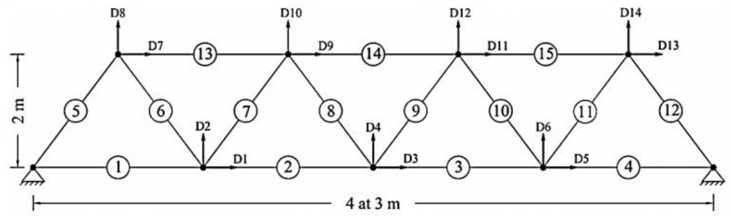


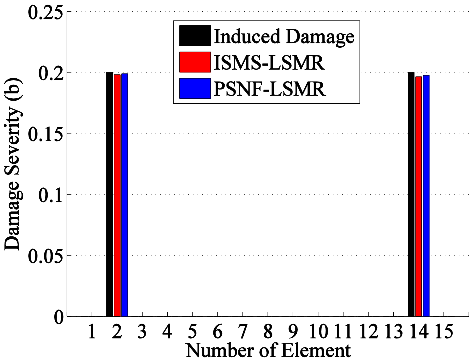
Fig. 8. The planner truss considered as continuous dynamic system

This 2-D truss is a continuous dynamic system and its mass and stiffness matrices can be determined by basic concepts of finite element method [38]. After calculating the physical properties of undamaged truss, generalized eigenvalue problem is used to identify modal data. It is assumed that proportional damping dominates structure’s behavior and thus the modal parameters are extracted as real data. Subsequently, four damage cases are considered to investigate effectiveness of proposed methods for damage detection. The damage scenarios and their corresponding inflicted damage are summarized in Table 3.

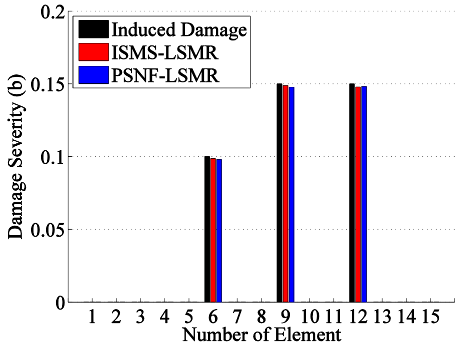
Table 3. Damages in the planner truss

Case number	Element no.	Damage
		Stiffness reduction (%)
1	2	-20%
	14	-20%
2	6	-10%
	9	-15%
	12	-15%
3	1	-30%
	8	-30%
	15	-25%
4	7	-20%
	13	-40%

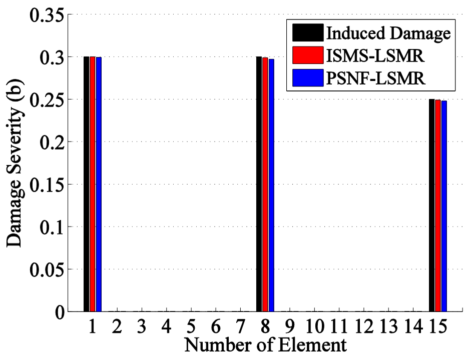
These induced damages are imposed on the truss's stiffness matrix and modal data of damaged structures are identified by generalized eigenvalue problem, one time more. In the previous section, the accuracy of the proposed sensitivity functions was verified by LSMR when only a limited number of modes are available. This result is definitely valid for the continuous systems such as planner truss. Indeed, the merit of sensitivity functions lies in the superiority, correctness and applicability of sensitivity formulations. Thus, type of dynamic system (discrete or continuous) and its properties does not effect on the accuracy of these functions. After determining the sensitivity matrices, the extent of induced damages are computed by LSMR in the several iterations. Results of structural damage diagnosis by LSMR in all damage scenarios are shown in Figs. 9a-d.



