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Fragmentation functions of neutral mesons π^0 and k^0 with Laplace transform approach

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With an analytical solutions of DGLAP evolution equations based on the Laplace transform method, we find the fragmentation functions (FFs) of neutral mesons, π^0 and k^0 at NLO approximation. We also calculated the total fragmentation functions of these mesons and compared them with experimental data and those from global fits. The results show a good agreement between our solutions and other models and they are compatible with experimental data.

Keywords: Laplace transform; fragmentation functions; natural meson.

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1. Introduction

Fragmentation process is the QCD process in which partons hadronize to colorless hadrons. In this transition, the parton fragmentation function (FF), $D_i^h(z, Q^2)$, represents the probability for a parton i to fragments into a particular hadron h carrying a certain fraction of the parton energy or momentum. Therefore, these FFs are essential inputs to study the hadron production in any processes like $p\bar{p}$, ep, γp and $\gamma\gamma$ scattering.

Fragmentation functions evolved with DGLAP evolution equations from a starting distribution at a defined energy scale.[1,2]

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Recently we have used Laplace transformation and provided an analytical solution to DGLAP equations to calculate the proton, pion and kaon FFs.[3] This method had provided analytical solutions to Polarized Parton Distribution Functions (PPDFs) too.[4,5]

In this paper, we will use the results of this new method introduced by Block *et al.*[6–11] to calculate the neutral mesons π^0 and k^0 FFs.

Therefore, our main task here is using our solutions for charged pion and kaon FFs,[3] for calculating the neutral mesons π^0 and k^0 FFs. These solutions enable us to use the neutral mesons data in a global fit, in addition to all experimental data for total FFs of charged mesons to determination of FFs.

The paper is organized as follows. In Sec. 2, we review the Laplace transform method of nonsinglet, singlet and gluon DGLAP evolution equations for extracting the FFs. These solutions led us to π^+ and k^+ FFs. Then, in Sec. 3, we utilize the charge conjugation symmetry to calculate the FFs of π^- and k^-. This led us to natural mesons FFs. Finally, in Sec. 4, we calculated the FFs of neutral mesons π^0 and k^0 and also compared them with available experimental data[26–29] and those from global fits.[12–16]

2. Fragmentation Functions via Decoupling of the DGLAP Evolution Equations by Laplace Transform Method

2.1. *Nonsinglet case*

The fragmentation of valence quarks into hadrons defined the nonsinglet FFs. The evolution of nonsinglet FF is given by DGLAP evolution equations at NLO approximation as:

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_{\rm ns}}{\partial \ln(Q^2)}(z,Q^2) = D_{\rm ns} \otimes \left[P_{qq}^{\rm LO,\, ns} + \frac{\alpha_s(\tau)}{4\pi} P_{qq}^{\rm NLO,\, ns} \right](z,Q^2)\,, \tag{1}$$

where

$$D_{\rm ns}^h(z,Q^2) = D_q^h(z,Q^2) - D_{\bar{q}}^h(z,Q^2)\,. \tag{2}$$

The $\otimes$ symbol in Eq. (1) refers to the convolution integral. In the new method introduced by Block *et al.*,[6–11] the DGLAP evolution equations can be solved by Laplace transform approach. To summarize, by introducing two variables $\nu \equiv \ln\left(\frac{1}{z}\right)$ and $\tau \equiv \frac{1}{4\pi}\int_{Q_0^2}^{Q^2} \alpha_s(Q'^2) d\ln Q'^2$, and two Laplace transforms from ν space to s space and from τ space to U space, the DGLAP evolution equations can be solved iteratively by a set of convolution integrals which are related to initial input FFs at scale of Q_0^2. Two inverse Laplace transforms will take us back to the usual space (z,Q^2).[3] We define $zD_{\rm ns}(z,Q^2) = F_{\rm ns}(z,Q^2)$, and find the solution of nonsinglet DGLAP evolution equation, Eq. (1) in s space as:[3]

$$f_{\rm ns}(s,\tau) = e^{\tau\Phi_{\rm ns}(s)} f_{{\rm ns}\,0}(s)\,, \tag{3}$$

where

$$\Phi_{\rm ns}(s) \equiv \Phi_{\rm ns}^{\rm LO}(s) + \frac{\tau_2}{\tau}\Phi_{\rm ns}^{\rm NLO}(s)\,. \tag{4}$$

$\Phi_{\rm ns}^{\rm LO}(s)$ and $\Phi_{\rm ns}^{\rm NLO}(s)$ are the Laplace transform of nonsinglet splitting functions and are given in App. A of Ref. 3. The τ_2 parameter is defined as

$$\tau_2 \equiv \frac{1}{4\pi}\int_0^\tau \alpha_s(\tau')d\tau' = \frac{1}{(4\pi)^2}\int_{Q_0^2}^{Q^2} \alpha_s^2(Q'^2)d\ln Q'^2\,. \tag{5}$$

$f_{\rm ns\,0}(s)$ in Eq. (3) is the Laplace transform of initial valence quark FFs at $Q_0 = 4.5$ GeV. They are selected from HKNS code[12] to be sure about our analytical solutions of DGLAP evolution equation. Finally, with an inverse Laplace transform of Eq. (3),[11] one can derive the valence quark FFs in (z, Q^2) space.

