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D-brane action at order α'^2

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We use the compatibility of D-brane action with linear T-duality, S-duality, and with S-matrix elements as guiding principles to find all world volume couplings of one massless closed string and two open strings at order α'^2 in type-II superstring theories. In particular, we find that the squares of second fundamental form appear only in world volume curvatures and confirm the observation that the dilaton appears in the string frame action via the transformation $\hat{R}_{\mu\nu} \to \hat{R}_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi$.

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I. INTRODUCTION AND RESULTS

The low-energy effective field theory of D_p-branes in type-II superstring theories consists of the Dirac–Born–Infeld (DBI) [1] and the Chern–Simons (CS) actions [2], i.e.,

$$S_p = S_p^{\rm DBI} + S_p^{\rm CS}. \tag{1}$$

The curvature corrections to the DBI action have been found in Ref. [3] by requiring the consistency of the effective action with the $O(\alpha'^2)$ terms of the corresponding disk-level scattering amplitude [4,5]. For totally geodesic embedding of the world volume in ambient spacetime in which second fundamental form is zero, the corrections in the string frame for a zero B-field and for a constant dilaton are[1]

$$S_p^{\rm DBI} \supset -\frac{\pi^2\alpha'^2 T_p}{48}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}\Big[R_{abcd}R^{abcd} - 2\hat{R}_{ab}\hat{R}^{ab} - R_{abij}R^{abij} + 2\hat{R}_{ij}\hat{R}^{ij}\Big], \tag{2}$$

where $\hat{R}_{ab} = \tilde{G}^{cd}R_{cadb}$, $\hat{R}_{ij} = \tilde{G}^{cd}R_{cidj}$, and $\tilde{G} = \det(\tilde{G}_{ab})$, where $\tilde{G}_{ab}$ is the pullback of the bulk metric onto the word volume, i.e.,

$$\tilde{G}_{ab} = \frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}G_{\mu\nu}.$$

The Riemann curvatures in (2) are the pullback of the spacetime curvature onto tangent and normal bundles [3].

The curvature corrections to the CS part can be found by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action [6–8]. These corrections involve the quadratic order of the curvatures at order α'^2. However, the consistency of the effective action with the S-matrix elements of one Neveu-Schwarz Neveu-Schwarz (NSNS) and one Ramond-Ramond (RR) vertex operator requires the CS part at this order to have linear curvature corrections as well [9], i.e.,

$$S_p^{\rm CS} \supset -\frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\Big[\frac{1}{(p+1)!}\nabla_j\mathcal{F}^{(p+2)}_{ia_0\ldots a_p}\hat{R}^{ij} + \frac{1}{2!p!}\nabla_a\mathcal{F}^{(p+2)}_{ija_1\ldots a_p}R_{a_0}{}^{aij}\Big], \tag{3}$$

where $\mathcal{F}^{n+1}$ is the field strength of the RR potential n-form. The S-matrix calculations produce also the couplings in the CS part which involve linear field strength of B-field [9], in which we are not interested in this paper.

For arbitrary embeddings, the couplings (2) have been extended in Ref. [3] to

$$S_p^{\rm DBI} \supset -\frac{\pi^2\alpha'^2 T_p}{48}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}\Big[(R_T)_{abcd}(R_T)^{abcd} - 2(\hat{R}_T)_{ab}(\hat{R}_T)^{ab} - (R_N)_{abij}(R_N)^{abij} + 2\bar{R}_{ij}\bar{R}^{ij}\Big], \tag{4}$$

where the world volume curvature $(R_T)_{abcd}$ and $(R_N)^{abij}$ obey the Gauss–Codazzi equations, i.e.,

$$\begin{aligned}(R_T)_{abcd} &= R_{abcd} + \delta_{ij}(\Omega_{ac}{}^i\Omega_{bd}{}^j - \Omega_{ad}{}^i\Omega_{bc}{}^j)\\ (R_N)_{ab}{}^{ij} &= R_{ab}{}^{ij} + g^{cd}(\Omega_{ac}{}^i\Omega_{bd}{}^j - \Omega_{ac}{}^j\Omega_{bd}{}^i),\end{aligned} \tag{5}$$

where $\Omega^i{}_{ab}$ is the second fundamental form [3][2] The relation between $(\hat{R}_T)_{ab}$ and the world volume curvature is then

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[1] Our index convention is that the Greek letters $(\mu,\nu,\ldots)$ are the indices of the spacetime coordinates, the Latin letters $(a,d,c,\ldots)$ are the world volume indices, and the letters $(i,j,k,\ldots)$ are the normal bundle indices.

[2] Note that there is a minus sign typo on the right-hand side of $(R_N)_{ab}{}^{ij}$ in Ref. [3]. For totally geodesic embedding, $(R_N)_{ab}{}^{ij}$ must be equal to $R_{ab}{}^{ij}$.

$$(\hat{R}_T)_{ab} = \hat{R}_{ab} + \delta_{ij}(\Omega_c{}^{ci}\Omega_{ab}{}^j - \Omega_{ca}{}^i\Omega_b{}^{cj}). \tag{6}$$

In Eq. (4), $\bar{R}^{ij} = \hat{R}^{ij} + g^{ab}g^{cd}\Omega_{ac}{}^i\Omega_{bd}{}^j + \cdots$, where dots stand for unknown terms which involve the trace of the second fundamental form. They could not be fixed in Ref. [3] because the couplings in Ref. [3] have been found by requiring the consistency of the corresponding couplings with the S-matrix element of one closed and two open string vertex operators for which the trace of the second fundamental form is zero. They may be fixed, however, by requiring the consistency of the couplings with dualities.

In the static gauge and to the linear order of fields, the second fundamental form has the simple form

$$\Omega_{ab}{}^i = \partial_a\partial_b\chi^i + \Gamma^i_{ab}, \tag{7}$$

where χ^i is the massless transverse scalar field and Γ^i_{ab} is the Levi-Civitá connection. The couplings of one graviton and two transverse scalars in (4) have been shown to be consistent with the corresponding S-matrix elements [3]. However, there are couplings in (4) which involve the trace of the second fundamental form which cannot be checked with the S-matrix element of one closed and two open string vertex operators. We will show, among other things, that the trace term in $(\hat{R}_T)_{ab}$ is required by the consistency of the couplings (4) with T-duality. Moreover, we will find that the duality fixes the dots in $\bar{R}^{ij}$ to be

$$\bar{R}^{ij} = \hat{R}^{ij} + g^{ab}g^{cd}(\Omega_{ac}{}^i\Omega_{bd}{}^j - \Omega_{ab}{}^i\Omega_{cd}{}^j), \tag{8}$$

where the last term is the trace of the second fundamental form.

It has been observed in Refs. [10,11] that the consistency of the closed string couplings with T-duality requires the couplings of nonconstant dilaton appear in the world volume action via the transformation

$$\hat{R}_{ab} \to \mathcal{R}_{ab} = \hat{R}_{ab} + \partial_a\partial_b\Phi \qquad \hat{R}_{ij} \to \mathcal{R}_{ij} = \hat{R}_{ij} + \partial_i\partial_j\Phi. \tag{9}$$

We will find that the transformation of the couplings (4) under the above replacement produces the couplings of one dilaton and two transverse scalars which are consistent with the dualities and with the corresponding S-matrix elements. In other worlds, the extension of the couplings (2) to include the curvature, the dilaton, and the second fundamental form are

$$S_p^{\rm DBI} \supset -\frac{\pi^2\alpha'^2 T_p}{48}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}[(R_T)_{abcd}(R_T)^{abcd} - 2(\hat{\mathcal{R}}_T)_{ab}(\hat{\mathcal{R}}_T)^{ab} - (R_N)_{abij}(R_N)^{abij} + 2\bar{\mathcal{R}}_{ij}\bar{\mathcal{R}}^{ij}], \tag{10}$$

where $(\hat{\mathcal{R}}_T)_{ab}$ and $\bar{\mathcal{R}}_{ij}$ are the same as $(\hat{R}_T)_{ab}$ and $\bar{R}_{ij}$, respectively, in which the replacement (9) has been performed. We will show that a similar extension exists for the couplings (3); i.e., the consistency of the couplings with dualities and with the S-matrix requires the following extension of (3):

$$S_p^{\rm CS} \supset -\frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\left[\frac{1}{(p+1)!}\nabla_j\mathcal{F}^{(p+2)}_{ia_0\ldots a_p}\bar{\mathcal{R}}^{ij} + \frac{1}{2!p!}\nabla_a\mathcal{F}^{(p+2)}_{ija_1\ldots a_p}(R_N)_{a_0}{}^{aij}\right]. \tag{11}$$

The coupling of the RR field strength and dilaton in the first term above has been already shown in Ref. [10] to be consistent with the linear T-duality and with the S-matrix.