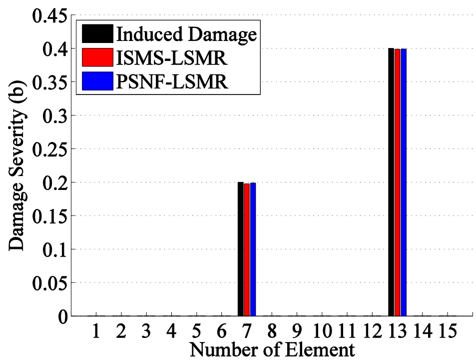
a) Damage scenario 1, truss



b) Damage scenario 2, truss



c) Damage scenario 3, truss



d) Damage scenario 4, truss

Fig. 9. Damage quantification in the planner truss by LSMR in all damage scenarios

It is clear from the results that the damage extents are accurately determined by LSMR. Particularly in the cases of low damage (less than 15%), the results of damage quantification by PSNF have been precisely estimated. It should be noted that damage detection in the planner truss has been performed as local damages. As mentioned before, one of the disadvantages of natural frequency lies in its disability to detect the local damage. Hence, the above figures clearly indicate to the accuracy of damage quantification results. It can be concluded that the PSNF can entirely deal with the disability of natural frequency for determining local damages and also damages of low extent. Moreover, ISMS method provides proper results in both cases of low and high damage extents. In addition, the computational errors present in the results of PSNF and ISMS are almost the same and so it can be concluded that LSMR and the proposed sensitivity functions are powerful methods for structural damage diagnosis. In order to allow a more ready comparison of the results obtained by the sensitivity functions, Figs. 10a-b show the relative error (difference between the predicted damage extents and induced damages) when the first five and the whole of measured modes are present, respectively.

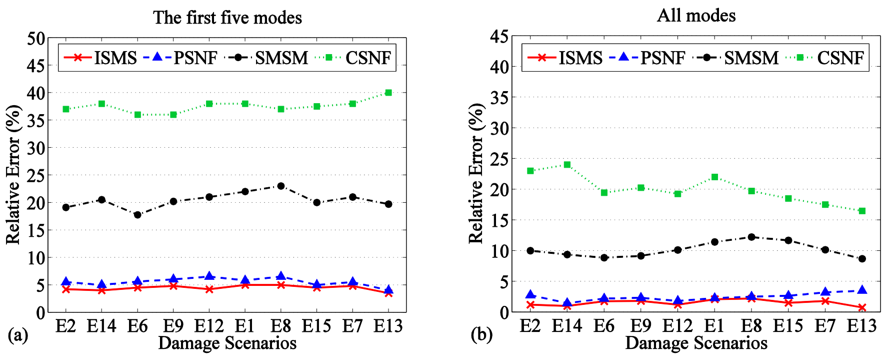
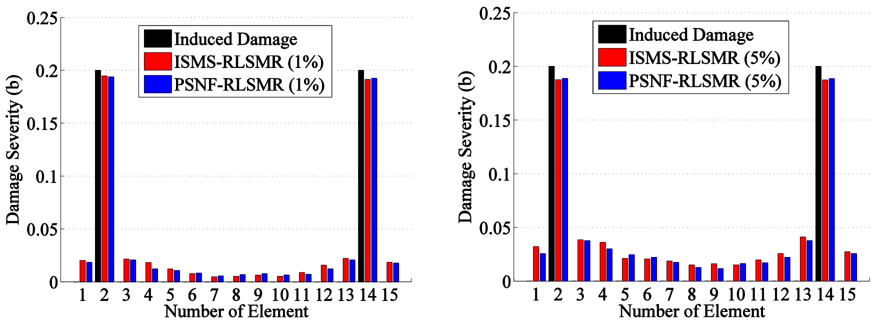


Fig. 10. The values of relative errors between sensitivity functions in the planner truss

