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THE COMPLEMENT OF SUBGROUP GRAPH OF A GROUP

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Abstract. Let G be a finite group and H a subgroup of G. In 2012, David F. Anderson et al. introduced the subgroup graph of H in G as a simple graph with vertex set consisting all elements of G and two distinct vertices x and y are adjacent if and only if $xy \in H$. They denoted this graph by $\Gamma_H(G)$. In this paper, we consider the complement of $\Gamma_H(G)$, denoted by $\overline{\Gamma_H(G)}$ and will give some graph properties of this graph, for instance diameter, girth, independent and dominating sets, regularity. Moreover, the structure of this graph, planerity and 1-planerity are also investigated in the paper.

1. Introduction

There are many papers on assigning a graph to a group, ring or other algebraic structures and investigation of algebraic properties of group or ring using the associated graph, for example non-commuting graph [1], power graph [8], prime graph [7], zero divisor graph [2] and so on. One of the graphs associated to a group is the subgroup graph denoted by $\Gamma_H(G)$ which was introduced by D. F. Anderson, et. al. in [3]. They defined the graph $\Gamma_H(G)$ as a directed simple graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$. Some properties of this graph are investigated by the above authors, for instance the structure of the connected components of $\Gamma_H(G)$ when $|H|$ is either two or three and also when H is a normal subgroup and G/H is a finite abelian group. In this paper, we investigated the complement of $\Gamma_H(G)$ which is denoted by $\overline{\Gamma_H(G)}$. It is easy to see that if H is a normal subgroup of G, then $\Gamma_H(G)$ is

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undirected simple graph. Moreover, if $H = G$, then we have an empty graph. For this reason, we assumed that H is always a proper normal subgroup and so $\overline{\Gamma_H(G)}$ is undirected, too. Throughout the paper, all graphs are simple and the notations and terminologies about graphs and groups are standard (for instance see [5, 4]).

2. Some properties of $\overline{\Gamma_H(G)}$

As we mentioned earlier in the previous section, the subgroup graph $\Gamma_H(G)$ is defined as the following:

Definition 1. *Let G be a group and H be a subgroup of G. The subgroup graph $\Gamma_H(G)$ is a directed simple graph with vertex set G; and two distinct elements x and y are adjacent if and only if $xy \in H$.*

It is clear that if x is adjacent to y, then it not necessary that y is adjacent to y. But, if H is normed, then $xy \in H$ will imply that $yx \in H$. Because, $yx = x^{-1}(xy)x \in H$. Thus, we assume that H is normal, then the complement of $\Gamma_H(G)$ is an undirected simple graph whose vertices are all elements of G, and two distinct vertices x and y are adjacent if $xy \notin H$.

One can see that if $\{x, y\}$ is an edge in $\overline{\Gamma_H(G)}$, then both x and y can not be in H. So, we have no edge in $\overline{\Gamma_H(G)}$ with end vertices are in H. Moreover, if $h \in H$ is a vertex in $\overline{\Gamma_H(G)}$, then h will be adjacent to every element in $G \setminus H$. Also, there is no vertex in H such that it is adjacent to h. Thus $\deg(h) = |G| - |H|$. Furthermore, if $x \in G \setminus H$, then x can not be adjacent to x^{-1}. we know that $x^{-1} \in G \setminus H$ and so $\deg(x) = |H| + |A|$ where $A = \{y \in G \setminus H; y \text{ adjacent to } x\}$.
In the following theorem, we determine the diameter of $\overline{\Gamma_G(H)}$.

