# ON THE CAPABILITY AND SCHUR MULTIPLIER OF NILPOTENT LIE ALGEBRA OF CLASS TWO

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Abstract. Recently, the authors in a joint paper obtained the structure of all capable nilpotent Lie algebras with derived subalgebra of dimension at most 1. The paper is devoted to characterize all capable nilpotent Lie algebras of class two with derived subalgebra of dimension 2. It develops and generalizes the result due to Heineken for the group case.

## 1. Motivation and preliminaries

According to Beyl and Tappe a $p$-group $G$ is called capable if $G$ is isomorphic to $H/Z(H)$ for a group $H$. There are some fundamental known results concerning capability of $p$-groups. For instance, in [9, Corollary 4.16], it is shown the only capable extra-special $p$-groups (the $p$-group with $Z(G) = G'$ and $|G'| = p$) are those of order $p^3$ and exponent $p$. In the case that $G' = Z(G)$ and $Z(G)$ is elementary abelian $p$-group of rank 2, Heineken in [15] proved that the capable ones have order at most $p^7$.
Lie algebras and groups have similarities in the structures, so some authors tried to make analogies between them. But in this way not every thing is the same and there are differences between groups and Lie algebras so that most of the time the proofs are different. Also the results in the field of Lie algebras are sometimes stronger than that for groups. For instance, in [23], the authors obtained the structure of a capable nilpotent Lie algebra $L$ when $\dim L^2 \leq 1$. It developed the result of [9, Corollary 4.16] for groups to the case of Lie algebras. Recall that a Lie algebra is capable provided that $L \cong H/Z(H)$ for a Lie algebra $H$. In the same scene of research, we are going to characterize the structure of all capable Lie algebras that are nilpotent of class two with with derived subalgebra of dimension 2. It obviously develops and generalizes the result of Heineken [15] for groups to the area of Lie algebras. As an application, we exactly obtain $\mathcal{M}(L)$, the Schur multiplier of those Lie algebras. Recall that if $L$ is a Lie algebra and $F$ a free Lie algebra such that $L \cong F/R$, then $\mathcal{M}(L)$, is isomorphic to $R \cap F^2/[R, F]$. The reader can find some literatures about the Schur multiplier of groups and Lie algebras for instance in [1, 4, 5, 7, 9, 11, 19, 20, 21, 22, 23, 24, 25, 27, 28, 26, 29].

Throughout the paper, we assume that all Lie algebras have finite dimensions on an algebraically closed field, and we use the symbol $H(m)$ for the Heisenberg algebra of dimension $2m+1$ that is a Lie algebra $L$ with $L^2 = Z(L)$ and $\dim L^2 = 1$.

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Such algebras are odd dimensional with basis $v_1, \ldots, v_{2m}, v$ and the only non–zero multiplication between basis elements is $[v_{2i-1}, v_{2i}] = -[v_{2i}, v_{2i-1}] = v$ for $i = 1, 2, \ldots, m$. We also denote an abelian Lie algebra of dimension $n$ by $A(n)$.

The following lemma gives the structure of the Schur multiplier of a direct sums of Lie algebras.

**Lemma 1.1.** *(See* [4, Proposition 3]*) Let $A$ and $B$ be Lie algebras. Then*

$$\mathcal{M}(A \oplus B) \cong \mathcal{M}(A) \oplus \mathcal{M}(B) \oplus (A/A^2 \otimes B/B^2).$$

The following is another easy consequence. It can also be deduced in different ways, using homological methods of abstract algebra in [32].

**Corollary 1.2.** *(See* [23, Corollary 2.5]*) We have $\mathcal{M}(A(n)) = A(n) \wedge A(n)$.*

Schur multipliers of abelian and Heisenberg algebras are well-known.

**Lemma 1.3.** *(See* [23, Lemma 2.6]*) We have*

(*i*) $\dim \mathcal{M}(A(n)) = \frac{1}{2}n(n-1)$.
(*ii*) $\dim \mathcal{M}(H(1)) = 2$.
(*iii*) $\dim \mathcal{M}(H(m)) = 2m^2 - m - 1$ *for all* $m \geq 2$.

Recall from [23], the concept of the exterior center $Z^{\wedge}(L)$, the set of all elements $l$ of $L$ for which $l \wedge l_1 = 0_{L \wedge L}$ for all $l_1 \in L$. Following [23], a Lie algebra $L$ is capable if and only if $Z^{\wedge}(L) = 0$. It gives a criteria for detecting capable Lie algebras.