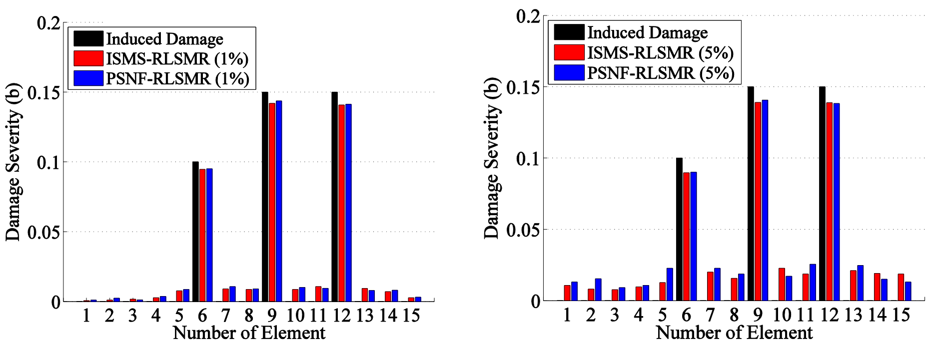
By comparing the results obtained using sensitivity functions in Figs. 10a-b, it can be observed that the ISMS and PSNF have better results than other sensitivity functions in any condition of measured modes. As shown in Fig 10-a, in the case of incompleteness of measured modes, the amounts of relative errors in CSNF are unreliable and also SMSM have considerable errors. On the other hand, when all of the vibration modes are measured the SMSM performs better in comparison with the case of incomplete measured modes. In such circumstance, increasing the number of the measured modes leads to decrease in the amount of relative error in SMSM whereas both PSNF and ISMS have negligible error. Moreover, it is clear from Fig. 10-b, that PSNF and ISMS are again more accurate than SMSM and CSNF when all of the modes are measured.

To detect the induced damage extents in the case of contaminated data, 1% and 5% random noises are imposed on the identified modal data in the next step. Once again, the generalized cross validation is performed by determining the regularization parameter at all scenarios. Figs 11-14, illustrate the structural damage extents in the case of incomplete noisy modal data.



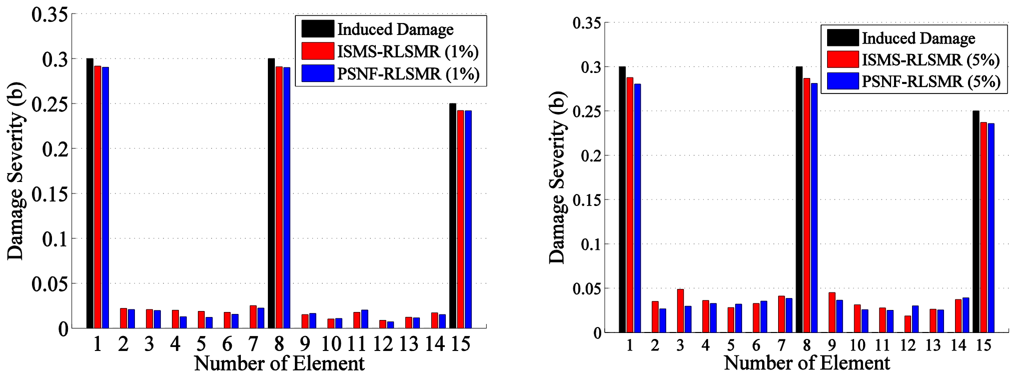
a) Damage scenario 1, truss b) Damage scenario 1, truss

Fig. 11. Damage quantification in the planner truss by RLSMR, damage scenario 1,
a) 1% noisy data, b) 5% noisy data



