On the Classification of Covering Spaces

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Abstract

For a connected, locally path connected space X, let H be a subgroup of the fundamental group of X, $\pi_1(X,x)$. We show that there exists an open cover $\mathscr{U}$ of X such that H contains the Spanier group $\pi(\mathscr{U},x)$ if and only if the core of H in $\pi_1(X,x)$ is open in the quasitopological fundamental group $\pi_1^{qtop}(X,x)$ or equivalently it is open in the topological fundamental group $\pi_1^{\tau}(X,x)$. As a consequence, using the relation between the Spanier groups and covering spaces, we give a classification for connected covering spaces of X based on the conjugacy classes of subgroups with open core in $\pi_1^{qtop}(X,x)$.

Keywords: Covering space, Spanier group, Quasitopological fundamental group, Topological fundamental group, Semilocally small generated.

1 Introduction and motivation

The motivation of this paper is the following interesting classical result of Spanier [8, §2.5 Theorems 12, 13] for the existence of covering spaces:

Theorem 1.1. *Let X be a connected and locally path connected space and $H \leq \pi_1(X,x)$, for $x \in X$. Then there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$ if and only if there exist an open cover $\mathscr{U}$ of X in which $\pi(\mathscr{U},x) \leq H$.*

Since for a locally path connected and semilocally simply connected space X there exists an open cover $\mathscr{U}$ such that $\pi(\mathscr{U},x) = 1$, for a point $x \in X$, the existence of simply connected universal covering follows from the above theorem.

We recall from [7] that the Spanier group $\pi(\mathscr{U},x)$ with respect to the open cover $\mathscr{U} = \{U_i | i \in I\}$ is defined to be the subgroup of $\pi_1(X,x)$ which contains all homotopy classes having representatives of the following type:

$$\prod_{j=1}^{n} u_j v_j u_j^{-1},$$

where u_j are arbitrary paths starting at x and each v_i is a loop inside one of the open sets $U_i \in \mathscr{U}$.

One of the main objects of this paper is to find conditions on a subgroup H of $\pi_1(X,x)$ under which there is an open cover $\mathscr{U}$ of X such that $\pi(\mathscr{U},x) \leq H$. In Section 2 without using Theorem 1.1, we show that every open normal subgroup H of the quasitopological fundamental group $\pi_1^{qtop}(X,x)$ satisfies the above property. Then using this fact we obtain a weaker condition on the subgroup H in order to satisfies the above property. In fact, we prove that for every subgroup H of $\pi_1(X,x)$ with open core in $\pi_1^{qtop}(X,x)$ there is an open cover $\mathscr{U}$ of X such that $\pi(\mathscr{U},x) \leq H$. We also give an example to show that locally path connectedness of X is essential for the above result. Later in Section 3, we show that if the subgroup H contains a Spanier group, then its core, $H_{\pi_1(X,x)}$, is open in $\pi_1^{qtop}(X,x)$.

We recall that the quasitopological fundamental group $\pi_1^{qtop}(X,x)$ is the quotient space of the loop space $\Omega(X,x)$ equipped with the compact-open topology with respect to the function $\Omega(X,x) \longrightarrow \pi_1(X,x)$ identifying path components (see [2]). It should be mentioned that $\pi_1^{qtop}(X,x)$ is a quasitopological group in the sense of [1] and it is not always a topological group (see [3, 6]). Also we recall from [5] that the topological fundamental group $\pi_1^{\tau}(X,x)$ is the fundamental group $\pi_1(X,x)$ with the finest group topology on $\pi_1(X,x)$ such that the canonical function $\Omega(X,x) \longrightarrow \pi_1(X,x)$ identifying path components is continuous. It should be mentioned that $\pi_1^{\tau}(X,x)$ and $\pi_1^{qtop}(X,x)$ have the same open subgroups [5, Proposition 4.4] and every open set of $\pi_1^{\tau}(X,x)$ is also an open set in $\pi_1^{qtop}(X,x)$.

Biss [2, Theorem 5.5] showed that for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of open subgroups of its fundamental group $\pi_1(X,x)$. There is a misstep in the proof of the above theorem. In fact, Biss assumed that every fibration with discrete fiber is a covering map which is not true in general. We give an example to show that the above classification of connected covering spaces does not hold.

