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Separation properties of topological fundamental groups

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Abstract

In this talk, we discuss on the topological properties of the Spanier groups which are new subgroups of fundamental group, constructed by an inverse limit method. Also, by using these results, we establish an algebraic criteria for the Hausdorffness of topological fundamental groups.

1 Introduction

In 2002, a work of Biss initiated the development of a theory in which the familiar fundamental group $\pi_1(X,x)$ of a topological space X becomes a topological space denoted by $\pi_1^{top}(X,x)$ by endowing it with the quotient topology inherited from the path components of based loops in X with the compact-open topology. Among other things, Biss claimed that $\pi_1^{top}(X,x)$ is a topological group. However, there is a gap in his proof. Brazas discovered some interesting counterexamples for continuity of multiplication in $\pi_1^{top}(X,x)$ (for more details, see [1]).

In fact, $\pi_1^{top}(X,x)$ was a quasitopological group, that is, a group with a topology such that inversion and all translations are continuous. After this obstacle, Brazas [1] by removing some open sets of $\pi_1^{top}(X,x)$, make it a topological group and denote

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it by $\pi_1^\tau(X,x)$. Indeed, the functor π_1^τ removes the smallest number of open sets from the topology of $\pi_1^{top}(X,x)$ so as to make it a topological group. Here, by the topological fundamental group we mean $\pi_1^\tau(X,x)$.

In the sequel, we study the topology of topological fundamental group from some separation axioms viewpoint. The main idea is working with the Spanier groups with respect to open covers of a given space X which have been introduced in [2] and named in [2]. The importance of these groups and their intersection which is named Spanier group, $\pi_1^{sp}(X,x)$, is studied by H. Fischer et al. in [2] in order to modification of the definition of semi-locally simply connectedness.

Throughout this article, all the homotopies between two paths are relative to end points, X is a connected and locally path connected space with the base point $x \in X$, and $p : \widetilde{X} \longrightarrow X$ is a covering of X with $\tilde{x} \in p^{-1}(\{x\})$ as the base point of $\widetilde{X}$. For a space X and any $H \leq \pi_1(X,x)$, by $\widetilde{X}_H$ we mean a covering space of X such that $p_*\pi_1(\widetilde{X}, tildex) = H$, where $\tilde{x} \in p^{-1}(x)$ and $p : \widetilde{X}_H \longrightarrow X$ is the corresponding covering map.

2 Definitions and preliminaries

E.H. Spanier [2, §2.5] classified path connected covering spaces of a space X using some subgroups of the fundamental group of X, recently named Spanier groups (see [3]). If $\mathcal{U}$ is an open cover of X, then the subgroup of $\pi_1(X,x)$ consisting of all homotopy classes of loops that can be represented by a product of the following type

$$\prod_{j=1}^{n} \alpha_j * \beta_j * \alpha_j^{-1},$$

where the α_j's are arbitrary paths starting at the base point x and each β_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$, is called the *Spanier group with respect to* $\mathcal{U}$, and denoted by $\pi(\mathcal{U},x)$ [2]. For two open covers $\mathcal{U},\mathcal{V}$ of X, we say that $\mathcal{V}$ refines $\mathcal{U}$ if for every $V \in \mathbb{V}$, there exists $U \in \mathbb{U}$ such that $V \subseteq U$.

Definition 2.1. [2] The Spanier group of a topological space X, denoted by $\pi_1^{sp}(X,x)$ for an $x \in X$ is

$$\pi_1^{sp}(X,x) = \bigcap_{open\ covers\ \mathcal{U}} \pi(\mathcal{U},x).$$

Also, we can obtain the Spanier groups as follows: Let $\mathcal{U},\mathcal{V}$ be open coverings of X, and let $\mathcal{U}$ be a refinement of $\mathcal{V}$. Then since $\pi(\mathcal{U},x) \subseteq \pi(\mathcal{V},x)$, there exists an inverse limit of these Spanier groups, defined via the directed system of all open covers with respect to refinement and it is $\pi_1^{sp}(X,x)$.

