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ON THE REGULAR COVERING OF A GRAPH

HAMID TORABI*

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad,
P.O.Box 1159-91775, Mashhad, Iran.
email: hamid_torabi86@yahoo.com

Abstract. In this talk, we discuss on the covering and immersion of graphs. Based on some properties, we establish some results about relation between immersion and covering. In particular, we give some condition in which an immersion $f : X \longrightarrow Y$ is a regular covering.

1. Introduction

A graph X consists of two sets E and V (directed edges and vertices), with three functions $^{-1} : E \longrightarrow E$ and $s, t : E \longrightarrow V$ such that $(e^{-1})^{-1} = e, e^{-1} \neq e, s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. We say that the (directed) edge $e \in E$ has initial vertex $s(e)$ and terminal vertex $t(e)$. A path p in X of length $n = |p|$, with initial vertex u and terminal vertex v, is an n-tuple of edges of X of the form $p = e_1 ... e_n$ such that for $i = 1, ..., n-1$, we have $t(e_i) = s(e_{i+1})$ and $s(e_1) = u$ and $t(e_n) = v$. For $n = 0$, given any vertex v, there is a unique path c_v of length 0

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* Speaker.

whose initial and terminal vertices coincide and are equal to v. A graph is connected if any pair of vertices is joined by some path. A path p is called a closed path if its initial and terminal vertices coincide.
If p and q are paths in X and the terminal vertex of p equals the initial vertex of q, they may be concatenated to form a path pq, whose initial vertex is that of p and whose terminal vertex is that of q. A round-trip is a path of the form ee^{-1}. An elementary reduction is insertion or deletion a round-trip in a path. Two paths p and q are homotopic iff there is a finite sequence of elementary reductions taking one path to the other. Homotopy is an equivalence relation on the set of paths in X. Let v be a fix vertex in X, $\pi_1(X, v)$ is defined to be the set of all homotopy classes of closed paths with initial and terminal vertex v. Then $\pi_1(X, v)$ together with the product $[p][q] := [pq]$ forms a group with identity $[c_v]$.
For a fix vertex v in X, the star of v in X is defined as follows:

$$St(v, X) = \{e \in E : s(e) = v\}$$

A map of graphs $f : f : X \longrightarrow Y$ is a function which maps edges to edges and vertices to vertices. Also we have $f(e^{-1}) = f(e)^{-1}, f(s(e)) = s(f(e))$ and $f(t(e)) = t(f(e))$. A map $f : X \longrightarrow Y$ yields, for each vertex $v \in X$, a function $f_v : St(v, X) \longrightarrow St(f(v), Y)$. If for each vertex $v \in X$, f_v is injective, f is called an immersion. If each f_v is bijective, f is called a covering. The theory of coverings of graphs is almost completely analogous to the topological theory of coverings. Immersions have some of the properties of coverings. A basic account of coverings and immersion may be found in [1] and [2].

2. Main results

In this section we can establish our main results. All the graphs in this section are connected.

Lemma 2.1. *For a connected graph Y, every covering $f : X \longrightarrow Y$ is a surjective map.*

Proof. Since Y is connected by definition of map of graphs, it suffices to show that for a edge e in Y such that $s(e) \in Im(f)$, we have $e \in Im(f)$. For this let e be an edge in Y with $s(e) = f(q)$ for some vertex q of X. Since the function $f_q : St(q, X) \longrightarrow St(f(q), Y)$ is bijective and $e \in St(f(q), Y)$, hence there is an edge l in X such that $s(l) = q$ and $f(l) = e$. Therefore $e \in Im(f)$ which implies that f is surjective. □

Proposition 2.2. *An immersion $f : X \longrightarrow Y$ is a covering if and only if f has the path lifting property i.e. for every path $p = e_1...e_n$ in Y*

and $r \in f^{-1}(s(e_1))$ *there is a path* $q = l_1...l_n$ *in* Y *such that* $f(l_i) = e_i$ *and* $s(l_1) = r$.

Remark 2.3. Note that the path q in above Proposition is called a lifting of p.

Theorem 2.4. *An immersion* $f : X \longrightarrow Y$ *is an isomorphism of graphs if and only if* f *is surjective and for every closed path* p *in* Y *if* p *has a lifting, then* p *has a closed path lifting.*

Definition 2.5. Let $f : X \longrightarrow Y$ be a covering of graphs. Then we call f a regular covering if and only if $Im(\pi_1(f))$ is a normal subgroup of $\pi_1(Y, y)$.

Proposition 2.6. *A covering* $f : X \longrightarrow Y$ *is a regular covering if and only if for every path* p *with the initial vertex* s *in* Y *if* p *has a closed path lifting, then every lifting* q *of* p *with initial vertex* $v \in f^{-1}(s)$ *is a closed path.*

Theorem 2.7. *An immersion* $f : X \longrightarrow Y$ *is a regular covering if and only if* f *is surjective, for every path* p *with the initial vertex* s *in* Y *if* p *has a closed path lifting, then every lifting* q *of* p *with initial vertex* $v \in f^{-1}(s)$ *is a closed path and for every path* $p = e_1...e_n$ *in* Y *with* $s(e_2) = t(e_{n-1})$ *if* p *has a lifting, then the path* e_1e_n *has a lifting.*

References

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