2.2. *The singlet and gluon case*

At NLO approximation, the singlet and gluon FFs are given by these two coupled DGLAP evolution equations:

$$\frac{4\pi}{\alpha_s(Q^2)}\frac{\partial D_s}{\partial \ln Q^2}(z,Q^2) = D_s \otimes \left(P_{qq}^0 + \frac{\alpha_s(Q^2)}{4\pi}P_{qq}^1\right)(z,Q^2) + D_g \otimes \left(P_{gq}^0 + \frac{\alpha_s(Q^2)}{4\pi}P_{gq}^1\right)(z,Q^2)\,, \tag{6}$$

$$\frac{4\pi}{\alpha_s(Q^2)}\frac{\partial D_g}{\partial \ln Q^2}(z,Q^2) = D_s \otimes \left(P_{qg}^0 + \frac{\alpha_s(Q^2)}{4\pi}P_{qg}^1\right)(z,Q^2) + D_g \otimes \left(P_{gg}^0 + \frac{\alpha_s(Q^2)}{4\pi}P_{qg}^1\right)(z,Q^2)\,, \tag{7}$$

where the singlet FF, $D_s^h(z, Q^2)$, is defined as

$$D_s^h(z,Q^2) = \sum_{q=u,d,s,c,b}\left[D_q^h(z,Q^2) + D_{\bar q}^h(z,Q^2)\right]. \tag{8}$$

By definition of $zD_s(z,Q^2) \equiv F_s(z,Q^2)$ and $zD_g(z,Q^2) \equiv G(z,Q^2)$, the solutions of these coupled DGLAP evolution equations in Laplace (s, U) space can be calculated as:[3]

$$[U - \Phi_f(s)]\mathcal{F}(s,U) - \Theta_g(s)\mathcal{G}(s,U) = f_0(s) + a_1\left[\Phi_f^{\rm NLO}(s)\mathcal{F}(s,U+b_1) + \Theta_g^{\rm NLO}(s)\mathcal{G}(s,U+b_1)\right], \tag{9}$$

$$-\Theta_f(s)\mathcal{F}(s,U) + [U - \Phi_g(s)]\mathcal{G}(s,U) = g_0(s) + a_1\left[\Theta_f^{\rm NLO}(s)\mathcal{F}(s,U+b_1) + \Phi_g^{\rm NLO}(s)\mathcal{G}(s,U+b_1)\right]. \tag{10}$$

Here the $\mathcal{F}(s, U)$ and $\mathcal{G}(s, U)$ are the Laplace transform of singlet and gluon FFs in (s, U) space. Initial input FFs are denoted by $f_0(s)$ and $g_0(s)$. As we mentioned before, these initial inputs are selected from HKNS code[12] at initial scale of $Q_0 = 4.5$ GeV. The parameters $a_1 = 0.025$ and $b_1 = 10.7$ are the best fit parameters to $a(\tau) = \frac{\alpha_s(\tau)}{4\pi} \approx a_0 + a_1 e^{-b_1\tau}$ at NLO approximation.[6] The functions $\Phi_{f,g}$ and

$\Theta_{f,g}$ specified the Laplace transformation of splitting functions and can be found in Ref. 3 and also given in App. A:

$$\Phi_f(s) \equiv \Phi_f^{\rm LO}(s) + a_0 \Phi_f^{\rm NLO}(s)\,, \quad \Phi_g(s) \equiv \Phi_g^{\rm LO}(s) + a_0 \Phi_g^{\rm NLO}(s)\,, \tag{11}$$

$$\Theta_f(s) \equiv \Theta_f^{\rm LO}(s) + a_0 \Theta_f^{\rm NLO}(s)\,, \quad \Theta_g(s) \equiv \Theta_g^{\rm LO}(s) + a_0 \Theta_g^{\rm NLO}(s)\,. \tag{12}$$

With the initial input functions for $f_0(s)$ and $g_0(s)$, their evolved solutions in the Laplace s space are given by[10]

$$\begin{aligned} f(s,\tau) &= k_{ff}(s,\tau) f_0(s) + k_{fg}(s,\tau) g_0(s)\,, \\ g(s,\tau) &= k_{gg}(s,\tau) g_0(s) + k_{gf}(s,\tau) f_0(s)\,, \end{aligned} \tag{13}$$

where the k's in Eq. (13) are given in App. B of Ref. 3 for the first iteration. Finally, the singlet and gluon FFs in (z, Q^2) space can be calculated with known inverse Laplace transform.[11] Our results in Ref. 3 show a nice agreement between these analytical solution and other global fit results for charged mesons π^+ and k^+.

3. Natural Pions and Kaons Fragmentation Functions and the Role of Charge Conjugation Symmetry

The FF of total sea quarks is defined as follows:

$$D_s(z,Q^2) - D_{\rm ns}(z,Q^2) = D_{\bar q}(z,Q^2)\,, \tag{14}$$

where

$$\begin{aligned} D_{\bar q}(z,Q^2) &= 2D_{\bar u}(z,Q^2) + 2D_{\bar d}(z,Q^2) \\ &\quad + 2D_s(z,Q^2) + 2D_c(z,Q^2) + 2D_b(z,Q^2)\,. \end{aligned} \tag{15}$$