In general, one expects that the consistency of the world volume couplings with full nonlinear T-duality and S-duality would fix all couplings at order α'^2 [11,12]; e.g., the T-duality would relate the couplings (11) to the standard CS couplings $\mathcal{C}^{p-3}(R_T\wedge R_T - R_N\wedge R_N)$ at order α'^2. They would involve also the world volume gauge field, the spacetime B-field, and other RR-fields. In this paper, however, we will use only linear T-duality and S-duality. As a result, we will find many couplings which are consistent with such simplified dualities. We are interested in the couplings of one closed and two open string states in this paper. Even the coefficients of such couplings cannot be fully fixed by the linear dualities. To reduce the number of arbitrary coefficients, we use the consistency of the couplings with the corresponding S-matrix elements as well. This latter condition fixes all unknown coefficients of the couplings in the DBI part; i.e., we will find the couplings (10) and the following couplings in the string frame,

$$\begin{aligned} S_p^{\rm DBI} \supset -\frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}\Big[&\mathcal{R}_{bd}(\partial_aF^{ab}\partial_cF^{cd} - \partial_aF_c{}^d\partial^cF^{ab}) + \frac{1}{2}R_{bdce}\partial^cF^{ab}\partial^eF_a{}^d \\ &+ \frac{1}{4}\mathcal{R}_d{}^d(\partial_aF^{ab}\partial_cF_b{}^c + \partial_bF_a{}^c\partial_cF^{ab}) + \Omega_a{}^{ai}\partial_dH_c{}^d{}_i\partial_bF^{bc} \\ &- \Omega^{bai}\left(\partial_bF_a{}^c\partial_dH_c{}^d{}_i + \partial^dF_a{}^c\partial_iH_{bcd} - \frac{1}{2}\partial^dF_a{}^c\partial_cH_{bdi}\right)\Big], \end{aligned} \tag{12}$$

where the scalar curvature $\mathcal{R}_a{}^a \equiv \tilde{G}^{ab}\hat{R}_{ab} + 2\partial^a\partial_a\Phi$ is invariant under linear T-duality as the Ricci curvatures $\mathcal{R}_{ab}$ and $\mathcal{R}_{ij}$ in (9). The consistency of the couplings with the dualities and with the S-matrix elements fixes also the couplings in the CS part to be those in (11) and the following couplings in the string frame:

$$S_p^{\rm CS} \supset \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\bigg[\frac{1}{2!(p-2)!}\partial^a F_{a_1a_2}\partial_b F_{aa_0}\partial^b \mathcal{F}^{(p-2)}_{a_3a_4\ldots a_p} - \frac{1}{(p-1)!}\Omega_{a_0}{}^{ai}\partial_a F_{ba_1}\partial^b\mathcal{F}^{(p)}_{ia_2a_3\ldots a_p} + \frac{1}{2!(p-1)!}\Omega^{bai}\partial_a F_{a_0a_1}\partial_b\mathcal{F}^{(p)}_{ia_2a_3\ldots a_p} - \frac{1}{2!(p-1)!}\Omega_a{}^{ai}\partial^b F_{a_0a_1}\partial_i\mathcal{F}^{(p)}_{ba_2a_3\ldots a_p} + \frac{1}{p!}\Omega_a{}^{ai}\partial^b F_{ba_0}\partial_i\mathcal{F}^{(p)}_{a_1a_2\ldots a_p} - \frac{1}{p!}\Omega^{bai}\partial_a F_{ba_0}\partial_i\mathcal{F}^{(p)}_{a_1a_2\ldots a_p} + \frac{1}{(p-1)!}\Omega_{a_0}{}^{ai}\partial^b F_{ba_1}\partial_i\mathcal{F}^{(p)}_{aa_2a_3\ldots a_p}\bigg]. \tag{13}$$

In the CS part, there is another multiplet of which the coefficient cannot be fixed by the linear dualities and by the S-matrix elements of one closed and two open strings. It involves, however, the square of the second fundamental form. On the other hand, as the couplings (10) and (11) indicate, the square of the second fundamental form combines with the appropriate curvatures to form world volume curvatures R_T and $\bar{R}$. Since the coefficients of the curvature terms are already fixed in (3), we expect the coefficient of this multiplet to be zero.

An outline of the paper is as follows. In the next section, we review the constraints that linear T-duality and S-duality may impose on an effective world volume action. In Sec. III, we review the contact terms of the S-matrix element of one closed and two open strings at order α'^2. In Sec. IV, we construct all couplings of one NSNS and two NS strings with arbitrary coefficients and find the coefficients by requiring the consistency of the couplings with the linear dualities and with the S-matrix elements. In Sec. V, we construct all couplings of one RR and two NS strings with arbitrary coefficients and find the coefficients by requiring the consistency of the couplings with the linear dualities and with the S-matrix elements.

II. LINEAR DUALITY CONSTRAINTS

The T-duality and S-duality transformations on a massless field are in general nonlinear. Constraining the effective actions to be invariant under these nonlinear transformations, which may fix all couplings of bosonic fields including the nonperturbative effects [13], would be a difficult task (see Refs. [11,12,14] for nonlinear T-duality). In this paper, however, we are interested only in the world volume couplings of one massless closed and two open string states at order α'^2. Using the fact that the world volume couplings of one closed string and the couplings of one closed and one open string have no higher-derivative corrections in the superstring theory, one realizes that the higher-derivative couplings of one closed and two open string states must be invariant under linear duality transformations.

The full set of nonlinear T-duality transformations has been found in Refs. [15–19]. We consider a background consisting of a constant dilaton ϕ_0 and a metric which is flat in all directions except the Killing direction y, which is a circle with radius ρ. Assuming quantum fields are small perturbations around this background, e.g., $G_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$ and $G_{yy} = \frac{\rho^2}{\alpha'}(1+2h_{yy})$ where $\mu,\nu \neq y$, the T-duality transformations for the background are $e^{2\tilde{\phi}_0} = \frac{\alpha' e^{2\phi_0}}{\rho^2}$, $\tilde{G}_{\mu\nu} = \eta_{\mu\nu}$, and $\tilde{G}_{yy} = \frac{\alpha'}{\rho^2}$, and the quantum fluctuations at the linear order take the following form[3]:

$$\tilde{\phi} = \phi - \frac{1}{2}h_{yy}, \qquad \tilde{h}_{yy} = -h_{yy}, \qquad \tilde{h}_{\mu y} = B_{\mu y},$$
$$\tilde{B}_{\mu y} = h_{\mu y}, \qquad \tilde{h}_{\mu\nu} = h_{\mu\nu}, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu}$$
$$\tilde{C}^{(n)}_{\mu\ldots\nu y} = C^{(n-1)}_{\mu\ldots\nu}, \qquad \tilde{C}^{(n)}_{\mu\ldots\nu} = C^{(n+1)}_{\mu\ldots\nu y}. \tag{14}$$

The T-duality transformation of the world volume gauge field when it is along the Killing direction is $\tilde{A}_y = \chi_y$, where χ_y is the transverse scalar. Similarly, $\tilde{\chi}_y = A_y$. When the gauge field and the transverse scalar field are not along the Killing direction, they are invariant under the T-duality. We are interested in applying the above linear T-duality transformations on the quantum fluctuations and apply the full nonlinear T-duality on the background. The latter requires the CS part to have no overall dilaton factor and the DBI part to have the overall factor $e^{-\Phi}\sqrt{-\tilde{G}}$.

[3]Note that if one considers full T-duality transformation for background and quantum fluctuations then the effective action would contain all couplings at order α'^2, e.g., H^4 or $(\partial F)^2H^2$. However, in this paper, we are interested only in the couplings consisting of one closed and two open string fields, and hence we consider only linear T-duality.

Following Ref. [20], the effective couplings which are invariant under the above linear T-duality can be constructed as follows. We first write, in the static gauge, all couplings on the world volume of the D_p-brane involving one massless closed and two open string states, in terms of the world volume indices $a, b, \ldots$ and the transverse indices $i, j, \ldots$. We call this action S_p. Then, we reduce the action to the nine-dimensional space. It produces two different actions. In one of them, the Killing direction y is a world volume direction, i.e., $a = (\tilde{a}, y)$, which we call S_p^w, and in the other one, the Killing direction y is a transverse direction, $i = (\tilde{i}, y)$, which we call S_p^t. The transformation of S_p^w under the linear T-duality (14), which we call S_{p-1}^{wT}, must be equal to S_{p-1}^t up to some total derivative terms, i.e.,

$$S_{p-1}^{wT} - S_{p-1}^t = 0. \tag{15}$$

This constrains the unknown coefficients in the original action S_p.

The S-duality of type-IIB theory produces another set of constraints on the coefficients of S_p. Under the S-duality, the graviton in the Einstein frame, i.e., $G_{\mu\nu}^E = e^{-\Phi/2} G_{\mu\nu}$; the transverse scalar fields; and the RR 4-form are invariant, and the following objects transform as doublets [21–23],

$$\begin{aligned} \mathcal{B} &\equiv \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} \\ \mathcal{F} &\equiv \begin{pmatrix} *F \\ G(F) \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} *F \\ G(F) \end{pmatrix}, \end{aligned} \tag{16}$$

where the matrix $\Lambda \in SL(2,Z)$ and $G(F)$ is a nonlinear function of F, Φ, C. To the linear order of the quantum fluctuations and nonlinear background which we call linear S-duality,[4] $G(F) = e^{-\phi_0} F$, where ϕ_0 is the constant dilaton background [21]. In the above equation, $(*F)_{ab} = \epsilon_{abcd} F^{cd}/2$. The transformation of the dilaton and the RR scalar C appears in the transformation of the $SL(2,Z)$ matrix $\mathcal{M}$,

$$\mathcal{M} = e^{\phi} \begin{pmatrix} |\tau|^2 & C \\ C & 1 \end{pmatrix}, \tag{17}$$

where $\tau = C + ie^{-\Phi}$. This matrix transforms as [21]

$$\mathcal{M} \to \Lambda \mathcal{M} \Lambda^T. \tag{18}$$

[4]Note that we consider finite $SL(2,Z)$ transformation but infinitesimal quantum fluctuations.

To the zeroth and the first orders of quantum fluctuations and the nonlinear order of the background field ϕ_0, the matrix $\mathcal{M}$ is

$$\mathcal{M}_0 = \begin{pmatrix} e^{-\phi_0} & 0 \\ 0 & e^{\phi_0} \end{pmatrix}, \qquad \delta\mathcal{M} = \begin{pmatrix} -e^{-\phi_0}\phi & e^{\phi_0} C \\ e^{\phi_0} C & e^{\phi_0}\phi \end{pmatrix}. \tag{19}$$

They transform as (18) under the $SL(2,Z)$ transformations.