Theorem 1. *$diam(\overline{\Gamma_G(H)}) \leqslant 2$.*

Proof. Since H is a proper normal subgroup of G, there are the following three cases:
Case 1. If x and y are two vertices in H, then there exists an element $z \in G \setminus H$ and so $x - z - y$.
Case 2. If x and y are two vertices in $G \setminus H$, then we have $x - e - y$.
Case 3. If $x \in H$ and $y \in G \setminus H$, then $x - y$. Hence. the above cases imply that $diam(\overline{\Gamma_H(G)}) \leqslant 2$. It is interesting to see that whenever $diam(\overline{\Gamma_H(G)}) = 1$. The following theorem states this fact. □

Theorem 2. *$\overline{\Gamma_H(G)}$ is complete if and only if G is an elementary abelian 2-group and H is trivial subgroup.*

Proof. Assume that $\overline{\Gamma_H(G)}$ is complete since, there is no edge between any two elements in H, we should have $H = \{e\}$. Now, if G has a nontrivial

element x in which $|x| \neq 2$; then $x \neq x^{-1}$ and x can not be adjacent to x^{-1} which is a contradiction. Thus $\exp(G) = 2$ and G is abelian. Hence G is elementary abelian 2-group. Conversely, if $H = \{e\}$ and G is elementary abelian 2-group. Then we claim that for every two distinct vertices $x \neq y$, x should be joined to y by an edge. If x or y is identity element, then trivially x adjacent to y. If $x, y \in G \setminus H$ and x is not adjacent to y then we have $xy = e$ or equivalently $y = x^{-1} = x$ which is a contradiction. Therefore $\overline{\Gamma_H(G)}$ is complete as required.
Let $[G : H] = n$ and $\{x_1H, x_2H, \ldots, x_nH\}$ be the set of all distinct left cosets of H in G and $\{e = x_1, x_2, \ldots, x_n\}$ be the set of representative left transverals of H in G. Then we may state the following simple lemmas which play an important role to find the clique number and girth of $\overline{\Gamma_H(G)}$. □

Lemma 3. *Let $[G : H] = n$ and $\{x_1H, x_2H, \ldots, x_nH\}$ be the set of all representative distinct left casets of H. If $x_i^2 \notin H$, for some $1 \leqslant i \leqslant n$, then all elements in x_iH form a clique of size $|H|$.*

Proof. Suppose that $a, b \in x_iH$, where $x_i^2 \notin H$. Then there elements $h_1, h_2 \in H$ such that $a = x_ih_1$; $b = x_ih_2$.
Thus; $ab = x_ih_1x_ih_2 = x_ih_1x_i^{-1}x_i^2h_2$. Since $x_1h_1x_i^{-1} \in H$, $h_2 \in H$ and $x_i^2 \notin H$, so we have $ab \notin H$. Therefore a is adjacent to b and we have a clique of size $|H|$. □

Lemma 4. *By the notation as in the previous lemma, if x_i; adjacent to x_j, for some $1 \leqslant i \neq j \leqslant n$, then every element in x_iH is adjacent to every element in x_jH.*

Proof. Assume that $a \in x_iH$ and $b \in x_jH$. Then $a = x_ih_1$ and $b = x_jh_2$, for some $h_1, h_2 \in H$. Since $x_ix_j \notin H$, $x_ihx_i^{-1} \in H$ and $h_2 \in H$, so we have $ab = x_ih_1x_jh_2 = x_ih_1x_i^{-1}x_ix_jh_2 \notin H$. Thus a is adjacent to b. □

Theorem 5. *Let $[G : H] = n$. As the notation in Lemma 3, if X is a subset of $\{x_1, x_2, \ldots, x_n\}$ such that X has the property that $x_ix_j \notin H$, for every $x_i, x_j \in X$, then $\omega(\overline{\Gamma_H(G)}) = |X||H| + 1$.*

Proof. By Lemma 4 it is clear that there is an edge between any two vertices inside left coset xH and also any two vertices on in xH and other in yH, where $x, y \in X$. Thus a complete graph on $|X||H|$ exists. Since identity element is adjacent to every vertices in $G \setminus H$, so we will have a clique of size $|X||H| + 1$. One can easily cheek that one more vertex can not be added to this clique and there is no clique of size bigger than $|X||H| + 1$. Thus $\omega(\overline{\Gamma_H(G)}) = |X||H| + 1$. It is obvious that $|X|$ is at most $[G : H] - 1$, because $e \notin X$. So, if $\overline{\Gamma_H(G)}$ is a complete graph, then we should have $|X||H| + 1 = |G|$ or $([G : H] - 1)|H| + 1 = |G|$ or $|H| = 1$ which confirms the result given in Theorem 2 □

Theorem 6. *Girth* $(\overline{\Gamma_H(G)}) = 4$ *when* $[G : H] = 2$. *Otherwise girth* $(\overline{\Gamma_H(G)}) = 3$.