In Section 3, first we introduce a *path open cover* $\mathscr{V}$ of a pointed space (X,x) and also the *path Spanier group* $\widetilde{\pi}(\mathscr{V},x)$ with respect to $\mathscr{V}$. Then we show that $\widetilde{\pi}(\mathscr{V},x)$ is an open subgroup of $\pi_1^{qtop}(X,x)$ and using this fact we give a necessary and sufficient condition for a subgroup of $\pi_1(X,x)$ to be open in $\pi_1^{qtop}(X,x)$. Second, using these facts and the main results of Section 2, we present a suitable true classification of connected covering spaces with respect to the Biss's ones which is the second main object of the paper as follows.

For a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the cojugacy classes of subgroups of its fundamental group $\pi_1(X,x)$*, with open core in* $\pi_1^{qtop}(X,x)$.

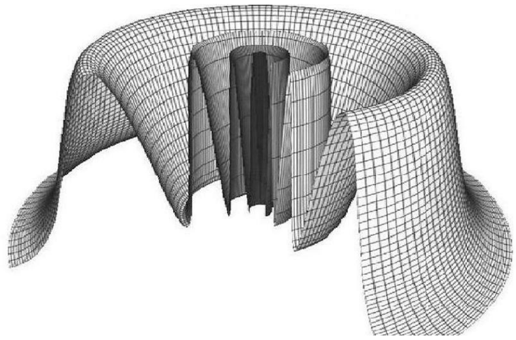

Figure 1:

2 Relation between open subgroups and Spanier groups

The following theorem gives a sufficient condition on an open subgroup in order to contain a Spanier group.

Theorem 2.1. *If X is a locally path connected space and H is an open normal subgroup of $\pi_1^{qtop}(X,x)$, then there exists an open cover $\mathscr{U}$ such that $\pi(\mathscr{U},x) \leq H$.*

In the following example, we show that locally path connectedness is an essential hypothesis in Theorem 2.1.

Example 2.2. Let X be a subspace of $\mathbb{R}^3$ which is obtained by taking the "surface" obtained by rotating the topologists' sine curve about its limiting arc (see Figure. 1 [7]). Put $A = \{(0,0,s) \in \mathbb{R}^3 | -1 \leq s \leq 1\}$, then $Y = X \backslash A$ is connected, locally path connected and semilocally simply connected. Hence $\pi_1^{qtop}(Y,x)$ is discrete where $x = (\frac{1}{\pi},0,0)$. Therefore $\pi_1^{qtop}(X,x)$ is a discrete topological group which implies that $\{[c_x]\}$ is open. But it is easy to see that for every open cover $\mathscr{U}$ of X, $\pi(\mathscr{U},x)$ is non-trivial (see [7]).

We recall from group theory that for any subgroup H of a group G, the core of H in G, denoted by H_G, is defined to be the join of all the normal subgroups of G that are contained in H. It is easy to see that $H_G = \bigcap_{g \in G} g^{-1}Hg$ which is the largest normal subgroup of G contained in H. Using this notion the following corollary is a consequence of Theorem 2.1.

Corollary 2.3. *If X is a locally path connected space and H is a subgroup of $\pi_1(X,x)$ such that the core of H in $\pi_1(X,x)$ is open in $\pi_1^{qtop}(X,x)$, then there exists an open cover $\mathscr{U}$ such that $\pi(\mathscr{U},x) \leq H$.*

3 A classification of covering spaces

In order to present a suitable classification of covering spaces we introduce the following concepts.

Definition 3.1. Let (X,x) be a pointed space. By a *path open cover* of (X,x) we mean an open cover

$\mathscr{V} = \{V_\alpha | \alpha \in P(X,x)\}$ of the path component of X involve x such that $\alpha(1) \in V_\alpha$ for every $\alpha \in P(X,x)$. We also define the *path Spanier group* $\widetilde{\pi}(\mathscr{V},x)$ with respect to the path open cover $\mathscr{V}$ to be the subgroup of $\pi_1(X,x)$ which contains all homotopy classes having representatives of the following type:

$$\prod_{j=1}^{n} u_j v_j u_j^{-1},$$

where u_j are arbitrary paths starting at x and each v_j is a loop inside V_{u_j} for all $i \in \{1,2,...,n\}$.