The following lemmas provide a condition under which we can decide whether $Z^{\wedge}(L) = 0$.

**Lemma 1.4.** *(See* [1, Theorem 4.4] *and* [23, Lemma 2.1]*) Let $K$ be a central ideal in a Lie algebra $L$. Then $K \subseteq Z^{\wedge}(L)$ if and only if $\mathcal{M}(L) \to \mathcal{M}(L/K)$ is a monomorphism.*

**Lemma 1.5.** *(See* [23, Theorem 2.7]*) Let $A$ and $B$ be Lie algebras. Then*

$$Z^{\wedge}(A \oplus B) \subseteq Z^{\wedge}(A) \oplus Z^{\wedge}(B).$$

We also need the following lemma.

**Lemma 1.6.** *(See* [11, Proposition 13] *and* [1, Proposition 4.1(iii)]*) Let $L$ be a Lie algebra and $N$ be a central ideal of $L$. Then the following sequences are exact.*

(*i*) $L \wedge N \to L \wedge L \to L/N \wedge L/N \to 0$.
(*ii*) $\mathcal{M}(L) \to \mathcal{M}(L/N) \to N \cap L^2 \to 0$.

The following lemma gives the structure of all capable Lie algebras in the class of abelian and Heisenberg Lie algebras.

**Lemma 1.7.** *(See* [23, Theorem 3.5]*) We have*

(i) *$A(n)$ is capable if and only if $n \geq 2$.*
(ii) *$H(m)$ is capable if and only if $m = 1$.*

The direct sum of two Heisenberg Lie algebras has the derived subalgebra of dimension 2. The following corollary gives a necessary condition for capability of such Lie algebras.

**Corollary 1.8.** *Let $L = H(t) \oplus H(m)$. If $t \geq 2$, then $L$ is non-capable for all $m \geq 1$.*

*Proof.* We show that $H(t)^2 \subseteq Z^\wedge(L)$. By Lemma 1.1, we have

$$\dim \mathcal{M}(L) = \dim \mathcal{M}(H(t)) + \dim \mathcal{M}(H(m)) + \dim(\frac{H(t)}{H(t)^2} \otimes \frac{H(m)}{H(m)^2}),$$

and

$$\dim \mathcal{M}(L/H(t)^2) = \dim \mathcal{M}(\frac{H(t)}{H(t)^2}) + \dim \mathcal{M}(H(m)) + \dim(\frac{H(t)}{H(t)^2} \otimes \frac{H(m)}{H(m)^2}).$$

Lemmas 1.6 and 1.7$(ii)$ show that

$$\dim \mathcal{M}(L) = \dim \mathcal{M}(L/H(t)^2) - 1.$$

Therefore by Lemma 1.4, $H(t)^2 \subseteq Z^\wedge(L)$ and hence $L$ is non-capable. □

The following lemma determines the structure of a capable Lie algebra $L$ of dimension $n$ with $\dim L^2 = 1$. In the next section, we follow the same line of research to characterize the structure of all nilpotent Lie algebra $L$ of class 2 with $\dim L^2 = 2$.

**Lemma 1.9.** *(See* [22, Lemma 3.3]*) Let $L$ be a nilpotent Lie algebra of dimension $n$ such that $\dim L^2 = 1$. Then $L \cong H(m) \oplus A(n-2m-1)$ for some $m \geq 1$ and $L$ is capable if and only if $m = 1$, that is, $L \cong H(1) \oplus A(n-3)$.*

## 2. Main results

We are going to classify all capable nilpotent Lie algebras of class two with derived subalgebra of dimension 2. Furthermore, we give the structure of Schur multiplier of such Lie algebras as an application. It gives a vast generalization of a result due to Heineken's result on capability in the class of $p$-groups in [15].

We need the notion of central product of Lie algebras.

**Definition 2.1.** The Lie algebra $L$ is a central product of $A$ and $B$, if $L = A + B$, where $A$ and $B$ are ideals of $L$ such that $[A, B] = 0$ and $A \cap B \subseteq Z(L)$. We denote the central product of two Lie algebras $A$ and $B$ by $A \dotplus B$.

On the capability of a central product we have the following proposition.