Proof. First assume that $[G : H] = 2$. So $G = H \cup xH$. We note that $x^2 \in H$; because if $x^2 \notin H$, then $x^2H = xH$ which implies that $x \in H$, it is a contradiction. Hence $x^2 \in H$. Since $|H| > 1$; there exists elements $h_1, h_2 \in H$. Thus $xh_1, xh_2 \in xH$ and we have a cycle

$$h_1 \text{ —— } xh_1 \text{ —— } h_2 \text{ —— } xh_2 \text{ —— } h_2$$

of length 4, by Lemma 4. Thus girth $(\overline{\Gamma_H(G)}) = 4$.
If $[G : H] \geqslant 3$; then there are at least two left cosets xH and yH such that $x, y \notin H$. It is clear that x is adjacent to y^{-1}, because $xH \neq yH$. Thus identity element, x and y^{-1} will construct a triangle. Hence girth $(\overline{\Gamma_H(G)}) = 3$ and the proof of theorem is completed.
In the following theorem, we determine the independence number of the graph. Note that the independence number of a graph X, denoted by $\alpha(X)$ is the maximum size of independent sets of X. A subset M of $V(X)$ is called an independent number if there is no edge with two ends in M. □

Theorem 7. *Let* $[G : H] = n$ *and* $X = \{x_1, x_2, \ldots, x_n\}$ *be a subset of vertex set such that* $\{x_1H, x_2H, \ldots, x_nH\}$ *is the set of representative distinct left cosets of* H *in* G.
Assume that $A = \{x_i \in X; x_i \text{ is not adjacent to } x_j, \text{ for all } j; 1 \leqslant i \neq j \leqslant n\}$ *and* $B = \{x_i \in X; x_i^2 \in H\}$, *then* $\alpha(\overline{\Gamma_H(G)}) = |A \cap B|(|H| - 1) + |A|$.

Proof. It is clear that if $x_i, x_j \in A \cap B$, then $x_i^2, x_j^2 \in H$ and x_i is not adjacent to x_j. By lemmas 3 and 4, there is no edge between vertices in x_iH, x_jH and also no edge between vertices in x_iH and vertices in x_jH. Hence $\bigcup_{x_i \in A \cap B} x_iH$ is an independent set.
Now, if $x_k \in A \setminus (A \cap B)$, then all elements in x_kH forms a clique, by Lemma 3 and also x_k is not adjacent to any vertex in $A \cap B$. Hence, we can add x_k to the above independent set.
Therefore $(\bigcup_{i \in A \cap B} x_iH) \bigcup (A \setminus A \bigcap B)$ is the largest independent set. Thus $\alpha(\overline{\Gamma_H(G)}) = \sum_{x_i \in A \cap B} |x_iH| + (|A| - |A \cap B|) = |A \cap B|(|H| - 1) + |A|$ as required.
Now, we are going to find an upper bound for the chromatic number of $\overline{\Gamma_H(G)}$. Remind that chromatic number of a graph X, denoted by $\chi(X)$ is the minimum number of colors that required for labeling vertices such that there is no edge with two ends vertices have the same color. □

Theorem 8. *With the same notation as in Theorem 7,* $\chi(\overline{\Gamma_H(G)}) \leqslant (n + 1 - |A \cup B|)|H| + |A \cup B| - |A|$

Proof. One can easily see that for all vertices in $\bigcup_{x_i \in A} x_iH$, we need $|H|$ colors. Vertices in $\bigcup_{x_j \in B \setminus A} x_jH$ need at most $|B| - |A \cap B|$ colors. For the reminding vertices, at most $(n-|A\cup B|)|H|$ are required. Thus $\chi(\overline{\Gamma_H(G)}) \leqslant |H|+|B|-|A\cap B|+(n-|A\cup B|)|H| = (n+1-|A\cup B|)|H|+|A\cup B|-|A|$. $\square$