Because the heavier sea quarks can produce hadrons with higher probability, we simply parametrized the fraction of different kinds of sea quarks FFs as their mass ratio and then we have:[3]

$$D_{\rm quark}(z,Q^2) = \frac{D_{\bar q}(z,Q^2)}{B^A}\,. \tag{16}$$

The parameters A and B are summarized in Table 1 in Ref. 3. We also used these flavor symmetries between different kinds of FFs in π^+, K^+:[12]

$$\begin{aligned} D_{\bar u}^{\pi^+}(z,Q^2) &= D_d^{\pi^+}(z,Q^2) \neq D_s^{\pi^+}(z,Q^2)\,, \\ D_u^{\pi^+}(z,Q^2) &= D_{\bar d}^{\pi^+}(z,Q^2)\,, \\ D_s^{\pi^+}(z,Q^2) &= D_{\bar s}^{\pi^+}(z,Q^2)\,, \\ D_c^{\pi^+}(z,Q^2) &= D_{\bar c}^{\pi^+}(z,Q^2)\,, \\ D_b^{\pi^+}(z,Q^2) &= D_{\bar b}^{\pi^+}(z,Q^2)\,, \end{aligned} \tag{17}$$

$$
\begin{aligned}
&D_{\bar{u}}^{K^+}(z,Q^2) \neq D_d^{K^+}(z,Q^2) \neq D_s^{K^+}(z,Q^2)\,,\\
&D_d^{K^+}(z,Q^2) = D_{\bar{d}}^{K^+}(z,Q^2)\,,\\
&D_c^{K^+}(z,Q^2) = D_{\bar{c}}^{K^+}(z,Q^2)\,,\\
&D_b^{K^+}(z,Q^2) = D_{\bar{b}}^{K^+}(z,Q^2)\,.
\end{aligned}
\tag{18}
$$

To calculate the natural mesons FFs, we first used the charge conjugation symmetries related the $\pi^+(K^+)$ FFs to those of $\pi^-(K^-)$ to derive the $\pi^-(K^-)$ FFs:

$$
\begin{aligned}
&D_{\bar{q}}^{\pi^+(K^+)}(z,Q^2) = D_q^{\pi^-(K^-)}(z,Q^2)\,,\\
&D_g^{\pi^+(K^+)}(z,Q^2) = D_g^{\pi^-(K^-)}(z,Q^2)\,.
\end{aligned}
\tag{19}
$$

Finally the neutral mesons FFs can be obtained by:

$$
\begin{aligned}
D_i^{\pi^0}(z,Q^2) &= \frac{1}{2}\left[D_i^{\pi^+}(z,Q^2) + D_i^{\pi^-}(z,Q^2)\right],\\
D_i^{K^0}(z,Q^2) &= \frac{1}{2}\left[D_i^{K^+}(z,Q^2) + D_i^{K^-}(z,Q^2)\right].
\end{aligned}
\tag{20}
$$

Figures 1 and 2 show the results of FFs of neutral pions and kaons at $Q^2 = M_z^2$. We also compared our results with those of AKK, DSS and HKNS codes[12,15,16] to be sure about our analytical solutions.

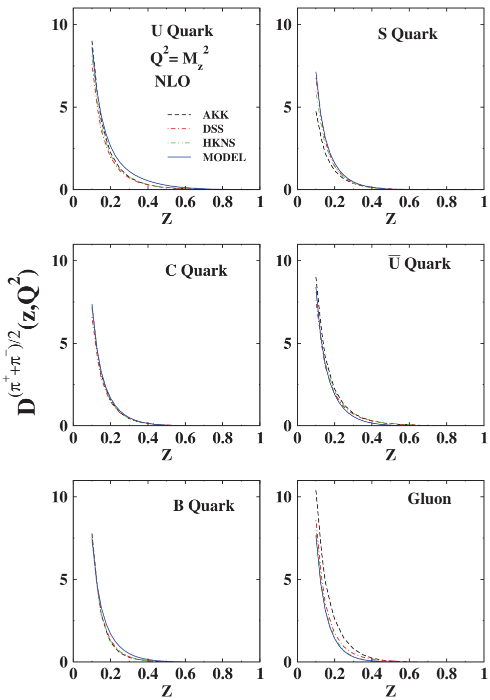

Fig. 1. Fragmentation functions of natural pion and comparison with AKK, DSS and HKNS global fits.

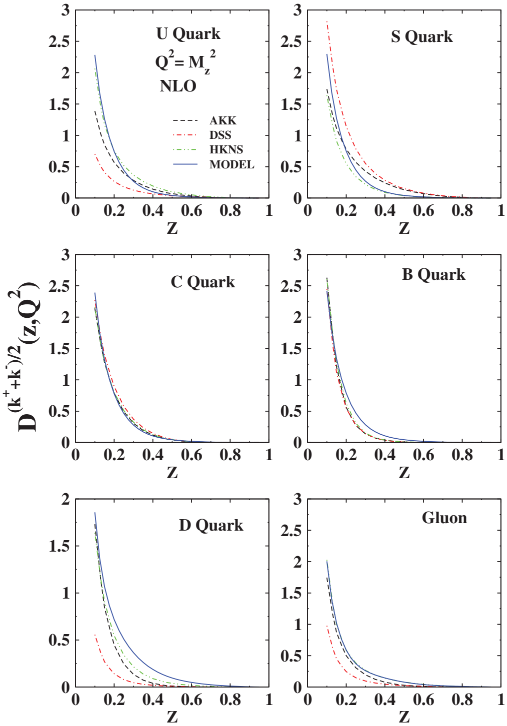

Fig. 2. Fragmentation functions of natural kaon and comparison with AKK, DSS and HKNS global fits.