Using the above transformations, it is obvious that there must be no couplings in the Einstein frame between one dilaton and two transverse scalars because it is impossible to construct $SL(2,Z)$ invariant from $\mathcal{M}_0$ and one $\delta\mathcal{M}$, i.e., $\mathrm{Tr}(\mathcal{M}_0^{-1}\delta\mathcal{M}) = 0$. This produces a set of constraints on the coefficients of the effective action S_p.

One can easily find that the following structures are invariant under the linear S-duality transformation:

$$\begin{aligned} \partial(*\mathcal{F}^T)\mathcal{M}_0\partial^2\mathcal{B} &= e^{-\phi_0}\partial F\partial^2 B - \partial(*F)\partial^2 C^{(2)} \\ \partial\mathcal{F}^T\mathcal{M}_0\partial\mathcal{F} &= e^{-\phi_0}[\partial(*F)\partial(*F) + \partial F\partial F] \\ \partial\mathcal{F}^T\partial^2\mathcal{M}\partial\mathcal{F} &= e^{-\phi_0}\partial^2\Phi\partial F\partial F - e^{-\phi_0}\partial^2\Phi\partial(*F)\partial(*F) \\ &\quad + \partial^2 C\partial F\partial(*F) + \partial^2 C\partial(*F)\partial F. \end{aligned} \tag{20}$$

Up to total derivative terms then, the couplings of one closed and two open string states on the world volume of the D_3-brane should appear in the structures $R\Omega\Omega$, $\partial^2 C^{(4)}\Omega\Omega$, $\Omega\partial(*\mathcal{F}^T)\mathcal{M}_0\partial^2\mathcal{B}$, $R\partial\mathcal{F}^T\mathcal{M}_0\partial\mathcal{F}$, and $\partial\mathcal{F}^T\partial^2\mathcal{M}\partial\mathcal{F}$, which are invariant under the linear S-duality. They constrain the coefficients of the couplings in S_p.

III. S-MATRIX CONSTRAINTS

Another set of constraints on the coefficients of S_p is produced by comparing the couplings with the S-matrix element of one closed and two open string states at order α'^2. This S-matrix element has been calculated in [5]

$$A \sim \frac{\Gamma[-2t]}{\Gamma[1-t]^2} K(1,2,3), \tag{21}$$

where K is the kinematic factor and $t = -\alpha' k_1 \cdot k_2$ is the only Mandelstam variable in the amplitude. k_1 and k_2 are the open string momenta. The low-energy expansion of the gamma functions is $\frac{\Gamma[-2t]}{\Gamma[1-t]^2} = -\frac{1}{2t} - \frac{\pi^2 t}{12} + \cdots$. The first term produces the couplings which is consistent with the corresponding couplings in DBI and CS actions at order α'^0 [24]. The second term produces the on-shell couplings in the Einstein frame when the closed string is a NSNS state [24],

$$
\begin{aligned}
A(\chi,\chi,h) &\sim (2k_1\cdot k_2\zeta_1\cdot\varepsilon_3\cdot\zeta_2 + k_1\cdot k_2\zeta_1\cdot\zeta_2\varepsilon_{3a}{}^a + \zeta_1\cdot p_3\zeta_2\cdot p_3\varepsilon_{3a}{}^a \\
&\quad - 2k_1\cdot\varepsilon_3\cdot k_2\zeta_1\cdot\zeta_2 + 4\zeta_1\cdot\varepsilon_3\cdot k_1\zeta_2\cdot p_3 + (1\longleftrightarrow 2))(k_1\cdot k_2)^2 \\
A(\chi,\chi,\phi) &\sim \frac{p-3}{2\sqrt{2}}(k_1\cdot k_2\zeta_1\cdot\zeta_2 + \zeta_1\cdot p_3\zeta_2\cdot p_3 + (1\longleftrightarrow 2))(k_1\cdot k_2)^2 \\
A(\chi,a,b) &\sim -2i(2k_1^a\zeta_{1i}f_{2ab}\varepsilon_3^{bi} - \zeta_1\cdot p_3 f_{2ab}\varepsilon_3^{ab})(k_1\cdot k_2)^2 \\
A(a,a,h) &\sim 2\left(\varepsilon_{3ab}f_1{}^{ac}f_2{}^b{}_c - \frac{1}{4}f_{1ab}f_2{}^{ab}\varepsilon_{3a}{}^a + (1\longleftrightarrow 2)\right)(k_1\cdot k_2)^2 \\
A(a,a,\phi) &\sim -\frac{p-7}{4\sqrt{2}}(f_{1ab}f_2{}^{ab} + (1\longleftrightarrow 2))(k_1\cdot k_2)^2,
\end{aligned}
$$

where ζ_1, ζ_2 are the polarizations of the open string states and ε_3 is the polarization of the closed string. For the RR state, the couplings in the momentum space are [24]

$$
\begin{aligned}
A(\chi,\chi,c_{(p+1)}) &\sim -\frac{2}{(p+1)!}(\zeta_1\cdot p_3\zeta_2\cdot p_3\varepsilon_3{}^{a_0\dots a_p} + 2(p+1)\zeta_1^i k_1^{a_0}\zeta_2\cdot p_3\varepsilon_{3i}{}^{a_1\dots a_p} \\
&\quad + p(p+1)\zeta_1^i\zeta_2^j k_1^{a_0}k_2^{a_1}\varepsilon_{3ij}{}^{a_2\dots a_p})\epsilon^v_{a_0\dots a_p}(k_1\cdot k_2)^2 + (1\longleftrightarrow 2) \\
A(\chi,a,c_{(p-1)}) &\sim -\frac{2}{(p-1)!}(\zeta_1\cdot p_3 f_2^{a_0a_1}\varepsilon_3{}^{a_2\dots a_p} + (p-1)\zeta_1^i f_2^{a_0a_1}k_1^{a_2}\varepsilon_{3i}{}^{a_3\dots a_p})\epsilon^v_{a_0\dots a_p}(k_1\cdot k_2)^2 \\
A(a,a,c_{(p-3)}) &\sim -\frac{1}{2(p-3)!}f_1{}^{a_0a_1}f_2{}^{a_2a_3}\varepsilon_3{}^{a_4\dots a_p}\epsilon^v_{a_0\dots a_p}(k_1\cdot k_2)^2 + (1\longleftrightarrow 2).
\end{aligned}
$$

Compatibility of the couplings with the above amplitudes constrains the coefficients in S_p.

It has been argued in Ref. [12] that to construct the effective action for probe branes one has to impose the bulk equations of motion at order α'^0 into S_p. Since we are interested in the world volume couplings which have linear closed string fields, we have to impose the supergravity equations of motion at linear order, i.e.,

$$
\begin{aligned}
R + 4\nabla^2\Phi &= 0 \\
R_{\mu\nu} + 2\nabla_{\mu\nu}\Phi &= 0 \\
\nabla^\rho H_{\rho\mu\nu} &= 0 \\
\nabla^{\mu_1}\mathcal{F}^{(n)}_{\mu_1\mu_2\dots\mu_n} &= 0,
\end{aligned} \tag{22}
$$

where μ, ν, ρ are the bulk indices. Using these equations, one finds

$$
\begin{aligned}
R_\mu{}^i{}_{\nu i} &= -2\nabla_{\mu\nu}\Phi - R^c_{\mu\ \nu c} \\
\nabla^i{}_i\Phi &= -\nabla^a{}_a\Phi \\
\nabla^i H_{i\mu\nu} &= -\nabla^a H_{a\mu\nu} \\
\nabla^i\mathcal{F}^{(n)}_{i\mu_2\dots\mu_n} &= -\nabla^a\mathcal{F}^{(n)}_{a\mu_2\dots\mu_n},
\end{aligned} \tag{23}
$$

which indicates that the terms on the left-hand side are not independent. In other words, the coefficients of the couplings in S_p which involve the terms on the left-hand side above must be zero.

IV. DBI COUPLINGS

In this section, using the mathematica package "xAct" [25], we are going to write all couplings of one closed string NSNS state and two open strings with unknown coefficients. We then constrain the coefficients by imposing the consistency of the couplings with the linear dualities and with the corresponding S-matrix element. Since all such couplings are too many to write at once, we consider the couplings with a specific closed string NSNS state and open string Neveu-Schwarz (NS) states.

A. One graviton and two transverse scalar fields

We begin with the couplings of one graviton and two transverse scalar fields. The transverse scalar fields should appear in the action through the pullback of bulk tensors, through the Taylor expansion of bulk tensors, or through the second fundamental form. Since there is no higher-derivative correction to the couplings of one closed string and one open string in type-II superstring theories, e.g., there is no coupling with structure $DR\Omega$ or $RD\Omega$, the pullback operator and Taylor expansion would produce no coupling between two scalars and one curvature from $DR\Omega$ or $RD\Omega$. Therefore, the only possibility for the two transverse scalars is through the second fundamental form. All such couplings at order α'^2 are

$$S_{h\chi\chi} = \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}[w_1 R^{bc}{}_{bc}\Omega_a{}^{ai}\Omega_d{}^d{}_i + 2w_2 R^{bj}{}_{bj}\Omega_a{}^{ai}\Omega_c{}^c{}_i + w_3 R^{ij}{}_{ij}\Omega_a{}^{ak}\Omega_b{}^b{}_k$$
$$+ w_4 R^b{}_{ibj}\Omega_a{}^{ai}\Omega_c{}^{cj} + w_5 R_i{}^j{}_{kj}\Omega_a{}^{ai}\Omega_b{}^{bk} + w_6 R^{bc}{}_{bc}\Omega_{dai}\Omega^{dai}$$
$$+ 2w_7 R^{bj}{}_{bj}\Omega_{cai}\Omega^{cai} + w_8 R^{kj}{}_{kj}\Omega_{bai}\Omega^{bai} + w_9 R^b{}_{ibj}\Omega_{ca}{}^j\Omega^{cai}$$
$$+ w_{10} R_i{}^j{}_{kj}\Omega_{ba}{}^k\Omega^{bai} + w_{11} R_d{}^c{}_{bc}\Omega^b{}_{ai}\Omega^{dai} + w_{12} R_c{}^j{}_{bj}\Omega^b{}_{ai}\Omega^{cai}$$
$$+ w_{13} R_{cbij}\Omega^b{}_a{}^j\Omega^{cai} + w_{14} R_{cibj}\Omega^b{}_a{}^j\Omega^{cai} - w_{15} R_{cjbi}\Omega^b{}_a{}^j\Omega^{cai}$$
$$+ w_{16} R_d{}^c{}_{bc}\Omega_a{}^{ai}\Omega^{bd}{}_i + w_{17} R_c{}^j{}_{bj}\Omega_a{}^{ai}\Omega^{bc}{}_i + w_{18} R_{cibj}\Omega_a{}^{ai}\Omega^{bcj}$$
$$+ w_{19} R_{abdc}\Omega^{cb}{}_i\Omega^{dai}], \tag{24}$$

where w_i with $i = 1, 2, \ldots, 19$ are the unknown constants that must be determined by imposing various constraints.