Example 1. *Let $[G:H]=2$. Then we have $X=\{e,x\}$ and so $A=\{x\}$ and $B=\{e\}$. Hence $\chi(\overline{\Gamma_H(G)}) \leqslant (2+1-2)|H|+2-1=|H|+1$. But, we know that $\overline{\Gamma_H(G)}$ consists a complete graph on vertices in xH and $|H|$ isolated vertices on vertices in H. Thus $|H|+1$ colors is the minimum required colors therefore $\chi(\overline{\Gamma_H(G)})=|H|+1$ and it shows that the upper bound given in Theorem 8 is sharp.*
The following theorem will state the dominating number of $\overline{\Gamma_H(G)}$. The dominating number of a graph X, denoted by $\chi(X)$; is the minimum size of dominating set. A subset D of $V(X)$ is called a dominating set if for every vertex $a\in V(X)\setminus D$ there exists a vertex $b\in D$ such that $\{a,b\}\in E(x)$.

Theorem 9. $\gamma(\overline{\Gamma_H(G)})=|H|$.

Proof. It is obvious that H is a dominating set, because $e\in H$ and e is adjacent to all other vertices outside of H. So $\gamma(\overline{\Gamma_H(G)})\leqslant |H|$. But, one can easily cheek that there is no dominating set of smaller size than $|H|$. Hence $\gamma(\overline{\Gamma_H(G)})=|H|$.
In the next lemma, we give some necessary conditions for the graph $\overline{\Gamma_H(G)}$ to be a planner graph. Note that a graph X is said to be planner if it can be embeded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. There is a know result that states a finite graph is planar if and only if it does not contain a subgraph that is division of K_5 or $K_{3,3}$ (see [5]) $\square$

Lemma 10. *For each of the following cases, $\overline{\Gamma_H(G)}$ is not planar.*

(i) $|H|=1, |G|\geqslant 5$ *and G is an elementary abelian 2-group.*
(ii) $|H|\geqslant 2$ *and* $|X-A\cup B|\geqslant 2$.
(iii) $|H|\geqslant 4$ *and* $|X-B|\geqslant 1$.

Note that sets X, A and B are defined in Theorem 7.

Proof. (i) By Theorem 2, $\overline{\Gamma_H(G)}$ is complete and has at least 5 vertices, thus it contains K_5 and can not be planar. For (ii), if $|X-A\cup B|\geqslant 2$, then there are non-trivial elements x_1 and x_2 in $X-A\cup B$. Since $x_1,x_2\notin A\cup B$ so $x_1H\cup x_2H\cup\{e\}$ consists a complete graph K_5 and so it does not planar.
(iii) Suppose that there exists an element $x\in X-B$. Then $x\notin B$ implies that xH is a complete graph. Thus $xH\cup\{e\}$ contains K_5 and it can not be planar. One can see that the converse of the above lemma is true. So we can state the following corollary. $\square$

Corollary 11. *By Lemma 10, we can state that $\overline{\Gamma_H(G)}$ is planar if and only if conditions (i), (ii) and (iii) can not be happen.*

By the similar method as in Lemma 10, we may consider 1-planar property for $\overline{\Gamma_H(G)}$. Remind that a graph is called 1-planar if it can be drawn in the plane in such a way that each edges has at most one crossing point, where it crosses a single additional edge. It is known that in a 1-planar graph we have some forbidden subgraphs, for instance complete graph K_7, complete bipartite graph $K_{4,5}$ and complete multipartite graph $K_{2,3,3}$ (see [6] for more details). Thus we may state the following result here. The proof is very similar to the proof of Theorem 7 and so we omit here.

Theorem 12. $\overline{\Gamma_H(G)}$ *is 1-planar if and only if each of the following conditions can not occured:*

(i) $|H| = 1, |G| \geqslant 7$ *and* G *is an elementary abelian* 2*-group.*
(ii) $|H| = 3, [G : H] \geqslant 3$ *and* $X - A \neq \emptyset$*.*
(iii) $|H| = 4$ *and* $|X - A \cup B| \geqslant 2$*.*
(iv) $|H| \geqslant 5$ *and* $X - A \neq \emptyset$*.*

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