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A DENSE SUBGROUP OF TOPOLOGICAL FUNDAMENTAL GROUP OF QUOTIENT SPACES

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ABSTRACT. In this talk, we show that the image of the topological fundamental group of a given space X is dense in the topological fundamental group of the quotient space X/A under the induced homomorphism of the quotient map, where A is a suitable subspace of X with some conditions on X. Also, we give some applications to find out some properties for $\pi_1^{top}(X/A,*)$. In particular, we give some conditions under which $\pi_1^{top}(X/A,*)$ is an indiscrete topological group.

1. INTRODUCTION

Let X be a topological space and $\sim$ be an equivalent relation on X. Then one can consider the quotient topological space $X/\sim$ and the quotient map $p : X \longrightarrow X/\sim$. By applying the fundamental group functor on p there exists the induced homomorphism

$$p_* : \pi_1(X) \longrightarrow \pi_1(X/\sim).$$

It seems interesting to determine the image of the above homomorphism. Recently, Calcut, Gompf, and Mccarthy [2] proved that if X is a locally path connected topological space partitioned into connected subsets and if the associated quotient space $X/\sim$ is semi-locally simply connected, then the induced homomorphism, p_*, of fundamental groups is surjective for each choice of base point $x \in X$. They inspired the problem by one of Arnold's problems on orbit spaces of vector fields on manifolds [2, Section 3.6].

D. Biss [1] introduced the topological fundamental group $\pi_1^{top}(X,x)$ of a based space (X,x) as the fundamental group $\pi_1(X,x)$ with the quotient topology of the loop space $\Omega(X,x)$ with respect to the canonical function $\pi : \Omega(X,x) \longrightarrow \pi_1(X,x)$ identifying path components. It is known that this construction gives rise a homotopy invariant functor $\pi_1^{top} : htop_* \longrightarrow QTG$ from the homotopy category of based spaces to the category of quasitopological groups and continuous homomorphism [3]. By applying the above functor on the quotient map $p : X \longrightarrow X/\sim$, we have a continuous homomorphism $p_* : \pi_1^{top}(X) \longrightarrow \pi_1^{top}(X/\sim)$. For a pointed topological space (X,x), by a path we mean a continuous map $\alpha : [0,1] \longrightarrow X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α, respectively. A loop α is a path with $\alpha(0) = \alpha(1)$. For a path $\alpha : [0,1] \longrightarrow X$, α^{-1} denotes a path such that $\alpha^{-1}(t) = \alpha(1-t)$, for all $t \in [0,1]$. Denote $[0,1]$ by I, two paths $\alpha, \beta : I \longrightarrow X$ with the same initial and terminal points are called homotopic relative to end

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points if there exists a continuous map $F: I\times I \longrightarrow X$ such that

$$F(t,s)=\begin{cases}\alpha(t) & s=0\\ \beta(t) & s=1\\ \alpha(0)=\beta(0) & t=0\\ \alpha(1)=\beta(1) & t=1.\end{cases}$$

The homotopy class containing a path α is denoted by $[\alpha]$. Since most of the homotopies that appear in this paper have this property and end points are the same, we drop the term "relative homotopy" for simplicity. For paths $\alpha,\beta: I \longrightarrow X$ with $\alpha(1)=\beta(0)$, $\alpha*\beta$ denotes the concatenation of α and β, that is, a path from I to X such that $(\alpha*\beta)(t)=\alpha(2t)$, for $0\le t\le 1/2$ and $(\alpha*\beta)(t)=\beta(2t-1)$, for $1/2\le t\le 1$.