**Proposition 2.2.** *Let $L = A \dotplus B$ with $A^2 \cap B^2 \neq 0$. Then $L$ is non-capable.*

*Proof.* Assume that $0 \neq x \in A^2 \cap B^2$ and $l = a + b$ be an arbitrary element of $L$ such that $a \in A, b \in B$. We have $x = \sum_{j=1}^{n} \alpha_j [a'_j, a_j] = \sum_{i=1}^{m} \beta_i [b'_i, b_i]$ in which $\alpha_j, \beta_i$ are scalers such that $a'_j, a_j \in A, b'_i, b_i \in B$ . Since $[A, B] = 0$, we have

$$a \wedge x = a \wedge \sum_{i=1}^{m} \beta_i [b'_i, b_i] = \sum_{i=1}^{m} \beta_i ([a, b'_i] \wedge b_i - [a, b_i] \wedge b'_i) = 0.$$

and

$$b \wedge x = b \wedge \sum_{j=1}^{n} \alpha_j [a'_j, a_j] = \sum_{j=1}^{n} \alpha_j ([b, a'_j] \wedge a_j - [b, a_j] \wedge a'_j) = 0,$$

which shows $x \wedge l = 0_{L \wedge L}$, for all $l \in L$. Therefore $Z^\wedge(L) \neq 0$, and the result holds. □

**Definition 2.3.** A Lie algebra $H$ is called generalized Heisenberg of rank $n$ if $H^2 = Z(H)$ and $\dim H^2 = n$.

The following proposition states a close relationship between generalized Heisenberg Lie algebras and those of class 2 with derived subalgebra of dimension 2. This relationship allows us to work only on generalized Heisenberg Lie algebras when working on the capability of mentioned Lie algebras of class 2.

**Proposition 2.4.** *Let $L$ be a nilpotent Lie algebra of nilpotency class 2. Then $L = H \oplus A$ and $Z^\wedge(L) = Z^\wedge(H)$, where $A$ is abelian and $H$ is generalized Heisenberg.*

*Proof.* Since $Z(L) = L^2 + A$, where $A$ is abelian, we have

$$L/L^2 = H/L^2 \oplus (A + L^2)/L^2$$

for some subalgebra $H$ of $L$. Therefore $L = H + A$ and $H \cap A = L^2 \cap A = 0$. This shows $L = H \oplus A$ and $Z(H) = L^2 = H^2$. By Lemma 1.5, we have $Z^\wedge(L) \subseteq Z^\wedge(H) \oplus Z^\wedge(A)$. First assume that $\dim A \geq 2$. Now Lemma 1.7($i$) shows $Z^\wedge(L) = 0$ and so $Z^\wedge(L) \subseteq Z^\wedge(H)$. Now let $\dim A = 1$. Using Lemma 1.1,

$$\mathcal{M}(L) = \mathcal{M}(H \oplus A(1)) = \mathcal{M}(H) \oplus (H/H^2 \otimes A(1)).$$

By Lemma 1.6($ii$) and the fact that $A(1) \cap L^2 = 0$, we have $\mathcal{M}(L) \longrightarrow \mathcal{M}(L/A(1))$ is an epimorphism. Thus $\mathcal{M}(L) \longrightarrow \mathcal{M}(L/A(1))$ is not a monomorphism. Therefore $A(1) \nsubseteq Z^\wedge(L)$, using Lemma 1.4, we should have $Z^\wedge(L) \subseteq Z^\wedge(H)$. Now we show that $Z^\wedge(H) \subseteq Z^\wedge(L)$. Note that by Lemma 1.7($i$) $H/H^2$ is capable so $Z^\wedge(H) \subseteq H^2$, therefor for getting the result we consider two cases as follows

(i) $Z^\wedge(H) = 0$,
(ii) $0 \neq Z^\wedge(H) \subseteq H^2$.

Case ($i$) is trivial.
Case ($ii$), by invoking Lemma 1.1, we have

$$\mathcal{M}(L) \cong \mathcal{M}(H) \oplus \mathcal{M}(A) \oplus (H/H^2 \otimes A)$$

and

$$\mathcal{M}(L/Z^\wedge(H)) \cong \mathcal{M}(H/Z^\wedge(H)) \oplus \mathcal{M}(A) \oplus (H/H^2 \otimes A).$$

Now Lemma 1.6($ii$) shows that, $\dim \mathcal{M}(H) = \dim \mathcal{M}(H/Z^\wedge(H)) - \dim(Z^\wedge(H)$. From Lemma 1.4, the homomorphism $\mathcal{M}(L) \to \mathcal{M}(H/Z^\wedge(H))$ is a monomorphism, and so $Z^\wedge(H) \subseteq Z^\wedge(L)$, as required. □

Heineken in [15] proved that for a finite capable $p$-groups with

$$\mathbb{Z}_p \oplus \mathbb{Z}_p \cong G' \subseteq Z(G)$$

we have

$$p^2 < |G/Z(G)| < p^6.$$

Similarly, if $L$ is a finite dimensional nilpotent Lie algebra of class 2 with $\dim L^2 = 2$, then the capability of $L$ implies

$$2 < \dim L/Z(L) < 6.$$

In the sequel we prove the above result.