All the above couplings are not independent. In fact, by applying the cyclic symmetry of the Riemann curvature, one can neglect some of the constants. For example, one finds the couplings in (24) with coefficients w_{13}, w_{14}, and w_{15} are not independent, i.e.,

$$w_{13} R_{cbij}\Omega^b{}_a{}^j\Omega^{cai} + w_{14} R_{cibj}\Omega^b{}_a{}^j\Omega^{cai} - w_{15} R_{cjbi}\Omega^b{}_a{}^j\Omega^{cai}$$
$$= (w_{13} + w_{15}) R_{cbij}\Omega^b{}_a{}^j\Omega^{cai}$$
$$+ (w_{14} - w_{15}) R_{cibj}\Omega^b{}_a{}^j\Omega^{cai}, \tag{25}$$

so the coupling with coefficient w_{15} is not independent and may be ignored from the list (24) before imposing various constraints. Alternatively, one may keep all couplings in (24) and impose the constraints to find appropriate relations between the coefficients and at the end impose the cyclic symmetry. The final result of course must be identical in both methods. However, we find the latter method is easier to apply by computer, so we do it in this paper. In fact, after imposing the constraints, we write the Riemann curvature in terms of the metric. Then, all terms that are related by the cyclic symmetry would be canceled. So the coefficients of all such terms can easily be set to zero.

By comparing the above couplings with (4), we find $w_9 = 1$, $w_{11} = 1$, $w_{16} = -1$, and $w_{19} = 1$. These constraints are in fact the S-matrix constraints because the couplings in (4) are fixed in Ref. [3] by comparing them with the corresponding S-matrix elements. Furthermore, the constraint that the bulk equations of motion (23) have to be imposed on the brane couplings fixes the coefficients $w_2 = w_3 = w_5 = w_7 = w_8 = w_{10} = w_{12} = w_{17} = 0$.

B. One graviton and two gauge fields

Under T-duality, the transverse scalar field along the Killing direction transforms to the gauge field; i.e., Ω transform to ∂F. So consistency of the couplings (24) with T-duality requires the couplings of one graviton and two gauge fields to have structure $R\partial F\partial F$. All such couplings are

$$S_{haa} = \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}[z_1 R^{cd}{}_{cd}\partial_a F^{ae}\partial_b F_e{}^b + 2z_2 R^{ci}{}_{ci}\partial_a F^{ad}\partial_b F_d{}^b$$
$$+ z_3 R^{ij}{}_{ij}\partial_a F^{ac}\partial_b F_c{}^b + z_4 R_e{}^d{}_{cd}\partial_a F^{ae}\partial_b F^{bc} + z_5 R_d{}^i{}_{ci}\partial_a F^{ad}\partial_b F^{bc} + z_6 R_e{}^d{}_{cd}\partial_a F_b{}^c\partial^b F^{ae}$$
$$+ z_7 R_d{}^i{}_{ci}\partial_a F_b{}^c\partial^b F^{ad} + z_8 R^{cd}{}_{cd}\partial^b F^{ae}\partial_e F_{ab} + 2z_9 R^{ci}{}_{ci}\partial^b F^{ad}\partial_d F_{ab} + z_{10} R^{ij}{}_{ij}\partial^b F^{ac}\partial_c F_{ab}$$
$$+ z_{11} R^{cd}{}_{cd}\partial_b F_{ae}\partial^b F^{ae} + 2z_{12} R^{ci}{}_{ci}\partial_b F_{ad}\partial^b F^{ad} + z_{13} R^{ij}{}_{ij}\partial_b F_{ac}\partial^b F^{ac} + z_{14} R_e{}^d{}_{cd}\partial_b F_a{}^c\partial^b F^{ae}$$
$$+ z_{15} R_d{}^i{}_{ci}\partial_b F_a{}^c\partial^b F^{ad} + z_{16} R_{aecd}\partial_b F^{cd}\partial^b F^{ae} + z_{17} R_{aced}\partial_b F^{cd}\partial^b F^{ae} + z_{18} R_b{}^d{}_{cd}\partial^b F^{ae}\partial^c F_{ae}$$
$$+ z_{19} R_b{}^i{}_{ci}\partial^b F^{ad}\partial^c F_{ad} + z_{20} R_e{}^d{}_{cd}\partial^b F^{ae}\partial^c F_{ab} + z_{21} R_d{}^i{}_{ci}\partial^b F^{ad}\partial^c F_{ab} + z_{22} R_b{}^d{}_{cd}\partial_a F^{ae}\partial^c F_e{}^b$$
$$+ z_{23} R_b{}^i{}_{ci}\partial_a F^{ad}\partial^c F_d{}^b + z_{24} R_{ebcd}\partial^b F^{ae}\partial^d F_a{}^c + z_{25} R_{ecbd}\partial^b F^{ae}\partial^d F_a{}^c + z_{26} R_{edbc}\partial^b F^{ae}\partial^d F_a{}^c$$
$$+ z_{27} R_{aecd}\partial^b F^{ae}\partial^d F_b{}^c + z_{28} R_{aced}\partial^b F^{ae}\partial^d F_b{}^c + z_{29} R_{ebcd}\partial_a F^{ae}\partial^d F^{bc} + z_{30} R_{edbc}\partial_a F^{ae}\partial^d F^{bc}], \tag{26}$$

where z_i with $i = 1, 2, \ldots, 30$ are constants that must be determined by imposing the constraints and F^{ab} is field strength of the gauge field. Here also one may impose the cyclic symmetry and the Bianchi identity $dF = 0$ before imposing the constraints to cancel some of the couplings in (26) before. However, we prefer to impose the cyclic symmetry and the Bianchi identity after imposing the constraints. The bulk equations of motion (23) constrain $z_2 = z_3 = z_5 = z_7 = z_9 = z_{10} = z_{12} = z_{13} = z_{15} = z_{19} = z_{21} = z_{23} = 0$.

C. One dilaton and two transverse scalar fields

The same reason as in Sec. IV. A leads one to conclude that the couplings of one dilaton and two transverse scalar fields have structure $\partial\partial\Phi\Omega\Omega$. All such couplings are

$$S_{\Phi\chi\chi} = \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}[t_1\Omega^a{}_a{}^i\Omega^b{}_{bi}\partial_c\partial^c\Phi + t_2\Omega^a{}_a{}^i\Omega^{bc}{}_i\partial_c\partial_b\Phi + t_3\Omega_a{}^c{}_i\Omega^{abi}\partial_c\partial_b\Phi$$
$$+ t_4\Omega_{abi}\Omega^{abi}\partial_c\partial^c\Phi + t_5\Omega^a{}_a{}^i\Omega^b{}_{bi}\partial_j\partial^j\Phi + t_6\Omega_{abi}\Omega^{abi}\partial_j\partial^j\Phi$$
$$+ t_7\Omega^a{}_a{}^i\Omega^b{}_b{}^j\partial_j\partial_i\Phi + t_8\Omega_{ab}{}^j\Omega^{abi}\partial_j\partial_i\Phi], \tag{27}$$

where t_i with $i = 1, 2, \ldots, 8$ are the unknown constants that we must determined. The bulk equations of motion (23) constrain $t_5 = t_6 = 0$.

D. One dilaton and two gauge fields

The consistency of the couplings (27) with T-duality requires the couplings of one dilaton and two gauge fields to have the structure $\partial\partial\Phi\partial F\partial F$. All such couplings are

$$S_{\Phi aa} = \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}[x_1\partial_a F^{cd}\partial_b F_{cd}\partial^b\partial^a\Phi + x_2\partial^b\partial^a\Phi\partial_c F_a{}^c\partial_d F_b{}^d$$
$$+ x_3\partial_a\partial^a\Phi\partial_b F^{bc}\partial_d F_c{}^d + x_4\partial_b F_a{}^c\partial^b\partial^a\Phi\partial_d F_c{}^d + x_5\partial_b F_{cd}\partial^b\partial^a\Phi\partial^d F_a{}^c$$
$$+ x_6\partial^b\partial^a\Phi\partial_c F_{bd}\partial^d F_a{}^c + x_7\partial^b\partial^a\Phi\partial_d F_{bc}\partial^d F_a{}^c + x_8\partial_a\partial^a\Phi\partial_c F_{bd}\partial^d F^{bc}$$
$$+ x_9\partial_a\partial^a\Phi\partial_d F_{bc}\partial^d F^{bc} + x_{10}\partial_a F^{ab}\partial_c F_b{}^c\partial_i\partial^i\Phi + x_{11}\partial_b F_{ac}\partial^c F^{ab}\partial_i\partial^i\Phi$$
$$+ x_{12}\partial_c F_{ab}\partial^c F^{ab}\partial_i\partial^i\Phi], \tag{28}$$

where the constants x_i with $i = 1, 2, \ldots, 12$ must be determined by imposing the constraints. The bulk equations of motion (23) constrain $x_{10} = x_{11} = x_{12} = 0$.