For a space (X,x), let $\Omega(X,x)$ be the space of based maps from I to X with the compact-open topology. A subbase for this topology consists of neighborhoods of the form $\langle K,U\rangle=\{\gamma\in\Omega(X,x)\mid \gamma(K)\subseteq U\}$, where $K\subseteq I$ is compact and U is open in X. When X is path connected and the basepoint is clear, we just write $\Omega(X)$ and we will consistently denote the constant path at x by e_x. The topological fundamental group of a pointed space (X,x) may be described as the usual fundamental group $\pi_1(X,x)$ with the quotient topology with respect to the canonical map $\Omega(X,x)\longrightarrow \pi_1(X,x)$ identifying homotopy classes of loops, denoted by $\pi_1^{top}(X,x)$. A basic account of topological fundamental groups may be found in [1], [3].

2. Main Results

Theorem 2.1. *Let A be an open subset of X such that $\overline{A}$ is path connected, then for each $a\in A$ the image of p_* is dense in $\pi_1^{top}(X/A,*)$ i.e:*

$$\overline{p_*\pi_1^{top}(X,a)}=\pi_1^{top}(X/A,*).$$

Definition 2.2. *Let X be a topological space and $A_1,A_2,\ldots,A_n$ be any subsets of X, $n\in\mathbb{N}$. By the quotient space $X/(A_1,\ldots,A_n)$ we mean the quotient space obtained from X by identifying each of the sets A_i to a point. Also, we denote the associated quotient map by $p: X\longrightarrow X/(A_1,A_2,\ldots,A_n)$.*

Corollary 2.3. *Let $A_1,A_2,\ldots,A_n$ be open subsets of a path connected space X such that the A_i's are path connected for every $i=1,2,\ldots,n$. Then for every $a\in\bigcup_{i=1}^n A_i$,*

$$p_*\pi_1^{top}(X,a)=\pi_1^{top}(X/(A_1,A_2,\ldots,A_n),*).$$

Corollary 2.4. *Let $A_1,A_2,\ldots,A_n$ be open subsets of a connected, locally path connected space X such that the A_i's are path connected for every $i=1,2,\ldots,n$. If $X/(A_1,A_2,\ldots,A_n)$ is semi-locally simply connected, then for each $a\in\bigcup_{i=1}^n A_i$,*

$$p_*:\pi_1^{top}(X,a)\longrightarrow \pi_1^{top}(X/(A_1,A_2,\ldots,A_n,*)$$

is an epimorphism.

In the following example we show that with assumption of previous Theorem p_* is not necessarily onto.

Examples 2.5. *Let* $A_n = \{1/(2n-1), 1/2n\} \times [0, 1+1/2n] \bigcup [1/2n, 1/2n-1] \times \{1+1/2n\}$ *for each* $n \in \mathbb{N}$. *Consider* $X = (\bigcup_{n\in\mathbb{N}} A_n) \bigcup \{0\} \times [0,1] \bigcup [0,1] \times \{0\}$ *with* $a = (0,0)$ *as the base point and* $A = \{(x,y) \in X \mid y < 1\}$. *A is an open subset of X with path connected closure. Let* $I_n = (1/2 + 1/2(n+1), 1/2 + 1/2n]$ *and* f_n *be a homeomorphism from* I_n *to* $A_n - \{(1/2n, 0)\}$ *for every* $n \in \mathbb{N}$. *Define* $f : I \longrightarrow X$ *by*

$$f(t) = \begin{cases} \text{the point } (0,2t) & t \in [0,1/2], \\ f_n(t) & t \in I_n. \end{cases}$$

We claim that $\alpha = p \circ f$ *is a loop in* X/A *at* $*$. *It suffices to show that* α *is continuous on* $t = 1/2$ *and boundary points of* I_n*'s since f is continuous on* $[0,1/2)$ *and by gluing lemma on* $\bigcup int(I_n)$. *Since* α *is locally constant at* $t = 1/2 + 1/2n$ *for each* $n \in \mathbb{N}$, α *is continuous at boundary points of* I_n. *For each open neighborhood G of* $f(1/2) = (0,1)$ *in X, there exists* $n_0 \in \mathbb{N}$ *such that G contains* $A_n \bigcap A^c$ *for* $n > n_0$. *Therefore continuity at* $t = 1/2$ *follows from* $\alpha(1/2) \in \overline{\{*\}}$. *Now let* $B \subseteq \mathbb{N}$ *and define*

$$g_B(t) = \begin{cases} (p \circ f)(t) & t \in \bigcup_{m\in B} I_m, \\ * & otherwise. \end{cases}$$