**Lemma 2.5.** *Let $L$ be a finite dimensional nilpotent Lie algebra of nilpotency class 2 with $\dim L^2 = 2$. Then there exists two subalgebras $U$ and $V$ of $L$ such that $L = U + V$ and $U \cap V \subseteq Z(L)$.*

*Proof.* Consider $L$ as a vector space on the filed $S$. If $L^2 = \langle a\rangle \oplus \langle b\rangle$, then consider the mapping

$$\begin{aligned}\lambda : L/Z(L) \times L/Z(L) &\to L^2\\ (l_1 + Z(L), l_2 + Z(L)) &\mapsto [l_1, l_2].\end{aligned}$$

Now $[l_1, l_2] = n(l_1, l_2)a + m(l_1, l_2)b$, where $n(l_1, l_2), m(l_1, l_2) \in S$, so the mapping $\lambda$ induces the following two alternating bilinear forms

$$\begin{aligned}\lambda_1 : L/Z(L) \times L/Z(L) &\to S\\ (l_1 + Z(L), l_2 + Z(L)) &\mapsto n(l_1, l_2)\end{aligned}$$

and

$$\begin{aligned}\lambda_2 : L/Z(L) \times L/Z(L) &\to S\\ (l_1 + Z(L), l_2 + Z(L)) &\mapsto m(l_1, l_2);\end{aligned}$$

on $L/Z(L)$. By [15, Proposition 1], we have $L/Z(L) = U/Z(L) \oplus V/Z(L)$, for subalgebras $U$ and $V$ of $L$. The result follows. □

**Proposition 2.6.** *Let $L$ be a finite dimensional nilpotent Lie algebra of nilpotency class 2 such that $\dim L^2 = 2$. If $L$ is capable, then*

$$2 < \dim L/Z(L) < 6.$$

*Proof.* It is clear that $2 < \dim L/Z(L)$. Now we show that $\dim L/Z(L) < 6$. We suppose that $L \cong E/Z(E)$ for some Lie algebra $E$ and deduce restrictions on $L$. By Lemma 2.5, we have $L = U + V$ and $U \cap V \subseteq Z(L)$ for subalgebras $U$ and $V$ of $L$. First we show that if $L = U + V$ with $[U, V] = 0$ and $U \nsubseteq Z(L), V \nsubseteq Z(L)$, then $\dim L/Z(L) = 4$.
Since $L$ is capable so $U^2 \cap V^2 = 0$, by Proposition 2.2. We conclude

$$\dim\ U^2 = \dim\ V^2 = 1,\ L^2 = U^2 \oplus V^2.$$

Using Lemma 1.9, we have

$$U = H(m) \oplus A(n - 2m - 1),\ V = H(t) \oplus A(n_1 - 2t - 1).$$

Since $[U, V] = 0$, we have $Z(L) = Z(U) + Z(V)$ and $U \cap V \subseteq Z(L)$. Therefore

$$L = H(m) \oplus H(t) \oplus A,$$

where $A$ is abelian. Corollary 1.8 follows $L = H(1) \oplus H(1) \oplus A$. Therefore $U/Z(U)$ and $V/Z(V)$ are abelian of dimension 2. Since $L/Z(L) = U/Z(U) \oplus V/Z(V)$, $\dim L/Z(L) = 4$.

Let $\dim L/Z(L) > 4$. From now the proof is completely similar in techniques, to that of [15, Proposition 3] except that here it is stated for Lie algebras. □

Now considering the above proposition, for determining capable Lie algebras among nilpotent ones of class 2 with derived subalgebra of dimension 2, it is enough to consider those ones with

$$5 \leq \dim L \leq 7.$$

Following to Shirshov [31] for a free Lie algebra $L$ on the set $X = \{x_1, x_2, \ldots\}$. The basic commutator on the set $X$ defined inductively.

(i) The generators $x_1, x_2, \ldots, x_n$ are basic commutators of length one and ordered by setting $x_i < x_j$ if $i < j$.