4. Total Fragmentation Functions

According to the factorization theorem,[23] the total FF can be expressed in terms of the partonic hard scattering cross-sections and the nonperturbative FFs as:

$$F^H(z, Q^2) = \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma^h}{dz}(e^- e^+ \to HX)(z, Q^2)$$
$$= \sum C_i(z, Q^2) \otimes D_i^H(z, Q^2) \,, \tag{21}$$

where σ_{tot} is the total hadronic cross-section.[25] The $C_i(z, Q^2)$ is the Wilson coefficient function related to the partonic cross-section $e^- e^+ \to q\bar{q}$ and calculated in perturbative QCD as:[14,24]

$$C_q^1(z) = C_F\left[(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\frac{1}{(1-z)_+}\right.$$
$$\left. + 2\frac{1+z^2}{1-z}\ln(z) + \frac{3}{2}(1-z) + \left(\frac{3}{2}\pi^2 - \frac{9}{2}\right)\delta(1-z)\right], \tag{22}$$

$$C_g^1(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\ln(z^2(1-z)) - 2\frac{1-z}{z}\right], \tag{23}$$

$$C_q^L(z) = C_F\,, \tag{24}$$

$$C_g^L(z) = 4C_F\frac{(1-z)}{z}\,, \tag{25}$$

where $C_F = \frac{4}{3}$. The total FFs of natural pion π^0 and kaon K^0 are shown in Figs. 3 and 4. We compared our result with those from HKNS global fit and also with data from TASSO, ALEPH, TOPAZ and OPAL experiments.[26–29] The agreement between experimental data and our model is quite reasonable. This means that our analytical solutions are correct.

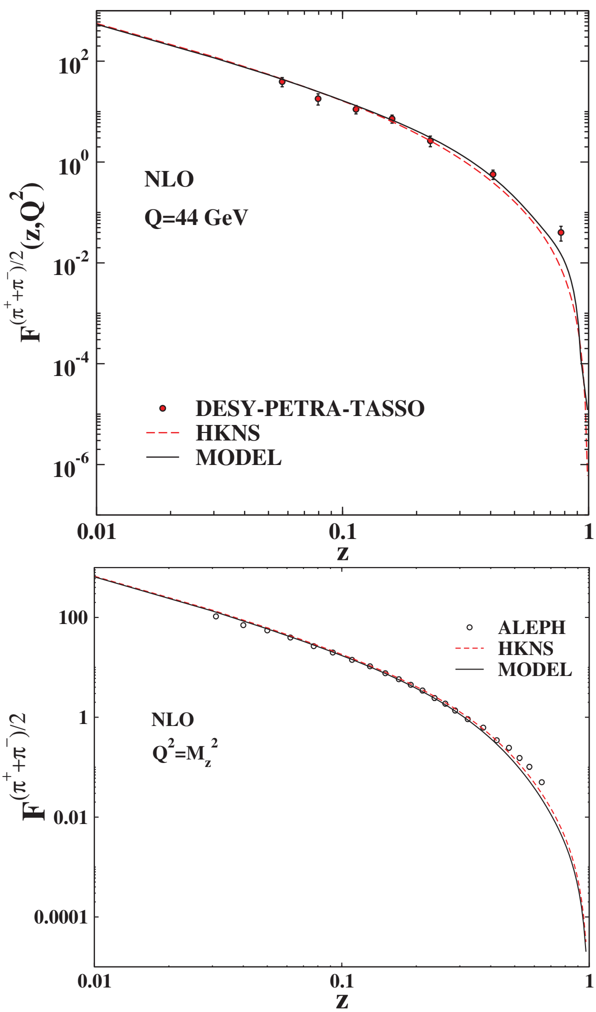

Fig. 3. Total FFs of natural pion and comparison with experimental data from TASSO and ALEPH Collaborations[26,27] at $Q = 44$ GeV and $Q^2 = M_z^2$. We also compared our results with HKNS global fit.

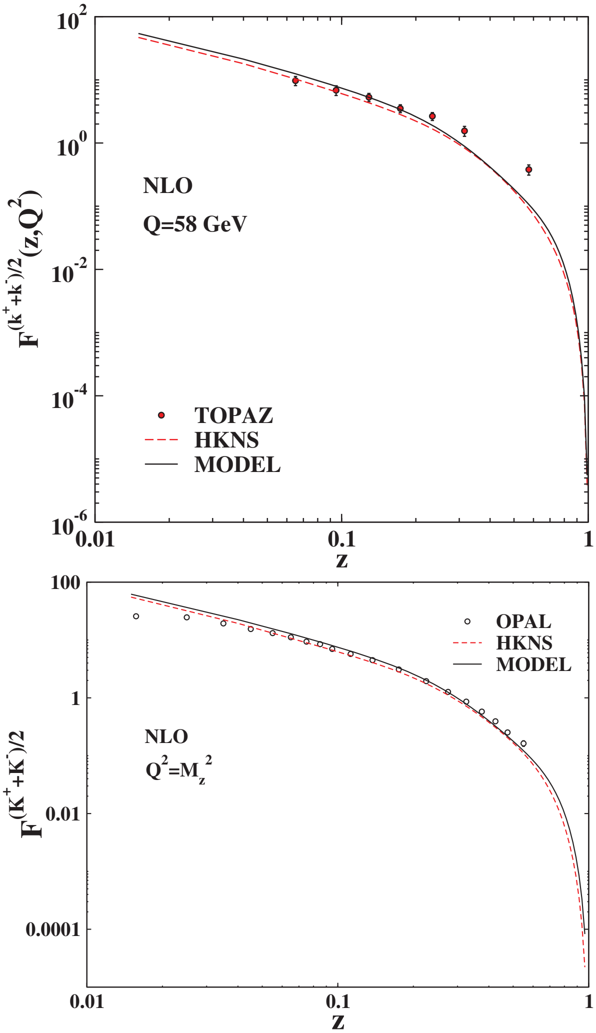

Fig. 4. Total FFs of natural kaon and proton and comparison with experimental data from TOPAZ and OPAL Collaborations[28,29] at $Q = 58$ GeV and $Q^2 = M_z^2$. We also compared our results with HKNS global fit.