Then g_B *is continuous and for* $B_1, B_2 \subseteq \mathbb{N}$ *such that* $B_1 \neq B_2$, $[g_{B_1}] \neq [g_{B_2}]$ *which implies that* $\pi_1(X/A, *)$ *is uncountable. But by compactness of I, a given path in X can traverse finitely many of the* A_n*'s and therefore* $\pi_1(X,a)$ *is a free group on countably many generators which implies that p does not induce a surjection of fundamental groups.*

3. Some Applications

After that Biss [1] equipped the fundamental group of a topological space and named it topological fundamental group, Fabel [5] showed that topological fundamental groups can distinguish spaces with isomorphic fundamental groups. Hence studying the topology of fundamental groups seems important.
By $(X, A_1, A_2, ..., A_n)$ we mean an $(n+1)$-*tuple* of spaces where the A_i's are open subsets of X with path connected closure.

Theorem 3.1. *For an* $(n+1)$-*tuple of spaces* $(X, A_1, A_2, ..., A_n)$, *if X is simply connected, then* $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ *is an indiscrete topological group.*

Theorem 3.2. *For an* $(n+1)$-*tuple of spaces* $(X, A_1, A_2, ..., A_n)$, *if* $\pi_1^{top}(X,a)$ *is compact and* $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ *is Hausdorff, then* p_* *is an epimorphism and so is a closed quotient map.*

Theorem 3.3. *For an* $(n+1)$-*tuple of spaces* $(X, A_1, A_2, ..., A_n)$, *if* $\pi_1^{top}(X,a)$ *is compact and* $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ *is Hausdorff, then either* $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ *is a discrete topological group or uncountable.*

Corollary 3.4. *For an* $(n+1)$-*tuple of spaces* $(X, A_1, A_2, ..., A_n)$, *if* $\pi_1^{top}(X,a)$ *is a compact, countable quasitopological group, then either* $X/(A_1, A_2, ..., A_n)$ *is semi-locally simply connected or* $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ *is not Hausdorff.*

Corollary 3.5. *For an* $(n+1)$*-tuple of spaces* $(X,A_1,A_2,...,A_n)$*, if* $\pi_1^{top}(X,a)$ *is finite and* $\pi_1^{top}(X/(A_1,A_2,\ldots,A_n\ ),*)$ *is Hausdorff, then* $\pi_1^{top}(X/(A_1,A_2,\ldots,A_n),*)$ *is finite and* $X/(A_1,A_2,\ldots,A_n)$ *is semi-locally simply connected.*

Definition 3.6. (cf. [4]) *A topological space is said to be* **irreducible** *if it cannot be expressed as a union of two proper closed subsets. Equivalently, it is irreducible if each of its nonempty open subsets is dense. A space is called* **semi-irreducible** *if every disjoint family of its nonempty open subsets is finite.*

A sober space is a topological space X such that every irreducible closed subset of X is the closure of exactly one point of X ([4]). For example every Hausdorff space is sober space, since the only irreducible subsets are points. In [4] it is proved that a sober space is semi-irreducible if and only if it has a finite dense subset.

Corollary 3.7. *For an* $(n+1)$*-tuple of spaces* $(X,A_1,A_2,...,A_n)$*, if* $\pi_1^{top}(X,a)$ *is finite and* $\pi_1^{top}(X/(A_1,A_2,...,A_n\ ),*)$ *is a sober topological space, then* $\pi_1^{top}(X/(A_1,A_2,...,A_n\ ),*)$ *is semi-irreducible.*

Proof. Note that since $\pi_1^{top}(X/(A_1,A_2,...,A_n),*)$ has a finite dense subset, the result holds. □

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