(ii) If all basic commutators $d_i$ of length less than $t$ have been defined and ordered, then we may define the basic commutators of length $t$ to be all commutators of the form $[d_i, d_j]$ such that the sum of lengths of $d_i$ and $d_j$ is $t$, $d_i > d_j$, and if $d_i = [d_s, d_t]$, then $d_j \geq d_t$. The basic commutators of length $t$ follow those of lengths less than $t$. The basic commutators of the same length can be ordered in any way, but usually the lexicographical order is used.

The number of all basic commutators on a set $X = \{x_1, x_2, \ldots x_d\}$ of length $n$ is denoted by $l_d(n)$. Thanks to [17], we have

$$l_d(n) = \frac{1}{n} \sum_{m|n} \mu(m) d^{\frac{n}{m}},$$

where $\mu(m)$ is the Möbius function, defined by $\mu(1) = 1, \mu(k) = 0$ if $k$ is divisible by a square, and $\mu(p_1 \ldots p_s) = (-1)^s$ if $p_1, \ldots, p_s$ are distinct prime numbers. Using the topside statement and looking [30, Lemma 1.1] and [31], we have the following theorem.

**Theorem 2.7.** *Let $F$ be a free Lie algebra on set $X$, then $F^c/F^{c+i}$ is an abelian Lie algebra with the basis of all basic commutators on $X$ of lengths $c, c+1, \ldots, c-i+1$ for all $0 \leq i \leq c$. In particular, $F^c/F^{c+1}$ is an abelian Lie algebra of dimension $l_d(c)$, where $F^c$ is the $c$-th term of the lower central series of $F$.*

*Proof.* See [3], page 72-74. □

By Propositions 2.4 and 2.6, it is enough to determine the capability of generalized Heisenberg Lie algebras of rank 2 whose dimensions are $5, 6$ or 7. According to the classification of nilpotent Lie algebras of dimension at most 7 in [15] and using notation and terminology in [10, 13, 14], the following Lie algebras must be considered.

$$L_{5,8} = \langle x_1, x_2, x_3, x_4, x_5 \big| [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle,$$
$$L_1 = 27A = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \big| [x_1, x_2] = x_6 = [x_3, x_4], [x_1, x_5] = x_7 = [x_2, x_3] \rangle,$$
$$L_2 = 27B = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \big| [x_1, x_2] = x_6, [x_1, x_4] = x_7, [x_3, x_5] = x_7 \rangle,$$
$$L_{6,22(0)} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \big| [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_3, x_4] = x_5 \rangle.$$

The following remark shows that $L_{6,22(0)}$ is isomorphic to a known Lie algebra as follows.

*Remark* 2.8. With the above notations and assumptions we have

$$L_{6,22(0)} = \langle y_1, y_2, y_3, y_4, y_5, y_6 \big| [y_1, y_2] = y_6, [y_3, x_4] = y_5 \rangle \cong H(1) \oplus H(1).$$

*Proof.* By taking $y_2 = x_3 - x_2, y_6 = x_6 - x_5, y_i = x_i, i = 1, 3, 4, 5$. We have

$$L_{6,22(0)} = \langle y_1, y_2, y_3, y_4, y_5, y_6 \big| [y_1, y_2] = y_6, [y_3, x_4] = y_5 \rangle = H(1) \oplus H(1),$$

and the result holds. □

For capability of these algebras we have

**Lemma 2.9.** *$H(1) \oplus H(1)$ is capable.*

*Proof.* Using Lemma 1.5, we have $Z^\wedge(H(1)\oplus H(1)) \subseteq Z^\wedge(H(1))\oplus Z^\wedge(H(1))$. But the latter is trivial because $H(1)$ is capable, so the result follows. □

In the next proposition, we determine the structure of the Schur multipliers of those Lie algebras whose capabilities are determined in the next investigation.