5. Conclusions and Remarks

Using the analytical solutions to DGLAP evolution equation based on the Laplace transforms, we find the FFs of the neutral pions and kaons. Finding these solutions enables us to use the natural mesons' experimental data for total FFs in a global fit to determine FFs. This technique has the facility that the analytical solution of the FFs is obtained more strictly by using the related kernels and the calculations controlled in a better way. We have used the HKNS code for initial input FFs to be sure about our analytical solutions.

Our results for natural pions and kaons are compared with those from global fits and also with experimental data and there is a reasonable agreements between them.

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Appendix A

We present here the results for the Laplace transforms of splitting functions, denoted by $\Phi^{\mathrm{LO,\,NLO}}$ and $\Theta^{\mathrm{LO,\,NLO}}$ at the NLO approximation.

$$\Phi_f^{\mathrm{LO}}(s) = 4 - \frac{8}{3}\left(\frac{1}{s+1} + \frac{1}{s+2} + 2(\psi(s+1)+\gamma_E)\right),$$

$$\Theta_g^{\mathrm{LO}}(s) = \frac{16}{3}n_f\left(\frac{2}{s} - \frac{2}{s+2} + \frac{2}{s+3}\right),$$

$$\Theta_f^{\mathrm{LO}}(s) = \frac{1}{s+1} - \frac{2}{s+2} + \frac{2}{s+3},$$

$$\Phi_g^{\mathrm{LO}}(s) = \frac{33-2n_f}{3} + 12\left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} - \frac{1}{s+3} - \psi(s+1) - \gamma_E\right),$$

$$\begin{aligned}
\Phi_{\mathrm{ns}\,qq}^{\mathrm{NLO}} &= C_F T_f\left(-\frac{2}{3(s+1)^2} - \frac{2}{9(s+1)} - \frac{2}{3(s+2)^2} + \frac{22}{9(s+2)} + \frac{4}{3}\psi'(s+1)\right)\\
&\quad + C_F^2\left(\frac{5}{(s+1)^3} + \frac{5}{(s+1)^2} - \frac{5}{s+1} + \frac{5}{(s+2)^3} + \frac{3}{(s+2)^2}\right.\\
&\quad + \frac{5}{s+2} - \frac{2}{(s+1)^2}\left(\gamma_E + \frac{1}{s+1}\psi(s+1) - (s+1)\psi'(s+2)\right)\\
&\quad - \frac{2}{(s+2)^2}\left(\gamma_E + \frac{1}{s+2}\psi(s+2) - (s+2)\psi'(s+3)\right)\\
&\quad \left. + 4\left(\left(\psi(s+1)+\gamma_E\right)\psi'(s+1) - \frac{1}{2}\psi''(s+1)\right) - 3\psi'(s+1) + 4\psi''(s+1)\right)\\
&\quad + C_A C_F\left(-\frac{1}{(s+1)^3} + \frac{5}{6(s+1)^2} + \frac{53}{18(s+1)} + \frac{\pi^2}{6(s+1)}\right.\\
&\quad - \frac{1}{(s+2)^3} + \frac{5}{6(s+2)^2} - \frac{187}{18(s+2)}\\
&\quad + \frac{\pi^2}{6(s+2)} - \frac{67}{9}(\psi(s+1)+\gamma_E) + \frac{1}{3}\pi^2(\psi(s+1)+\gamma_E)\\
&\quad \left. + 2\left(\frac{67}{18} - \frac{\pi^2}{6}\right)(\psi(s+1)+\gamma_E) - \frac{11}{3}\psi'(s+1) - \psi''(s+1)\right),
\end{aligned}$$