E. One B-field, one transverse scalar field, and one gauge field

The final list of couplings in the DBI part is the couplings of one B-field, one transverse scalar field, and one gauge field, which is

$$S_{ba\chi} = \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x e^{-\Phi}\sqrt{-\tilde{G}}[\gamma_1\Omega^{abi}\partial_a F^{cd}\partial_b H_{cdi} + \gamma_2\Omega^{abi}\partial_b H_{adi}\partial_c F^{cd}$$
$$+ \gamma_3\Omega^{abi}\partial_a F^{cd}\partial_d H_{bci} + \gamma_4\Omega^{abi}\partial_c F_a{}^c\partial_d H_b{}^d{}_i + \gamma_6\Omega^{abi}\partial_b F_a{}^c\partial_d H_c{}^d{}_i$$
$$+ \gamma_5\Omega^a{}_a{}^i\partial_b F^{bc}\partial_d H_c{}^d{}_i + \gamma_7\Omega^{abi}\partial_b H_{cdi}\partial^d F_a{}^c + \gamma_8\Omega^{abi}\partial_c H_{bdi}\partial^d F_a{}^c$$
$$+ \gamma_9\Omega^{abi}\partial_d H_{bci}\partial^d F_a{}^c + \gamma_{10}\Omega^a{}_a{}^i\partial_c H_{bdi}\partial^d F^{bc} + \gamma_{11}\Omega^a{}_a{}^i\partial_d H_{bci}\partial^d F^{bc}$$
$$+ \gamma_{12}\Omega^{abi}\partial_a F^{cd}\partial_i H_{bcd} + \gamma_{13}\Omega^{abi}\partial^d F_a{}^c\partial_i H_{bcd} + \gamma_{14}\Omega^a{}_a{}^i\partial^d F^{bc}\partial_i H_{bcd}$$
$$- \gamma_{15}\Omega^{abi}\partial_c F_a{}^c\partial_j H_{bi}{}^j - \gamma_{17}\Omega^{abi}\partial_b F_a{}^c\partial_j H_{ci}{}^j - \gamma_{16}\Omega^a{}_a{}^i\partial_b F^{bc}\partial_j H_{ci}{}^j], \tag{29}$$

where γ_i with $i = 1, 2, \ldots, 17$ are the unknown constants. The equations of motion (23) fix $\gamma_{15} = \gamma_{16} = \gamma_{17} = 0$.

We now consider the sum of the couplings in (24), (26), (27), (28), and (29), i.e.,

$$S_p^{\rm DBI} = S_{h\chi\chi} + S_{haa} + S_{\Phi\chi\chi} + S_{\Phi aa} + S_{ba\chi}, \tag{30}$$

and apply the T-duality constraint (15). It gives the following relations between the constants:

$$t_8 = 1, \qquad t_3 = -t_2, \qquad t_1 = -\frac{t_2}{2} - \frac{x_4}{2} - x_3, \qquad t_4 = \frac{t_2}{2} + \frac{x_4}{2} + x_8 - 2x_9$$
$$t_7 = -1 - z_{14} - z_4 - z_6, \qquad w_1 = \frac{1}{4} - \frac{x_3}{2} - \frac{x_4}{4}, \qquad w_{15} = 2 + w_{14} - 2\gamma_1 + \gamma_3 + 2\gamma_6 + \gamma_7 - \gamma_9$$

$$
\begin{aligned}
&w_{18}=2+2z_{14}-z_{29}+2z_{30}+2z_6, \qquad w_4=-1-z_{14}-z_4-z_6, \qquad x_5=2x_1-x_4+x_7\\
&z_{20}=-z_{14}-2z_{18}+z_{22}, \qquad \gamma_{13}=w_{14}+2\gamma_{12}+\gamma_3+\gamma_6-\gamma_9, \qquad \gamma_5=z_{14}+z_4+z_6-\gamma_6\\
&z_{28}=2z_{14}+4z_{16}+2z_{17}+z_{24}-z_{26}-2z_{27}+2z_6+2\gamma_1-\gamma_3-2\gamma_6-\gamma_7+\gamma_9\\
&\gamma_8=\frac{1}{2}-\gamma_4-\gamma_9, \qquad w_6=-\frac{1}{4}+\frac{x_4}{4}+\frac{x_8}{2}+x_9, \qquad z_1=\frac{x_3}{2}+\frac{x_4}{4}-\frac{z_{22}}{4}, \qquad z_{25}=\frac{1}{2}+\frac{z_{22}}{2}-z_{26}\\
&z_8=\frac{x_4}{4}+\frac{x_8}{2}-\frac{z_{22}}{4}+x_9-2z_{11}, \qquad \gamma_2=-\frac{z_{29}}{2}+z_{14}+z_{30}+z_6-\gamma_6\\
&\gamma_{11}=\frac{z_{14}}{2}-\frac{z_{29}}{4}+\frac{z_{30}}{2}+\frac{z_6}{2}-\frac{\gamma_{10}}{2}-\frac{\gamma_6}{2}, \qquad x_6=-x_2-x_7+z_{14}+z_4+z_6.
\end{aligned} \tag{31}
$$

As can be seen, not all coefficients of the DBI part are fixed by imposing the consistency of the couplings with the linear T-duality, so we need further constraints which may be the consistency with S-duality.

In general, S-duality connects the DBI couplings containing the NSNS states to the CS couplings containing RR states. However, the S-duality constrains even the couplings in the DBI part. For example, the world volume couplings of the D_3-brane in the Einstein frame must have no coupling with the structure $\Phi\Omega\Omega$. This produces the following constraints:

$$
\begin{aligned}
&\gamma_9=2-2\gamma_1+\gamma_3+2\gamma_6+\gamma_7, \quad x_9=\frac{1}{4}-\frac{x_4}{4}-\frac{x_8}{2}\\
&z_4=\frac{1}{2}+x_3+\frac{x_4}{2}-\frac{z_{29}}{2}+z_{30}, \quad z_6=-1-z_{14}-z_{30}+\frac{z_{29}}{2}.
\end{aligned} \tag{32}
$$

Another constraint from the S-duality in the DBI part is that up to total derivative terms the couplings of one graviton and two gauge fields in the D_3-brane action must appear in the S-duality invariant structure $R\partial\mathcal{F}^T\mathcal{M}_0\partial\mathcal{F}=e^{-\phi_0}R(\partial(*F)\partial(*F)+\partial F\partial F)$. This produces the following constraints:

$$
z_{14}=0, \qquad x_4=1-2x_3. \tag{33}
$$

The S-duality constrains the couplings of one dilaton and two gauge fields. It also connects them to the couplings of one RR scalar and two gauge fields. This results from the fact that the S-duality invariant structure which contains the couplings of one dilaton and two gauge fields is $\partial\mathcal{F}^T\partial^2\mathcal{M}\partial\mathcal{F}=e^{-\phi_0}\partial^2\Phi(-\partial(*F)\partial(*F)+\partial F\partial F)+\cdots$, where dots refer to the RR scalar couplings. This constraint on the couplings of one dilaton and two gauge fields produces the following relation:

$$
x_8=x_3.
$$

The S-duality connects the DBI couplings of one B-field, one gauge field, and one transverse scalar field to the CS couplings of one RR 2-form, one gauge field, and one transverse scalar. In the next section, we will write all couplings in the CS part and impose the T-duality condition (15). Then, we will impose the above S-duality condition. It produces the following relation between the coefficients in the DBI part,

$$
\gamma_6=-1+2\gamma_1-\gamma_3, \tag{34}
$$

and many relations between the coefficients in the CS part [see the constraints in (43)].

Imposing the above relations between the coefficients in $S_p^{\rm DBI}$, we find that the action (30) is consistent with the S-matrix elements in (22) except the following terms:

$$
w_{14}(R_{bicj}\Omega^{bai}\Omega^c{}_a{}^j-R_{bjci}\Omega^{bai}\Omega^c{}_a{}^j+\Omega^{bai}\partial^d F_a{}^c\partial_i H_{bcd}). \tag{35}
$$

They are not consistent with the couplings in (4) and with the corresponding S-matrix elements, so

$$
w_{14}=0. \tag{36}
$$

As can be seen, there are still many coefficients which are not fixed by the linear dualities and with the S-matrix elements.

We have considered all couplings in $S_p^{\rm DBI}$ which contain the Riemann curvature and the first derivative of the field strengths of the gauge field and the B-field. The Riemann curvature satisfies the cyclic symmetry, and the field strengths satisfy the Bianchi identities. So we have to impose these symmetries in $S_p^{\rm DBI}$. To perform this step, we write all field strengths in terms of their corresponding potentials and write the Riemann curvature in terms of

$$
R_{abcd}=\partial_b\partial_c h_{ad}+\partial_a\partial_d h_{bc}-\partial_b\partial_d h_{ac}-\partial_a\partial_c h_{bd}. \tag{37}
$$

Then, we find the coefficients γ_1, γ_3, γ_7, γ_9, x_4, x_8, x_9, z_4, z_6, z_{14}, disappear from the action. As a result, the terms with these coefficients represent only the cyclic symmetry and the Bianchi identity. So we ignore such terms in the DBI part. Finally, we find that the couplings with coefficients γ_4, t_2, x_2, x_3, z_{22}, z_{29}, z_{30} are total derivative terms, so they can be eliminated from the DBI part. too. The final result for the DBI part has no unknown coefficients. The couplings are those that appear in (10) and (12).