**Proposition 2.10.** *The Schur multiplier of Lie algebras $L_{6,22(0)}, L_{5,8}, L_1$ and $L_2$ are abelian Lie algebras of dimension $8, 6, 9$ and $10$, respectively.*

*Proof.* By Lemma 2.8, we have $L_{6,22(0)} \cong H(1)\oplus H(1)$. Using Lemmas 1.1 and 1.3 and 2.9,

$$\mathcal{M}(L_{6,22(0)}) = \mathcal{M}(H(1)\oplus H(1)) = \mathcal{M}(H(1))\oplus\mathcal{M}(H(1))\oplus(\frac{H(1)}{H(1)^2}\otimes\frac{H(1)}{H(1)^2}).$$

Therefore $\dim\mathcal{M}(L_{6,22(0)}) = 8$.
Now let $L\cong L_{5,8}$, $F$ be a free Lie algebra on the set $\{x_1,x_2,x_3,x_4,x_5\}$ note that here we order $\{x_1,\dots,x_5\}$ as $x_5<x_4<x_3<x_2<x_1$ and

$$R = \langle [x_1,x_2]-x_4, [x_1,x_3]-x_5, [x_1,x_4], [x_1,x_5], [x_i,x_j] \big| 2\le i,j\le 5\rangle^F.$$

Since $L_{5,8}$ is of class two, $F^3\subseteq R$ and hence $\mathcal{M}(L_{5,8}) = \dfrac{R\cap F^2/F^4}{[R,F]/F^4}$. On the other hand,

$$\frac{R\cap F^2}{F^4} = \frac{F^3+\langle [x_1,x_4],[x_1,x_5],[x_i,x_j]\big|2\le i,j\le 5\rangle}{F^4}.$$

Theorem 2.7 implies $\dim\dfrac{F^2}{F^4} = l_5(3)+l_5(2) = 40+10 = 50$, and also we have

$$R\cap F^2/F^4 \cong \frac{F^2/F^4}{\langle [x_1,x_3]+F^4, [x_1,x_2]+F^4\rangle},$$

so

$$\dim R\cap F^2/F^4 = \dim\frac{F^2/F^4}{\langle [x_1,x_3]+F^4, [x_1,x_2]+F^4\rangle} = 50-2 = 48.$$

Now $\dfrac{R}{F^4} = \dfrac{F^3}{F^4} + \dfrac{\langle [x_1,x_2]-x_4, [x_1,x_3]-x_5, [x_1,x_4], [x_1,x_5], [x_i,x_j]\big|2\le i,j\le 5\rangle}{F^4}$, and so

$$[R,F]/F^4 = [\langle [x_1,x_2]-x_4, [x_1,x_3]-x_5, [x_1,x_4], [x_1,x_5], [x_i,x_j]\big|2\le i,j\le 5\rangle, F]/F^4.$$

Putting

$$\begin{aligned} M_1 =& \langle [x_1,x_4,x_t], [x_1,x_5,x_k], \\ & [x_i,x_j,x_l]\big|2\le i<j\le 5, 1\le t\le 4, 1\le k\le 5, 1\le l\le j\rangle/F^4 \end{aligned}$$

and

$$\begin{gathered} M_2 = \langle [x_1,x_2,x_i]+[x_i,x_4], \\ [x_1,x_3,x_j]+[x_j,x_5], [x_3,x_5]-[x_3,x_4], [x_4,x_5]\big|1\le i\le 2, 1\le j\le 3\rangle/F^4. \end{gathered}$$

Invoking the jacobian identity and some calculations, we have

$$M_2 = \langle [[x_1,x_2]-x_4, f_1], [[x_1,x_3]-x_5, f_2]\big| f_1,f_2\in F\rangle/F^4$$

and

$$M_1 = \langle [x_1,x_4,f], [x_1,x_5,f], [x_i,x_j,f]\big|2\le i,j\le 5, f\in F\rangle/F^4.$$

It is easy to see that $[R,F]/F^4 = M_1 \oplus M_2$ and

$$M_1 \cong \frac{F^3/F^4}{\langle [x_1,x_2,x_i]+F^4, [x_1,x_3,x_j]+F^4 \big| 1 \le i \le 2, 1 \le j \le 3\rangle}.$$

By Theorem 2.7, we have

$$\dim M_1 = l_5(3) - 5 = 40 - 5 = 35.$$

Therefore

$$\dim\, [R,F]/F^4 = \dim M_1 + \dim M_2 = 35 + 7 = 42.$$

It follows

$$\dim \mathcal{M}(L_{5,9}) = \dim R \cap F^2/F^4 - \dim\, [R,F]/F^4 = 48 - 42 = 6.$$

Now assume that $L = L_1$, and let $F$ be the free Lie algebra on the set $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Note that here we order $\{x_1, \dots, x_7\}$ as $x_7 < x_6 < x_5 < x_4 < x_3 < x_2 < x_1$. Put

$$\begin{aligned} R =& \langle [x_1,x_2]-x_6, [x_3,x_4]-x_6, [x_1,x_5]-x_7, [x_2,x_3]-x_7, \\ & [x_1,x_2]-[x_3,x_4], [x_1,x_5]-[x_2,x_3], [x_t,x_d] \big| 1 \le t,d \le 7\rangle^F. \end{aligned}$$