Definition 2.2. [3] We call a topological space X the Spanier space if $\pi_1(X, x) = \pi_1^{sp}(X, x)$, for an arbitrary point $x \in X$.

Proposition 2.3. [3] If $p : \widetilde{X} \longrightarrow X$ is a covering and $x \in X$, then $\pi_1^{sp}(X, x) \leq p_*\pi_1(\widetilde{X}, tildex)$.

Theorem 2.4. ([2, §2.5 Theorems 12,13]) Let X be a connected, locally path connected space and $H \leq \pi_1(X, x)$, for $x \in X$. Then there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X}, tildex) = H$ if and only if there exists an open cover $\mathcal{U}$ of X in which $\pi(\mathcal{U}, x) \leq H$.

We immediately deduce that for every open cover $\mathcal{U}$ of X, $\widetilde{X}_{\pi(\mathcal{U},x)}$ exists.

3 Main results

By [5, Theorem 3.7], the connected coverings of a connected and locally path connected space X are classified by conjugacy classes of subgroups of $\pi_1^{qtop}(X, x)$ with open core, and since $\pi_1^{\tau}(X, x)$ and $\pi_1^{qtop}(X, x)$ have the same open subgroups [1, Corollary 3.9], we have the same result for open subgroups of $\pi_1^{\tau}(X, x)$. Using this fact and Theorem 1.2, for every open cover $\mathcal{U}$ of X, $\pi(\mathcal{U}, x)$ is an open subgroup of $\pi_1^{\tau}(X, x)$ and since $\pi_1^{\tau}(X, x)$ is a topological group, $\pi(\mathcal{U}, x)$ is a closed subgroup, which implies that $\pi_1^{sp}(X, x)$ is a closed subgroup of $\pi_1^{\tau}(X, x)$. Hence we have the following proposition.

Proposition 3.1. For a connected and locally path connected space X, $\pi_1^{sp}(X, x)$ is a closed subgroup of $\pi_1^{\tau}(X, x)$, for every $x \in X$.

Using the above proposition, the Spanier group of a connected and locally path connected space contains the closure of the trivial element of the topological fundamental group. Hence we have the following corollary.

Corollary 3.2. Let X be a connected and locally path connected space and $x \in X$. If $\pi_1^{\tau}(X, x)$ has an indiscrete topology, then X is a Spanier space.

Corollary 3.3. Let X be a connected and locally path connected space and $x \in X$. If $\pi_1^{sp}(X, x)$, then $\pi_1^{\tau}(X, x)$ has the T_1 topology.

We recall that a space X is called shape injective if the natural homomorphism $\varphi : \pi_1(X, x) \longrightarrow \check{\pi}_1(X, x)$ is injective, where $\check{\pi}_1(X, x)$ is the first shape group of (X, x). Also, an open cover $\mathcal{U}$ of X is called normal if it admits a partition of unity subordinated to $\mathcal{U}$. Also, every open cover of a paracompact space is normal. (see [3] for further details). In [1, Proposition 3.25] it is proved that the topological fundamental groups of shape injective spaces are Hausdorff. Since the spaces with a trivial

Spanier group are not necessarily shape injective, the triviality of Spanier groups can not certify the Hausdorffness of topological fundamental groups in general. In the following, we provide the conditions that guarantee this.

Definition 3.4. [3] A space X is small loop homotopically Hausdorff if for each $x \in X$ and each loop α based at x, if for each normal open cover $\mathcal{U}$ of X, $[\alpha] \in \pi(\mathcal{U}, x)$, then $[\alpha] = 1$.

Proposition 3.5. Suppose X is a connected, locally path connected and paracompact space. Then $\pi_1^{sp}(X, x)$ if and only if X is shape injective.

Corollary 3.6 (A criterion for the Hausdorffness of $\pi_1^{\tau}(X, x)$)**.** Let X be a connected, locally path connected and paracompact space. If $\pi_1^{sp}(X, x)$, then $\pi_1^{\tau}(X, x)$ is Hausdorff.

Note that since $\pi_1^{\tau}(X, x)$ is a topological group, by the assumption of the above corollary $\pi_1^{\tau}(X, x)$ is regular.

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