$$
\begin{aligned}
\Phi^{\mathrm{NLO}}_{\mathrm{ns}\,q\bar{q}} = C_F\left(-\frac{C_A}{2}+C_F\right)\Bigg(&\frac{2}{(s+1)^3}-\frac{2}{(s+1)^2}+\frac{4}{s+1}-\frac{\pi^2}{3(s+1)}\\
&-\frac{1.9968}{(s+2)^3}-\frac{2}{(s+2)^2}+\frac{3.3246}{s+2}+\frac{3.9404}{(s+3)^3}-\frac{7.1312}{s+3}\\
&-\frac{3.602}{(s+4)^3}+\frac{5.8861}{s+4}+\frac{2.6484}{(s+5)^3}+\frac{3.9432}{s+5}-\frac{1.2696}{(s+6)^3}-\frac{14.24}{s+6}\\
&+\frac{0.2796}{(s+7)^3}+\frac{20.43}{s+7}-\frac{19.77}{s+8}+\frac{13.05}{s+9}+\frac{6.286}{s+10}+\frac{1.997}{s+11}-\frac{0.3076}{s+12}\\
&-2\left(\frac{4}{(s+1)^3}-\frac{\ln(4)}{(s+1)^2}-\frac{\psi\left(\frac{s}{2}+1\right)}{(s+1)^2}+\frac{\psi\left(\frac{s+1}{2}\right)}{(s+1)^2}+\frac{\psi'\left(\frac{s}{2}+1\right)}{2s+2}-\frac{\psi'\left(\frac{s+1}{2}\right)}{2(s+1)}\right)\\
&-\frac{0.9984}{(s+2)^3}\Bigg(\frac{16}{(s+1)^2}+\frac{12s}{(s+1)^2}+(s+2)\ln(16)\\
&-2(s+2)\psi\left(\frac{s}{2}+1\right)+2(s+1)\psi\left(\frac{s+1}{2}\right)\\
&+(s+2)^2\psi'\left(\frac{s}{2}+1\right)-(s+2)^2\psi'\left(\frac{s+1}{2}\right)\Bigg)\\
&-\frac{1.9702}{(s+3)^3}\Bigg(\frac{164}{(s+1)^2(s+2)^2}+\frac{284s}{(s+1)^2(s+2)^2}+\frac{188s^2}{(s+1)^2(s+2)^2}\\
&+\frac{60s^3}{(s+1)^2(s+2)^2}+\frac{8s^4}{(s+1)^2(s+2)^2}-4(s+3)\ln(2)\\
&-2(s+3)\psi\left(\frac{s}{2}+1\right)+2(s+3)\psi\left(\frac{s+1}{2}\right)\\
&+(s+3)^2\psi'\left(\frac{s}{2}+1\right)-(s+3)^2\psi'\left(\frac{s+1}{2}\right)\Bigg)\\
&-\frac{1.801}{(s+4)^3}\Bigg(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2}+\frac{4392s}{(s+1)^2(s+2)^2(s+3)^2}\\
&+\frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2}+\frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2}\\
&+\frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2}+\frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2}\\
&+4(s+4)\ln(2)-2(s+4)\psi\left(\frac{s}{2}+1\right)+2(s+4)\psi\left(\frac{s+1}{2}\right)\\
&+(s+4)^2\psi'\left(\frac{s}{2}+1\right)-(s+4)^2\psi'\left(\frac{s+1}{2}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1.3242}{(s+5)^3}\left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\right.\\
&+ \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\\
&+ \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\\
&+ \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\\
&+ \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\\
&- 4(s+5)\ln(2) - 2(5+s)\psi\left(\frac{s}{2}+1\right) + 2(5+s)\psi\left(\frac{s+1}{2}\right)\\
&\left. + (s+5)^2\psi'\left(\frac{s}{2}+1\right) - (s+5)^2\psi'\left(\frac{s+1}{2}\right)\right)\\
&- \frac{0.6348}{(s+6)^2}\left(\ln(16) - 2\psi\left(\frac{s}{2}+4\right) + 2\psi\left(\frac{s+7}{2}\right)\right.\\
&\left. + (s+6)\psi'\left(\frac{s}{2}+4\right) - (s+6)\psi'\left(\frac{s+7}{2}\right)\right)\\
&+ \frac{0.1398}{(s+7)^2}\left(\ln(16) + 2\psi\left(\frac{s}{2}+4\right) - 2\psi\left(\frac{s+9}{2}\right)\right.\\
&\left.\left. - (s+7)\psi'\left(\frac{s}{2}+4\right) + (s+7)\psi'\left(\frac{s+9}{2}\right)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\Phi_q^{\mathrm{NLO}} = C_F T_f &\left(-\frac{40}{9s} + \frac{4}{(s+1)^3} + \frac{28}{3(s+1)^2} - \frac{146}{9(s+1)} + \frac{4}{(s+2)^3}\right.\\
&\left. + \frac{52}{3(s+2)^2} + \frac{94}{9(s+2)} + \frac{16}{3(s+3)^2} + \frac{112}{9(s+3)} + \frac{4}{3}\psi'(s+1)\right)\\
&+ C_F^2\left(\frac{7}{(s+1)^3} + \frac{3}{(s+1)^2} - \frac{1}{s+1} - \frac{\pi^2}{3(s+1)} + \frac{3.0032}{(s+2)^3}\right.\\
&+ \frac{1}{(s+2)^2} + \frac{8.3246}{s+2} + \frac{3.9404}{(s+3)^3} - \frac{7.1312}{s+3} - \frac{3.602}{(s+4)^3}\\
&+ \frac{5.886}{s+4} + \frac{2.6484}{(s+5)^3} + \frac{3.9432}{s+5} - \frac{1.2696}{(s+6)^3}\\
&- \frac{14.2478}{s+6} + \frac{0.2796}{(s+7)^3} + \frac{20.4376}{s+7} - \frac{19.7727}{s+8} + \frac{13.056}{s+9}
\end{aligned}
$$

$$- \frac{6.2862}{s+10} + \frac{1.9971}{s+11} - \frac{0.3075}{s+12} - \frac{8}{(s+1)^3} + \frac{2\ln(4)}{(s+1)^2}$$
$$+ \frac{2\psi\left(\frac{s}{2}+1\right)}{(s+1)^2} - \frac{2\psi\left(\frac{s+1}{2}\right)}{(s+1)^2} - \frac{\psi'\left(\frac{s}{2}+1\right)}{s+1} + \frac{\psi'\left(\frac{s+1}{2}\right)}{(s+1)^2}$$
$$- \frac{0.9984}{(s+2)^3}\left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi\left(\frac{s}{2}+1\right)\right.$$
$$\left. + 2(s+2)\psi\left(\frac{s+1}{2}\right) + (s+2)^2\psi'\left(\frac{s}{2}+1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$- \frac{1.9702}{(s+3)^3}\left(\frac{164}{(s+1)^2(s+2)^2} + \frac{284s}{(s+1)^2(s+2)^2}\right.$$
$$+ \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2}$$
$$- 4(s+3)\ln(2) - 2(s+3)\psi\left(\frac{s}{2}+1\right) + 2(s+3)\psi\left(\frac{s+1}{2}\right)$$
$$\left. + (s+3)^2\psi'\left(\frac{s}{2}+1\right) - (s+3)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$- \frac{1.801}{(s+4)^3}\left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2}\right.$$
$$+ \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2}$$
$$+ \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2}$$
$$+ 4(s+4)\ln(2) - 2(s+4)\psi\left(\frac{s}{2}+1\right) + 2(s+4)\psi\left(\frac{s+1}{2}\right)$$
$$\left. + (s+4)^2\psi'\left(\frac{s}{2}+1\right) - (s+4)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$- \frac{1.3242}{(s+5)^3}\left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\right.$$
$$+ \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$
$$+ \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$
$$+ \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$