$F^3 \subseteq R$, since $L_1$ is nilpotent of class two. We know $\mathcal{M}(L_1) = \dfrac{R \cap F^2/F^4}{[R,F]/F^4}$ and

$$\frac{R\cap F^2}{F^4} = \frac{F^3 + \langle [x_1,x_2]-[x_3,x_4], [x_1,x_5]-[x_2,x_3], [x_i,x_j] \big| 1 \le i,j \le 7\rangle}{F^4}.$$

Using Theorem 2.7, we have

$$\dim \frac{F^2}{F^4} = l_7(3) + l_7(2) = 112 + 21 = 133.$$

Therefore

$$\dim R \cap F^2/F^4 = l_7(3) + l_7(2) - 4 + 2 = 131.$$

Taking

$$\begin{aligned} M_3 =& \langle [x_1,x_2]-[x_3,x_4], [x_1,x_5]-[x_2,x_3], [x_1,x_2]-x_6, \\ & [x_1,x_5]-x_7, [x_i,x_j] \big| 1 \le i,j \le 7\rangle/F^4. \end{aligned}$$

Since $\dfrac{R}{F^4} = \dfrac{F^3}{F^4} + M_3$, $\dfrac{[R,F]}{F^4} = [M_3, F/F^4]$. Now define $M_4$ to be the subalgebra (subspace) of $F^3/F^4$ generated by all basic commutators of weight 3 except $[x_1,x_5,x_1], [x_2,x_3,x_1]$ and $[x_3,x_4,x_3]$.
Put

$$\begin{aligned} M_5 =& \langle [x_1,x_5,x_1]-[x_1,x_7], \\ & [x_2,x_3,x_1]-[x_1,x_7], [x_1,x_7]-[x_3,x_6], [x_3,x_4,x_3]-[x_1,x_7]\rangle/F^4. \end{aligned}$$

Using the jacobian identity, we have

$$\begin{aligned} M_5 =& \langle [[x_1,x_2]-x_4, f_1], [[x_1,x_5]-x_5, f_2], \\ & [[x_1,x_5]-[x_2,x_3], f], [[x_1,x_2]-[x_3,x_4], f] \big| f_1, f_2 \in F\rangle/F^4 \end{aligned}$$

It is easy to see that $\dfrac{[R,F]}{F^4} = M_4 \oplus M_5$. Now Theorem 2.7 shows

$$\dim M_4 = l_7(3) - 3 = 112 - 3 = 109.$$

Therefore

$$\dim[R,F]/F^4 = \dim M_1 + \dim M_2 = 109 + 13 = 122.$$

It implies

$$\dim \mathcal{M}(L_1) = \dim R \cap F^2/F^4 - \dim[R,F]/F^4 = 131 - 122 = 9.$$

By a similar way we can obtain that $\dim \mathcal{M}(L_2) = 10$. It completes the proof. $\square$

We are ready to decide about the capability of $L_{5,8}$, $L_1$ and $L_2$ in the following lemma.

**Lemma 2.11.** *$L_{5,8}$ and $L_1$ are capable while $L_2$ is not.*

*Proof.* By Lemmas 1.3 and 1.9, we have

$$\dim \mathcal{M}(L_{5,8}/L_{5,8}^2) = 3$$

and

$$\dim \mathcal{M}(L_{5,8}/\langle x_4\rangle) = \dim \mathcal{M}(L_{5,8}/\langle x_5\rangle) = \dim \mathcal{M}(H(1) \oplus A(1)) = 4.$$

Now using Proposition 2.10 and Lemma 1.4, we conclude $L_{5,8}$ is capable. Again by invoking Lemmas 1.3 and 1.9, we have

$$\dim \mathcal{M}(L_1/L_1^2) = 10,$$

$$\dim \mathcal{M}(L_1/\langle x_7\rangle) = \dim \mathcal{M}(L_2/\langle x_6\rangle) = \dim \mathcal{M}(H(2) \oplus A(1)) = 9$$

and

$$\dim \mathcal{M}(L_2/\langle x_7\rangle) = \dim \mathcal{M}(H(1) \oplus A(3)) = 11.$$

Proposition 2.10 and Lemma 1.4 show $L_1$ is capable while $L_2$ is not. It completes the proof. $\square$

By the following, we get the capability of all generalized Heisenberg Lie algebras of rank 2 and dimension $n$.