Theorem 3.2. *Let (X,x) be a locally path connected pointed space and $\mathscr{V} = \{V_\alpha | \alpha \in P(X,x)\}$ be a path open cover of (X,x). Then $\widetilde{\pi}(\mathscr{V},x)$ is a Spanier group $\pi(\mathscr{U},x)$ for some cover $\mathscr{U}$ of X if and only if $\widetilde{\pi}(\mathscr{V},x)$ is a normal subgroup of $\pi_1(X,x)$.*

The following theorem gives an important property of the path Spanier groups.

Theorem 3.3. *Let (X,x) be a locally path connected pointed space and $\mathscr{V} = \{V_\alpha | \alpha \in P(X,x)\}$ be a path open cover of (X,x). Then $\widetilde{\pi}(\mathscr{V},x)$ is an open subgroup of $\pi_1^{qtop}(X,x)$.*

Using the above theorem, we can give a necessary and sufficient condition for a subgroup of $\pi_1(X,x)$ to be open in $\pi_1^{qtop}(X,x)$.

Corollary 3.4. *If X is locally path connected, then for every subgroup H of $\pi_1^{qtop}(X,x)$, H is open if and only if there exists a path open cover $\mathscr{V} = \{V_\alpha | \alpha \in P(X,x)\}$ of X such that $\widetilde{\pi}(\mathscr{V},x) \leq H$.*

Proof. Let H be an open subgroup of $\pi_1^{qtop}(X,x)$. For every path α in X from x to any point y, there is an open set V_α of $\alpha(1) = y$ in X such that $[\alpha](i_*\pi_1(V,y))[\alpha]^{-1} \leq H$. Hence by putting $\mathscr{V} = \{V_\alpha | \alpha \in P(X,x)\}$ as a path open cover of X we have $\widetilde{\pi}(\mathscr{V},x) \leq H$. The Converse follows from Theorem 3.3. □

Remark 3.5. Note that if G is a quasitopological group and $H \leq K \leq G$ and H is open in G, then K is also an open subgroup of G since every translation in G is a homeomorphism.

Now, we can show that the Spanier groups are open subgroups.

Corollary 3.6. *If X is a locally path connected space and $\mathscr{U}$ is an open cover of X, then $\pi(\mathscr{U},x)$ is an open subgroup of $\pi_1^{qtop}(X,x)$.*

Proof. For every $y \in X$ let U_y be an element of cover $\mathscr{U}$ involve y. Consider $V_\alpha = U_y$ for every $\alpha \in P(X,x)$ with $\alpha(1) = y$. Hence $\widetilde{\pi}(\mathscr{V},x) \leq \pi(\mathscr{U},x)$ where $\mathscr{V} = \{V_\alpha | \alpha \in P(X,x)\}$. Therefore $\pi(\mathscr{U},x)$ is open since $\widetilde{\pi}(\mathscr{V},x)$ is an open subgroup. □

Now we are in a position to give a necessary and sufficient condition for a subgroup of the fundamental group X to contain a Spanier group of X.

Corollary 3.7. *If X is locally path connected and H is a subgroup of $\pi_1(X,x)$, then there exists an open cover $\mathscr{U}$ such that $\pi(\mathscr{U},x) \leq H$ if and only if the core of H in $\pi_1(X,x)$ is open in $\pi_1^{qtop}(X,x)$.*

The following classification of connected coverings of locally path connected spaces is a consequence of Corollary 3.7 and Theorem 1.1 which is the main result of this section.