V. CS COUPLINGS

In this section, using the mathematica package "xAct" [25], we are going to write all couplings of one closed string RR state and two open NS strings with unknown coefficients. We then constrain the coefficients by imposing the consistency of the couplings with the linear dualities and with the corresponding S-matrix element. The S-matrix elements (22) indicate that the world volume couplings of the D_p-brane in the CS part has three parts. One is the couplings of one C_{p-3} and two gauge fields, and another one is the couplings of one C_{p-1}, one gauge field, and one transverse scalar field, and the last one is the couplings of one C_{p+1} and two transverse scalar fields. Let us consider each case separately.

A. One RR and two gauge fields

In this section, we construct all possible couplings of one C_{p-3} and two gauge fields. Using the bulk equations of motion (23), one finds there are 23 nonzero couplings, i.e.,

$$\begin{aligned}
S_{caa} = {}& \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\bigg[\frac{1}{(p-2)!}\partial_a\mathcal{F}^{(p-2)}_{a_3a_4\ldots a_p}\bigg(\frac{1}{2!}\kappa_1\partial_bF_{a_1a_2}\partial^bF^a{}_{a_0}+\kappa_2\partial_{a_0}F^{ba}\partial_{a_2}F_{ba_1}\\
&+\kappa_3\partial_{a_2}F_{ba_1}\partial^aF^b{}_{a_0}+\kappa_4\partial_{a_2}F_{ba_1}\partial^bF^a{}_{a_0}+\frac{1}{2!}\kappa_5\partial_{a_2}F_b{}^a\partial^bF_{a_0a_1}\\
&+\frac{1}{2!}\kappa_6\partial^aF_{ba_2}\partial^bF_{a_0a_1}+\kappa_7\partial_bF^b{}_{a_0}\partial_{a_2}F^a{}_{a_1}+\frac{1}{2!}\kappa_8\partial_bF^b{}_{a_0}\partial^aF_{a_1a_2}\bigg)\\
&+\frac{1}{(p-3)!}\partial^b\mathcal{F}^{(p-2)}{}_{ba_4\ldots a_p}\bigg(\frac{1}{2!}\frac{1}{2!}\kappa_9\partial_aF_{a_2a_3}\partial^aF_{a_0a_1}\\
&+\kappa_{10}\partial_{a_1}F^a{}_{a_0}\partial_{a_3}F_{aa_2}+\frac{1}{2!}\kappa_{11}\partial^aF_{a_0a_1}\partial_{a_3}F_{aa_2}\bigg)\\
&+\frac{1}{(p-3)!}\partial_a\mathcal{F}^{(p-2)}_{ba_4\ldots a_p}\bigg(\kappa_{12}\partial_{a_1}F^b{}_{a_0}\partial_{a_3}F^a{}_{a_2}+\frac{1}{2!}\kappa_{13}\partial_{a_3}F^a{}_{a_2}\partial^bF_{a_0a_1}\\
&+\frac{1}{2!}\kappa_{14}\partial_{a_1}F^b{}_{a_0}\partial^aF_{a_2a_3}+\frac{1}{2!}\frac{1}{2!}\kappa_{15}\partial^bF_{a_0a_1}\partial^aF_{a_2a_3}\bigg)\\
&+\frac{1}{(p-3)!}\partial_{a_4}\mathcal{F}^{(p-2)}_{aa_3a_5\ldots a_p}\bigg(\frac{1}{2!}\kappa_{16}\partial_bF_{a_1a_2}\partial^bF^a{}_{a_0}+\kappa_{17}\partial_{a_2}F_{ba_1}\partial^bF^a{}_{a_0}\\
&+\kappa_{18}\partial_{a_0}F^{ba}\partial_{a_2}F_{ba_1}+\kappa_{19}\partial_{a_2}F_{ba_1}\partial^aF^b{}_{a_0}+\kappa_{20}\partial_bF^b{}_{a_0}\partial_{a_2}F^a{}_{a_1}\\
&+\frac{1}{2!}\kappa_{21}\partial_{a_2}F_b{}^a\partial^bF_{a_0a_1}+\frac{1}{2!}\kappa_{22}\partial^aF_{ba_2}\partial^bF_{a_0a_1}+\frac{1}{2!}\kappa_{23}\partial_bF^b{}_{a_0}\partial^aF_{a_1a_2}\bigg)\bigg],
\end{aligned} \tag{38}$$

where κ_i with $i=1,\ldots 23$ are the unknown constants that have to be found. In the above equation, $\mathcal{F}^{(p-2)}$ is the field strength of the RR potential C_{p-3}. One can easily verify that the above couplings are consistent with the T-duality transformations (14) when the Killing index y is a world volume index which is carried only by the RR field strength. When it is carried by the field strength of the gauge field, the consistency with T-duality requires the couplings of one C_{p-1}, one gauge field, and one transverse scalar field, which we consider next.

B. One RR, one gauge field, and one transverse scalar field

All possible nonzero couplings of one RR potential C_{p-1}-form, one gauge field, and one transverse scalar fields are

$$\begin{aligned}
S_{ca\chi} = {}& \frac{\pi^2\alpha'^2 T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\bigg[\frac{1}{(p-1)!}\partial_b\mathcal{F}^{(p)}_{ia_2a_3\ldots a_p}\bigg(\frac{1}{2!}\zeta_1\Omega^{bai}\partial_aF_{a_0a_1}+\zeta_2\Omega_{a_0}{}^{bi}\partial_aF^a{}_{a_1}\\
&+\zeta_3\Omega_{a_0}{}^{ai}\partial_aF^b{}_{a_1}+\zeta_4\Omega^{bai}\partial_{a_1}F_{aa_0}+\zeta_5\Omega_{a_0}{}^{ai}\partial_{a_1}F_a{}^b+\zeta_6\Omega_{a_0}{}^{ai}\partial^bF_{aa_1}
\end{aligned}$$

$$
\begin{aligned}
&+\zeta_7\Omega_a{}^{ai}\partial_{a_1}F^b{}_{a_0}+\frac{1}{2!}\zeta_8\Omega_a{}^{ai}\partial^bF_{a_0a_1}\bigg)\\
&+\frac{1}{(p-2)!}\partial^b\mathcal{F}^{(p)}{}_{iba_3a_4\ldots a_p}\left(\frac{1}{2!}\zeta_9\Omega_{a_0}{}^{ai}\partial_aF_{a_1a_2}+\zeta_{10}\Omega_{a_0}{}^{ai}\partial_{a_2}F_{aa_1}\right)\\
&+\frac{1}{(p-2)!}\partial_b\mathcal{F}^{(p)}_{iaa_3a_4\ldots a_p}\bigg(\zeta_{11}\Omega_{a_0}{}^{bi}\partial_{a_2}F^a{}_{a_1}+\frac{1}{2!}\zeta_{12}\Omega_{a_0}{}^{bi}\partial^aF_{a_1a_2}\\
&+\zeta_{13}\Omega_{a_0}{}^{ai}\partial_{a_2}F^b{}_{a_1}+\frac{1}{2!}\zeta_{14}\Omega_{a_0}{}^{ai}\partial^bF_{a_1a_2}\bigg)\\
&+\frac{1}{(p-3)!}\partial_{a_4}\mathcal{F}^{(p)}_{iaba_3a_5\ldots a_p}\left(\zeta_{15}\Omega_{a_0}{}^{ai}\partial_{a_2}F^b{}_{a_1}+\frac{1}{2!}\zeta_{16}\Omega_{a_0}{}^{ai}\partial^bF_{a_1a_2}\right)\\
&+\frac{1}{(p-2)!}\partial_{a_4}\mathcal{F}^{(p)}_{iba_2a_3a_5\ldots a_p}\bigg(\zeta_{17}\Omega_{a_0}{}^{bi}\partial_aF^a{}_{a_1}+\frac{1}{2!}\zeta_{18}\Omega^{bai}\partial_aF_{a_0a_1}\\
&+\zeta_{19}\Omega^{bai}\partial_{a_1}F_{aa_0}+\zeta_{20}\Omega_a{}^{ai}\partial_{a_1}F^b{}_{a_0}\zeta_{21}\Omega_{a_0}{}^{ai}\partial_aF^b{}_{a_1}+\zeta_{22}\Omega_{a_0}{}^{ai}\partial_{a_1}F_a{}^b\\
&+\zeta_{23}\Omega_{a_0}{}^{ai}\partial^bF_{aa_1}+\frac{1}{2!}\zeta_{24}\Omega_a{}^{ai}\partial^bF_{a_0a_1}\bigg)\\
&+\frac{1}{(p-1)!}\partial_{a_4}\mathcal{F}^{(p)}_{ia_1a_2a_3a_5\ldots a_p}(\zeta_{25}\Omega_a{}^{ai}\partial_bF^b{}_{a_0}+\zeta_{26}\Omega^{bai}\partial_bF_{aa_0}+\zeta_{27}\Omega_{a_0}{}^{bi}\partial_aF_b{}^a)\\
&+\frac{1}{(p-2)!}\partial_i\mathcal{F}^{(p)}_{aba_3a_4\ldots a_p}(\zeta_{28}\Omega_{a_0}{}^{ai}\partial_{a_2}F^b{}_{a_1}+\zeta_{29}\Omega_{a_0}{}^{ai}\partial^bF_{a_1a_2})\\
&+\frac{1}{(p-1)!}\partial_i\mathcal{F}^{(p)}_{ba_2a_3a_4\ldots a_p}\bigg(\zeta_{30}\Omega_{a_0}{}^{bi}\partial_aF^a{}_{a_1}+\zeta_{31}\Omega^{bai}\partial_aF_{a_0a_1}\\
&+\zeta_{32}\Omega^{bai}\partial_{a_1}F_{aa_0}+\zeta_{33}\Omega_a{}^{ai}\partial_{a_1}F^b{}_{a_0}+\zeta_{34}\Omega_{a_0}{}^{ai}\partial_aF^b{}_{a_1}+\zeta_{35}\Omega_{a_0}{}^{ai}\partial_{a_1}F_a{}^b\\
&+\zeta_{36}\Omega_{a_0}{}^{ai}\partial^bF_{aa_1}+\frac{1}{2!}\zeta_{37}\Omega_a{}^{ai}\partial^bF_{a_0a_1}\bigg)\\
&+\frac{1}{p!}\partial_i\mathcal{F}^{(p)}_{a_1a_2\ldots a_p}(\zeta_{38}\Omega_a{}^{ai}\partial_bF^b{}_{a_0}+\zeta_{39}\Omega^{bai}\partial_bF_{aa_0}+\zeta_{40}\Omega_{a_0}{}^{bi}\partial_aF_b{}^a)\bigg],
\end{aligned}
\tag{39}
$$