$$+ \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$

$$- 4(s+5)\ln(2) - 2(s+5)\psi\left(\frac{s}{2}+1\right) + 2(s+5)\psi\left(\frac{s+1}{2}\right)$$

$$+ (s+5)^2\psi'\left(\frac{s}{2}+1\right) - (s+5)^2\psi'\left(\frac{s+1}{2}\right)\bigg)$$

$$- \frac{2}{(s+1)^2}\left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)\right)$$

$$- \frac{2}{(s+2)^2}\left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)\right)$$

$$- \frac{0.6348}{(s+6)^2}\bigg(\ln(16) - 2\psi\left(\frac{s}{2}+4\right) + 2\psi\left(\frac{s+7}{2}\right)$$

$$+ (s+6)\psi'\left(\frac{s}{2}+4\right) - (s+6)\psi'\left(\frac{s+7}{2}\right)\bigg)$$

$$+ \frac{0.1398}{(s+7)^2}\bigg(\ln(16) + 2\psi\left(\frac{s}{2}+4\right) - 2\psi\left(\frac{s+9}{2}\right)$$

$$- (s+7)\psi'\left(\frac{s}{2}+4\right) + (s+7)\psi'\left(\frac{s+9}{2}\right)\bigg)$$

$$+ 4\left(\left((\psi(s+1)+\gamma_E)\psi'(s+1) - \frac{1}{2}\psi''(s+1)\right) - 3\psi'(s+1) + 4\psi''(s+1)\right)$$

$$+ C_A C_F\bigg(\frac{2}{(s+1)^3} + \frac{11}{6(s+1)^2} + \frac{17}{18(s+1)} + \frac{\pi^2}{3(s+1)}$$

$$- \frac{0.0016}{(s+2)^3} + \frac{11}{6(s+2)^2} - \frac{10.4062}{s+2} - \frac{1.9702}{(s+3)^3} + \frac{3.5656}{s+3}$$

$$+ \frac{1.801}{(s+4)^3} - \frac{2.9430}{s+4} - \frac{1.3242}{(s+5)^3} - \frac{1.9716}{s+5} + \frac{0.6348}{(s+6)^3}$$

$$+ \frac{7.1239}{s+6} - \frac{0.1398}{(s+7)^3} - \frac{10.2188}{s+7} + \frac{9.8863}{s+8} - \frac{6.5284}{s+9}$$

$$+ \frac{3.1431}{s+10} - \frac{0.9985}{s+11} + \frac{0.1537}{s+12} - \frac{67(\psi(s+1)+\gamma_E)}{9}$$

$$+ \frac{1}{3}\pi^2(\psi(s+1)+\gamma_E) + 2\left(\frac{67}{18} - \frac{\pi^2}{6}\right)(\psi(s+1)+\gamma_E) - \frac{\ln(4)}{(s+1)^2}$$

$$- \frac{\psi\left(\frac{s}{2}+1\right)}{(s+1)^2} + \frac{\psi\left(\frac{s+1}{2}\right)}{(s+1)^2} + \frac{\psi'\left(\frac{s}{2}+1\right)}{2s+2} - \frac{\psi'\left(\frac{s+1}{2}\right)}{2s+2}$$

$$+ \frac{0.4992}{(s+2)^3}\left(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi\left(\frac{s}{2}+1\right)\right.$$
$$\left. + 2(s+2)\psi\left(\frac{s+1}{2}\right) + (s+2)^2\psi'\left(\frac{s}{2}+1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$+ \frac{0.9851}{(s+3)^3}\left(\frac{164}{(s+1)^2(s+2)^2} + \frac{284s}{(s+1)^2(s+2)^2}\right.$$
$$+ \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2}$$
$$- 4(s+3)\ln(2) - 2(s+3)\psi\left(\frac{s}{2}+1\right) + 2(s+3)\psi\left(\frac{s+1}{2}\right)$$
$$\left. + (s+3)^2\psi'\left(\frac{s}{2}+1\right) - (s+3)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$+ \frac{0.9005}{(s+4)^3}\left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2}\right.$$
$$+ \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2}$$
$$+ \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2}$$
$$+ 4(s+4)\ln(2) - 2(s+4)\psi\left(\frac{s}{2}+1\right) + 2(s+4)\psi\left(\frac{s+1}{2}\right)$$
$$\left. + (s+4)^2\psi'\left(\frac{s}{2}+1\right) - (s+4)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$+ \frac{0.6621}{(s+5)^3}\left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\right.$$
$$+ \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$
$$+ \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$
$$+ \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$
$$+ \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5)\ln(2) - 2(s+5)\psi\left(\frac{s}{2}+1\right)$$
$$\left. + 2(s+5)\psi\left(\frac{s+1}{2}\right) + (s+5)^2\psi'\left(\frac{s}{2}+1\right) - (s+5)^2\psi'\left(\frac{s+1}{2}\right)\right)$$