**Theorem 2.12.** *Let $H$ be a generalized Heisenberg Lie algebra, $\dim H = n$ and $\dim H^2 = 2$. Then $H$ is capable if and only if $n = 5, 6, 7$ and $H = L_{5,8}$ or $H = L_{6,22(0)}$ or $H = L_1$.*

*Proof.* Using Lemmas 2.9 and 2.11, the result follows. $\square$

In the following corollary we classify all capable nilpotent Lie algebras of class 2 with derived subalgebra of dimension 2.

**Corollary 2.13.** *Let $L$ be a nilpotent Lie algebra of nilpotency class 2 and $\dim L^2 = 2$. Then $L$ is capable if and only if $L = L_{5,8} \oplus A$ or $L = L_{6,22(0)} \oplus A$ or $L = L_1 \oplus A$, where $A$ is abelian.*

*Proof.* This is an immediate consequence of Proposition 2.4 and Theorem 2.12. $\square$

## 3. Schur Multipliers of Generalized Heisenberg Lie Algebras

In this section using Proposition 2.10, we can compute the Schur multiplier of all generalized Heisenberg Lie algebras of rank 2 and dimension $n$. A usage of Lemma 1.1 enables us to compute the Schur multiplier of all nilpotent Lie algebras of nilpotency class 2 whose derived subalgebras have dimension 2.

**Proposition 3.1.** *Let $H$ be a non-capable generalized Heisenberg Lie algebra such that* $\dim H = n$. *Then*

$$\dim \mathcal{M}(H) = \frac{1}{2}(n-3)(n-2) - 2$$

*or*

$$\dim \mathcal{M}(H) = \frac{1}{2}(n-1)(n-4) + 1.$$

*Proof.* We divide the proof into two cases.

(i) $Z^\wedge(H) = H^2$,

(ii) $Z^\wedge(H) = K$, where $K \subset H^2$ and $\dim K = 1$.

In case $(i)$, since $H$ is non-capable, by Lemmas 1.4 and 1.6$(ii)$, we have

$$\dim \mathcal{M}(H) = \dim \mathcal{M}(H/H^2) - 2.$$

and so by Lemma 1.3,

$$\dim \mathcal{M}(H/H^2) = \dim \mathcal{M}(A(n-2)) = \frac{1}{2}(n-2)(n-3),$$

For the case $(ii)$, since $H$ is non-capable, Lemmas 1.4 and 1.6$(ii)$ imply that $\dim \mathcal{M}(H) = \dim \mathcal{M}(H/Z^\wedge(H)) - 1$.
Since $H/Z^\wedge(H)$ is capable and $\dim(H/Z^\wedge(H))^2 = 1$, Lemma 1.9 shows that $H/Z^\wedge(H) \cong H(1) \oplus A(n-4)$. Now by Lemma 1.1, we have

$$\begin{aligned}\dim \mathcal{M}(H/Z^\wedge(H)) &= \dim \mathcal{M}(H(1)) + \dim \mathcal{M}(A(n-4)) + \dim(\frac{H(1)}{H(1)^2} \otimes A(n-4)) \\ &= \frac{1}{2}(n-1)(n-4) + 2,\end{aligned}$$

using Lemma 1.9. The result follows. □

The following example show all cases in Proposition 3.1 can occur. Actually $L_2$ and $H(2) \oplus H(2)$ are examples.

*Example* 3.2.

Let $L = L_2$. By Lemma 2.11, $L_2$ is non-capable. We can show $Z^\wedge(L_2) = \langle x_7 \rangle \subset L_2^2$, since $\dim(L_2/\langle x_7 \rangle)^2 = 1$. From Lemma 1.9, we have $L_2/\langle x_7 \rangle \cong H(1) \oplus A(3)$. Thus

$$\dim \mathcal{M}(\frac{L_2}{\langle x_7 \rangle}) = \dim \mathcal{M}(H(1) \oplus A(3)) = 11.$$

This follows that $\dim \mathcal{M}(L_2) = \dim \mathcal{M}(L_2/\langle x_7 \rangle) - 1 = 10$.

Now let $L = H(2) \oplus H(2)$. Corollary 1.8 implies that $L$ is a non-capable generalized Heisenberg. Thus $Z^\wedge(L) = L^2$ and $\dim \mathcal{M}(L) = \dim \mathcal{M}(L/L^2) - 2 = 26$.

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