where we have also imposed the bulk equations of motion (23). In the above equation ζ_i with $i=1,\ldots 40$ are the unknown constants that have to be found by consistency with dualities and with the S-matrix elements. One can easily verify that the above couplings are consistent with the T-duality transformations (14) when the Killing index y is a world volume index which is carried only by the RR field strength. This index cannot be carried by the transverse scalar field. When it is carried by the field strength of the gauge field, the consistency with T-duality requires the couplings of one C_{p+1} and two transverse scalar fields, which we consider next.

C. One RR and two transverse scalar fields

All possible nonzero couplings of one RR potential C_{p+1}-form and two scalar fields after imposing the bulk equations of motion (23) are

$$
\begin{aligned}
S_{c\chi\chi}=\frac{\pi^2\alpha'^2T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\bigg[&\frac{1}{(p+1)!}\partial_b\mathcal{F}^{(p+2)}_{ca_0a_1\ldots a_p}(\rho_1\Omega^{cai}\Omega^b{}_{ai}+\rho_2\Omega_a{}^{ai}\Omega^{bc}{}_i)\\
&-\frac{1}{(p+1)!}\partial^b\mathcal{F}^{(p+2)}_{ba_0a_1\ldots a_p}(\rho_3\Omega_a{}^{ai}\Omega_c{}^c{}_i+\rho_4\Omega_{cai}\Omega^{cai})\\
&+\frac{1}{(p+1)!}\partial_j\mathcal{F}^{(p+2)}_{ia_0a_1\ldots a_p}(\rho_5\Omega_a{}^{ai}\Omega_c{}^{cj}+\rho_6\Omega_{ca}{}^j\Omega^{cai})\\
&+\frac{1}{p!}\partial_b\mathcal{F}^{(p+2)}_{ija_1\ldots a_p}(\rho_7\Omega_{a_0}{}^{ai}\Omega^b{}_a{}^j+\rho_8\Omega_a{}^{ai}\Omega^b{}_{a_0}{}^j)+\frac{\rho_9}{p!}\Omega^a{}_{a_0}{}^i\Omega^{cb}{}_i\partial_c\mathcal{F}^{(p+2)}_{aba_1\ldots a_p}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{p!}\partial^c\mathcal{F}^{(p+2)}_{bca_1a_2\ldots a_p}(\rho_{10}\Omega_{a_0}{}^{ai}\Omega^b{}_{ai}+\rho_{11}\Omega_a{}^{ai}\Omega^b{}_{a_0i})\\
&+\frac{1}{p!}\partial_i\mathcal{F}^{(p+2)}_{jba_1a_2\ldots a_p}(\rho_{12}\Omega_{a_0}{}^{ai}\Omega^b{}_a{}^j+\rho_{13}\Omega_a{}^{ai}\Omega^b{}_{a_0}{}^j)\\
&+\frac{1}{p!}\partial_i\mathcal{F}^{(p+2)}_{jba_1a_2\ldots a_p}(\rho_{14}\Omega_{a_0}{}^{aj}\Omega^b{}_a{}^i+\rho_{15}\Omega_a{}^{aj}\Omega^b{}_{a_0}{}^i)\\
&+\frac{1}{(p-1)!}(\rho_{16}\Omega^a{}_{a_1}{}^i\Omega^b{}_{a_0}{}^j\partial_a\mathcal{F}^{(p+2)}_{ijba_2\ldots a_p}+\rho_{17}\Omega^a{}_{a_1}{}^j\Omega^b{}_{a_0}{}^i\partial_j\mathcal{F}^{(p+2)}_{iaba_2\ldots a_p})\\
&+\frac{1}{(p-1)!}\partial_{a_4}\mathcal{F}^{(p+2)}_{ijba_1a_2a_3a_5\ldots a_p}(\rho_{18}\Omega_{a_0}{}^{ai}\Omega^b{}_a{}^j+\rho_{19}\Omega_a{}^{ai}\Omega^b{}_{a_0}{}^j)\\
&+\frac{\rho_{20}}{(p-1)!}\Omega_{a_1}{}^{ai}\Omega_{a_0a}{}^j\partial^c\mathcal{F}^{(p+2)}_{ijca_2\ldots a_p}+\frac{\rho_{21}}{(p-1)!}\Omega^a{}_{a_1i}\Omega^b{}_{a_0}{}^i\partial^c\mathcal{F}^{(p+2)}_{abca_2\ldots a_p}\Bigg],
\end{aligned}
\tag{40}
$$

where ρ_i with $i=1,\ldots 21$ are the unknown constants. One can easily verify that the above couplings are consistent with the T-duality transformations (14) when the Killing index y is a world volume index. So there is no D_p-brane coupling involving the RR potential C_{p+3}. The above couplings are also consistent with the S-duality of the D_3-brane action.

Now, consider the sum of couplings (38), (39), and (40), i.e.,

$$
S_p^{\rm CS}=S_{caa}+S_{ca\chi}+S_{c\chi\chi}. \tag{41}
$$

They are not invariant under the linear T-duality transformations (14) for arbitrary coefficients. Imposing the invariance under T-duality (15), one finds the following relations between the constants in the CS part:

$$
\begin{aligned}
&\rho_2=-\rho_1,\qquad \rho_9=\rho_{10},\qquad \rho_3=-\frac{\rho_1}{2},\qquad \rho_4=\frac{\rho_1}{2},\qquad \rho_{11}=-\rho_{10},\\
&\rho_{15}=\zeta_{17}+\zeta_{18}-\zeta_{19}-\zeta_{20}+\zeta_{21}+\zeta_{23}+\zeta_{24}-\zeta_{30}-\zeta_{31}+\zeta_{32}+\zeta_{33}-\zeta_{34}-\zeta_{36}-\zeta_{37}-\rho_{12}-\rho_{13}-\rho_{14}\\
&\zeta_8=-\zeta_1-\zeta_2-\zeta_{17}-\zeta_{18}+\zeta_{19}+\zeta_{20}-\zeta_{21}-\zeta_{23}-\zeta_{24}-\zeta_3+\zeta_4-\zeta_6+\zeta_7,\\
&\kappa_6=\zeta_9-\zeta_{10}-\kappa_1+2\kappa_9+2\kappa_{10}-2\kappa_{11}+\kappa_{12}-\kappa_{13}-\kappa_{14}+\kappa_{15}+\kappa_{16}-\kappa_{17}+\kappa_3-\kappa_{19}\\
&\qquad+\kappa_4+\kappa_{22}+\frac{1}{2}(\zeta_1-\zeta_{11}+\zeta_{12}-\zeta_{13}+\zeta_{14}+\zeta_{18}-\zeta_{19}-\zeta_{21}-\zeta_{23}-\zeta_3-\zeta_4-\zeta_6),\\
&\rho_{17}=\frac{1}{2}(-\zeta_{11}+\zeta_{12}+\zeta_{13}-\zeta_{14}-\zeta_{18}+\zeta_{19}-\zeta_{21}-\zeta_{23}-2\zeta_{28}+2\zeta_{29}+\zeta_{31}-\zeta_{32}+\zeta_{34}+\zeta_{36}+\rho_{12}+\rho_{14}),\\
&\rho_{19}=\frac{1}{2}(\zeta_{11}-\zeta_{12}+\zeta_{13}-\zeta_{14}-\zeta_{18}+\zeta_{19}+2\zeta_{20}\\
&\qquad-\zeta_{21}-\zeta_{23}-2\zeta_{24}+\zeta_{31}-\zeta_{32}-2\zeta_{33}+\zeta_{34}+\zeta_{36}+2\zeta_{37}+\rho_{12}+2\rho_{13}+\rho_{14}+2\rho_{16}),\\
&\rho_{20}=\frac{1}{4}(-2\zeta_{10}-\zeta_{11}+\zeta_{12}-\zeta_{13}+\zeta_{14}+\zeta_{18}-\zeta_{19}-\zeta_{21}-\zeta_{23}-\zeta_{31}+\zeta_{32}+\zeta_{34}+\zeta_{36}+2\zeta_9+\rho_{12}-\rho_{14}-2\rho_{16}-2\rho_{18}),\\
&\rho_5=\frac{1}{2}(\zeta_1-2\zeta_{25}+\zeta_3+\zeta_{31}-\zeta_{32}+\zeta_{34}+\zeta_{36}+2\zeta_{38}-\zeta_4+\zeta_6+\rho_{12}+\rho_{14}),\\
&\rho_6=\frac{1}{2}(-\zeta_1-2\zeta_{26}-\zeta_3-\zeta_{31}+\zeta_{32}-\zeta_{34}-\zeta_{36}+2\zeta_{39}+\zeta_4-\zeta_6-\rho_{12}-\rho_{14}),\\
&\rho_7=\frac{1}{2}(\zeta_1+\zeta_3-\zeta_{31}+\zeta_{32}+\zeta_{34}+\zeta_{36}+\zeta_4+\zeta_6+\rho_{12}-\rho_{14}),\\
&\rho_8=\frac{1}{2}(-\zeta 1-2\zeta_{17}-2\zeta_{18}+2\zeta_{19}-2\zeta_2+2\zeta_{20}-2\zeta_{21}-2\zeta_{23}-2\zeta_{24}-\zeta_3+\zeta_{31}-\zeta_{32}-2\zeta_{33}+\zeta_{34}+\zeta_{36}\\
&\qquad+2\zeta_{37}+\zeta_4-\zeta_6+\rho_{12}+2\rho_{13}+\rho_{14}),\\
&\kappa_8=\kappa_{12}-\kappa_{13}-\kappa_{14}+\kappa_{15}-\kappa_{20}+\kappa_{23}+\kappa_7-\zeta_2-\zeta_{17}\\
&\qquad+\frac{1}{2}(-\zeta_1-\zeta_{11}+\zeta_{12}-\zeta_{13}+\zeta_{14}-\zeta_{18}+\zeta_{19}-\zeta_{21}-\zeta_{23}-\zeta_3+\zeta_4-\zeta_6).
\end{aligned}
\tag{42}
$$