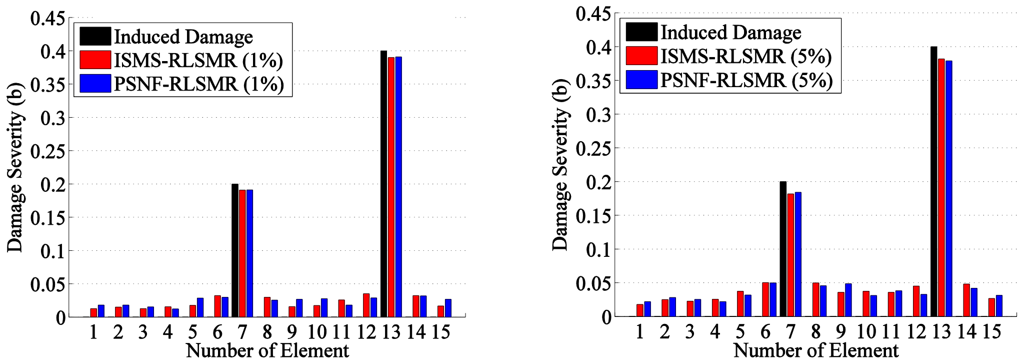
a) Damage scenario 2, truss b) Damage scenario 2, truss

Fig. 12. Damage quantification in the planner truss by RLSMR, damage scenario 2,
a) 1% noisy data, b) 5% noisy data



a) Damage scenario 3, truss b) Damage scenario 3, truss

Fig. 13. Damage quantification in the planner truss by RLSMR, damage scenario 3, a) 1% noisy data, b) 5% noisy data



a) Damage scenario 4, truss b) Damage scenario 4, truss

Fig. 14. Damage quantification in the planner truss by RLSMR, damage scenario 4, a) 1% noisy data, b) 5% noisy data

According to these figures, it can be noted that although incomplete noisy modal data have been used for damage detection, the RLSMR is considerably able to yield accurate results and decreases the effect of noises. At the low damage patterns, RLSMR and PSNF are appropriately capable to detect the extent of damage despite the presence of noisy data. Furthermore, the results obtained using RLSMR and ISMS indicate that these methods are potentially able to determine the damage extents in all of the damage scenarios. Thus, it can be deduced that the extent of damaged elements are estimated with acceptable accuracy. Moreover, an additional slight damage is detected in the undamaged elements due to the presence of noise in the modal data.

4. CONCLUSION

This paper presents new sensitivity-based methods for detecting structural damage using incomplete noisy modal data. Initially, a new sensitivity function of natural frequency (PSNF) is proposed applying Lagrange multipliers. Then, an improved sensitivity of mode shape (ISMS) is presented by developing the modal method and adding the flexibility matrix to sensitivity function. Two powerful mathematical methods were utilized to determine the damage quantity. At the first step, a new iterative method called Least-Square Minimal Residual (LSMR) method is presented to calculate the damage severities assuming that incomplete measured modes are present. Next, Regularized Least-Square Minimal Residual (RLSMR) is used to reduce the effect of perturbations on damage detection process. The RLSMR develops LSMR method by adding the regularization parameter to damage equation. Two numerical examples including a six-story shear building and a planner truss are presented for illustrating the capability and accuracy of proposed methods. The obtained relative errors, which is the ratio of amounts of the estimated and induced damages at each damage scenario, show that the proposed sensitivity equations have negligible values of error. In this regard, these insignificant amounts confirm that ISMS method is more accurate than the modal method and remains valid while the modal method fails. Furthermore, the precision of PSNF pertains to Lagrange multipliers and dynamic properties of structure. Thus, this approach can confidently be applied in LSMR and RLSMR for detecting the local damages with a high accuracy even when incomplete noisy modal data are available. In the condition of solving damage equation, when the modal parameters are incomplete, LSMR is potentially able to determine the damage severities in a number of iterations. Under such conditions, the sensitivity functions are rectangular matrices of singular values of almost zero. As a result, this process leads to unreliable results in the damage detection particularly when using direct methods. The iterative LSMR yields accurate results with negligible amounts of error in the damage quantification process when the incomplete modal data are not contaminated by noise. Furthermore, by comparing the computational error between determined and induced damages, it can be concluded that RLSMR decreases the effect of noise on structural damage detection. It can also be maintained that RLSMR is a more comprehensive method in comparison with LSMR. However, this technique depends directly on the regularization parameter so this parameter should be accurately determined.

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