Theorem 3.8. *For a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of subgroups of its fundamental group $\pi_1(X,x)$, with open core in $\pi_1^{qtop}(X,x)$.*

Proof. For a connected, locally path connected space X, if H is a subgroup of $\pi_1(X,x)$ with open core, then by Corollary 3.7 there exists an open cover $\mathscr{U}$ such that $\pi(\mathscr{U},x) \leq H$. Thus by Theorem 1.1 there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$. Note that if there is another connected covering $q : \widetilde{Y} \to X$, then by classical results in coverings, q and p are equivalent if and only if $q_*\pi_1(\widetilde{Y},\tilde{y}) = g^{-1}Hg$ for some $g \in \pi_1(X,x)$. Moreover, if there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$, then by choosing $\mathscr{U}$ consists of evenly covered open subsets of X we have $\pi(\mathscr{U},x) \leq H$. Hence by Corollary 3.7 the core of H is open in $\pi_1^{qtop}(X,x)$. □

The following two corollaries are immediate consequences of the proof of Theorem 3.8.

Corollary 3.9. *If X is connected, locally path connected and H is a subgroup of $\pi_1(X,x)$, then there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$ if and only if H contains an open normal subgroup in $\pi_1^{qtop}(X,x)$.*

Corollary 3.10. *If X is connected, locally path connected space and there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$, then H is open in $\pi_1^{qtop}(X,x)$.*

We recall from [9] that a space X is semilocally small generated if for every $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U,x) \leq \pi_1^{sg}(X,x)$, where i_* is the induced homomorphism from the inclusion $i : U \hookrightarrow X$ and the small generated subgroup $\pi_1^{sg}(X,x)$ is the subgroup generated by the following set

$$\{[\alpha * \beta * \alpha^{-1}] \mid [\beta] \in \pi_1^s(X,\alpha(1)),\ \alpha \in P(X,x)\},$$

where $P(X,x)$ is the space of all paths from I into X with initial point x and $\pi_1^s(X,\alpha(1))$ is the small subgroup of X at $\alpha(1)$ (see [10]).

Theorem 3.11. *Let X be a connected, locally path connected and semilocally small generated space and $H \leq \pi_1(X,x)$, for $x \in X$. Then there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$ if and only if H is open in $\pi_1^{qtop}(X,x)$.*

Proof. Let H be any open subgroup of $\pi_1^{qtop}(X,x)$. It suffices to show that there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$. By Corollary 3.7 there is a path open cover $\mathscr{V}$ of X such that $\widetilde{\pi}(\mathscr{V},x) \leq H$. By the definition of $\pi_1^{sg}(X,x)$ we can show that $\pi_1^{sg}(X,x) \leq \widetilde{\pi}(\mathscr{V},x)$. Also by the definition of semilocally small generatedness, there exists an open cover $\mathscr{U}$ of X such that $\pi(\mathscr{U},x) \leq \pi_1^{sg}(X,x)$. Therefore $\pi(\mathscr{U},x) \leq H$. Now Theorem 1.1 gives the desired result. □

In the following example, we show that the classification of connected coverings given by Biss [2, Theorem 5.5] which we mentioned in Section 1 does not hold.

Example 3.12. Let $X = \bigcup_{n\in\mathbb{N}}\{(x,y) \in \mathbb{R}^2 | (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ be the Hawaiian earing space. Brazas [4, Example 3.8] introduces a connected semicovering $p : \widetilde{X} \longrightarrow X$ say and hence a Serre fibration of X with discrete fiber which is not a covering. By classification of semicoverings [4, Corollary 7.20] $H =$

$p_*\pi_1(\widetilde{X},\tilde{x})$ is open in $\pi_1^{\tau}(X,x)$ and hence is open in $\pi_1^{qtop}(X,x)$. Now assume that the Biss's classification of connected coverings holds, then there exists a covering $q:\widetilde{Y}\longrightarrow X$ such that $p_*\pi_1(\widetilde{Y},\tilde{y})=H$. Since every covering is a semicovering, $q:\widetilde{Y}\longrightarrow X$ is a semicovering which implies that $p:\widetilde{X}\longrightarrow X$ and $q:\widetilde{Y}\longrightarrow X$ are equivalent as semicoverings. Therefore $p:\widetilde{X}\longrightarrow X$ is a covering which is a contradiction. Note that H is an open subgroup which does not contain an open normal subgroup in $\pi_1^{qtop}(X,x)$. Hence there exists a path Spanier group which is not a Spanier group.

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