The above constraints make the CS action to be consistent with the T-duality. There are still many constants that are not fixed yet.

Imposing the constraints (42), one finds the couplings (39) are not consistent with S-duality for the D_3-brane case. The S-duality requires, up to some total derivative terms, the couplings in (39) to be in the form of $\Omega\partial(*\mathcal{F}^T)\mathcal{M}_0\partial^2\mathcal{B}$. Using the expansion (20), one finds the relation between the constants in the CS part and the DBI part,

$$\begin{aligned}
\zeta_4 &= 1+\zeta_1+\zeta_{13}-\zeta_{14}+\zeta_2+\zeta_{17}-\zeta_{28}+\zeta_{29}+\zeta_{31}-\zeta_{32}\\
\zeta_5 &= 1-2\gamma_1+\zeta_{13}-\zeta_{14}+\zeta_2+\zeta_{17}+\zeta_3+\zeta_{27}-\zeta_{28}+\zeta_{29}+\zeta_{34}-\zeta_{35}-\zeta_{40}\\
\zeta_6 &= 1-\zeta_{11}+\zeta_{12}-\zeta_2-\zeta_{17}-\zeta_{18}+\zeta_{19}-\zeta_{21}-\zeta_{23}-\zeta_3-\zeta_{28}+\zeta_{29}+\zeta_{31}-\zeta_{32}\\
\zeta_9 &= \zeta_{10}+\zeta_{13}-\zeta_{14}-\zeta_{18}+\zeta_{19}-\zeta_{28}+\zeta_{29}+\zeta_{31}-\zeta_{32}\\
\zeta_{30} &= 1-2\gamma_1+\zeta_{13}-\zeta_{14}+\zeta_{17}-\zeta_{28}+\zeta_{29}\\
\zeta_{36} &= -2+2\gamma_1+\zeta_{11}-\zeta_{12}-\zeta_{13}+\zeta_{14}+\zeta_{18}-\zeta_{19}+\zeta_{21}+\zeta_{23}+2\zeta_{28}-2\zeta_{29}-\zeta_{31}+\zeta_{32}-\zeta_{34}\\
\zeta_{37} &= 1-\gamma_3-\zeta_{11}+\zeta_{12}-\zeta_{20}+\zeta_{24}-\zeta_{28}+\zeta_{29}+\zeta_{33}\\
\zeta_{38} &= 1-2\gamma_1+\zeta_{13}-\zeta_{14}+\zeta_2+\zeta_{17}+\zeta_{25}-\zeta_{28}+\zeta_{29}\\
\zeta_{39} &= -1+2\gamma_1-\zeta_{13}+\zeta_{14}-\zeta_2-\zeta_{17}+\zeta_{26}+\zeta_{28}-\zeta_{29},
\end{aligned} \tag{43}$$

as well as the constraint (34). Imposing the above constraints, one finds not only the couplings (39) but also the couplings (38) become consistent with the S-duality for D_3-brane; i.e., the couplings $\partial\partial C_0\partial F\partial F$ in (38) and the couplings $\partial\partial\Phi\partial F\partial F$ in the DBI part combine into the S-duality invariant structure (20).

We now compare the couplings with the S-matrix elements. Imposing the constraints (42) and (43) into the action $S_p^{\rm CS}$, one finds the resulting couplings are consistent with the S-matrix elements (22) provided that

$$\rho_{14}=-\rho_{12},\qquad \gamma_1=1. \tag{44}$$

The final step is to ignore the couplings which are total derivative terms or the couplings which can be eliminated by the Bianchi identities. Imposing the constraints (42), (43), and (44) into the action, we find the terms with coefficient ρ_{13} in (40) are total derivative terms, so ρ_{13} can be eliminated from the physical couplings. The terms with coefficients $\rho_1,\rho_{10},\rho_{12},\rho_{18},\rho_{21}$ in (40) can be canceled by the Bianchi identity. When we write the field strengths in (39) in terms of corresponding potentials, we find the terms with coefficients ζ_i with $i=1$, 3, 7, 10, 20, 23, 24, 25, 26, 33 disappear, so these constants can be eliminated from (39) by the Bianchi identity. Moreover, we find that terms with coefficients ζ_i with $i=2$, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 27, 28, 29, 31, 32, 34, 35, 40 are total derivative terms. As a result, these terms can be ignored, too. In the couplings (38), the constants κ_i with $i=7$, 10, 11, 20, 23 can be ignored by the Bianchi identities, and the constants κ_i with $i=1$, 2, 3, 4, 5, 6, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22 can be ignored by total derivative terms.

The final results for the CS part are the couplings which appear in (11), (13), and the couplings

$$\begin{aligned}
S_{\rm CS}\supset \frac{\pi^2\alpha'^2T_p}{12}\int d^{p+1}x\epsilon^{a_0a_1\ldots a_p}\bigg[&\frac{\gamma_3}{2!(p-1)!}\Omega_a{}^{ai}\partial^bF_{a_1a_0}\partial_i\mathcal{F}^{(p)}_{ba_2\ldots a_p}+\frac{\gamma_3}{p!}\Omega_a{}^{ai}\Omega_b{}^{bj}\partial_j\mathcal{F}^{(p+2)}_{ia_0\ldots a_p}\\
&-\frac{1-\gamma_3}{(p-1)!}\Omega_a{}^{ai}\Omega_{a_0}{}^{bj}\partial_{a_3}\mathcal{F}^{(p+2)}_{ijba_1a_2a_4\ldots a_p}+\frac{1-\gamma_3}{p!}\Omega_a{}^{ai}\Omega_{a_0}{}^{bj}\partial_b\mathcal{F}^{(p+2)}_{ija_1\ldots a_p}\bigg],
\end{aligned} \tag{45}$$

where we have also used the following identity in the second term:

$$\begin{aligned}
&(p+1)\epsilon^{a_0a_1\ldots a_p}\Omega_{a_0}{}^{bj}\partial_j\mathcal{F}^{(p+2)}_{iba_1a_2\ldots a_p}\\
&\quad=\epsilon^{a_0a_1\ldots a_p}\Omega_b{}^{bj}\partial_j\mathcal{F}^{(p+2)}_{ia_0a_1a_2\ldots a_p}.
\end{aligned} \tag{46}$$

In proving the above identity, we have used the totally antisymmetric property of the RR field strength which can be used to replace the world volume index b on the left-hand side by a_0. Using a similar relation and writing the RR field strength in terms of the RR potential, one can prove the following identity:

$$-p\Omega_a{}^{ai}\Omega_{a_0}{}^{bj}\partial_{a_3}\mathcal{F}^{(p+2)}_{ijba_1a_2a_4\ldots a_p}+\Omega_a{}^{ai}\Omega_{a_0}{}^{bj}\partial_b\mathcal{F}^{(p+2)}_{ija_1\ldots a_p}=0. \tag{47}$$

Using the above identity, one finds that couplings in the second line of (45) are zero. The couplings in the first line of (45) are consistent with the linear T-duality and the S-duality and are zero when the scalar fields are on shell. Note that the coupling in the first term for the case of the D_3-brane can be written as an S-dual multiplet because $\Omega_a{}^{ai}\partial^b F^{cd}\partial_i H^{(3)}_{bcd}$ is zero by the Bianchi identity of the gauge field strength. Therefore, the coefficient γ_3 cannot be fixed by the linear dualities and by the S-matrix element of one closed and two open strings. It may be fixed by the open string pole of the S-matrix element of two closed strings and one open string at order α'^2 or by the contact terms of the S-matrix element of three closed strings. We expect the square of the second fundamental form appears in the world volume curvatures as in (5), (6), and (8). The second fundamental forms in the second term of (45) cannot be extended to the curvature (8), so we speculate the coefficient of this term is zero, i.e.,

$$\gamma_3 = 0. \tag{48}$$

It would be interesting to analyze in details the S-matrix element of two closed strings and one open string or the S-matrix element of three closed strings to confirm the above relation.

Requiring the consistency of the D-brane effective action at order α'^2 with S-matrix and with the linear dualities, we have found the couplings of one NSNS and two NS states in the DBI part to be (10) and (12) and the couplings of one RR and two NS states in the CS part to be (11) and (13). On the other hand, the D-brane effective action at order α'^2 should be invariant under supersymmetry and κ symmetry. It would be interesting to verify the above couplings to be consistent with the supersymmetry and κ symmetry.

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