$$+\frac{0.3174}{(s+6)^2}\left(\ln(16)-2\psi\left(\frac{s}{2}+4\right)+2\psi\left(\frac{s+7}{2}\right)\right.$$
$$\left.+(s+6)\psi'\left(\frac{s}{2}+4\right)-(s+6)\psi'\left(\frac{s+7}{2}\right)\right)$$
$$-\frac{0.0699}{(s+7)^2}\left(\ln(16)+2\psi\left(\frac{s}{2}+4\right)-2\psi\left(\frac{s+9}{2}\right)\right.$$
$$\left.-(s+7)\psi'\left(\frac{s}{2}+4\right)+(s+7)\psi'\left(\frac{s+9}{2}\right)\right)-\frac{11}{3}\psi'(s+1)-\psi''(s+1)\Bigg),$$

$$\Theta_q^{\mathrm{NLO}}=T_f^2\left(\frac{8}{3(s+1)^2}-\frac{40}{9(s+1)}-\frac{16}{3(s+2)^2}+\frac{32}{9(s+2)}\right.$$
$$+\frac{16}{3(s+3)^2}-\frac{32}{9(s+3)}+\frac{8(\psi(s+2)+\gamma_E)}{3(s+1)}$$
$$\left.-\frac{16(\psi(s+3)+\gamma_E)}{3(s+2)}+\frac{16(\psi(s+4)+\gamma_E)}{3(s+3)}\right)$$
$$+C_AT_f\left(-\frac{40}{9s}+\frac{4}{(s+1)^3}+\frac{8}{3(s+1)^2}+\frac{26}{9(s+1)}\right.$$
$$+\frac{24}{(s+2)^3}+\frac{68}{3(s+2)^2}-\frac{33.231}{s+2}-\frac{4\pi^2}{3(s+2)}+\frac{8}{3(s+3)^2}$$
$$+\frac{96.875}{s+3}-\frac{67.644}{s+4}+\frac{83.04}{s+5}-\frac{82.976}{s+6}+\frac{56.16}{s+7}-\frac{22}{s+8}$$
$$-\frac{22(\psi(s+2)+\gamma_E)}{3(s+1)}+\frac{20(\psi(s+3)+\gamma_E)}{3(s+2)}-\frac{20(\psi(s+4)+\gamma_E)}{3(s+3)}$$
$$-2\left(\frac{4}{(s+1)^3}-\frac{\ln(4)}{(s+1)^2}-\frac{\psi\left(\frac{s}{2}+1\right)}{(s+1)^2}+\frac{\psi\left(\frac{s+1}{2}\right)}{(s+1)^2}+\frac{\psi'\left(\frac{s}{2}+1\right)}{2s+2}-\frac{\psi'\left(\frac{s+1}{2}\right)}{2(s+1)}\right)$$
$$+\frac{2}{(s+2)^3}\left(\frac{16}{(s+1)^2}+\frac{12s}{(s+1)^2}+(s+2)\ln(16)-2(s+2)\psi\left(\frac{s}{2}+1\right)\right.$$
$$\left.+2(s+2)\psi\left(\frac{s+1}{2}\right)+(s+2)^2\psi'\left(\frac{s}{2}+1\right)-(s+2)^2\psi'\left(\frac{s+1}{2}\right)\right)$$
$$-\frac{2}{(s+3)^3}\left(\frac{164}{(s+1)^2(s+2)^2}+\frac{284s}{(s+1)^2(s+2)^2}+\frac{188s^2}{(s+1)^2(s+2)^2}\right.$$
$$+\frac{60s^3}{(s+1)^2(s+2)^2}+\frac{8s^4}{(s+1)^2(s+2)^2}-4(s+3)\ln(2)-2(s+3)\psi\left(\frac{s}{2}+1\right)$$
$$\left.+2(s+3)\psi\left(\frac{s+1}{2}\right)+(s+3)^2\psi'\left(\frac{s}{2}+1\right)-(s+3)^2\psi'\left(\frac{s+1}{2}\right)\right)$$

$$
\begin{aligned}
&+\frac{2(\pi^2+6(\psi(s+2)+\gamma_E)^2-6\psi'(s+2))}{6s+6}\\
&-\frac{8}{(s+1)^2}\left(\gamma_E+\frac{1}{s+1}+\psi(s+1)-(s+1)\psi'(s+2)\right)\\
&-\frac{4(\pi^2+6(\psi(s+3)+\gamma_E)^2-6\psi'(s+3))}{6s+12}\\
&+\frac{16}{(s+2)^2}\left(\gamma_E+\frac{1}{s+2}+\psi(s+2)-(s+2)\psi'(s+3)\right)\\
&+\frac{4(\pi^2+6(\psi(s+4)+\gamma_E)^2-6\psi'(s+4))}{6s+18}\\
&-\frac{16}{(s+3)^2}\left(\gamma_E+\frac{1}{s+3}+\psi(s+3)-(s+3)\psi'(s+4)\right)\Bigg)\\
&+C_FT_f\Bigg(-\frac{2}{(s+1)^3}+\frac{7}{(s+1)^2}-\frac{12}{s+1}-\frac{2\pi^2}{3(s+1)}\\
&+\frac{4}{(s+2)^3}-\frac{8}{(s+2)^2}+\frac{39.16}{s+2}+\frac{4\pi^2}{3(s+2)}-\frac{8}{(s+3)^3}\\
&-\frac{65.856}{s+3}-\frac{4\pi^2}{3(s+3)}+\frac{77.872}{s+4}-\frac{81.216}{s+5}+\frac{80.128}{s+6}-\frac{51.968}{s+7}\\
&+\frac{17.6}{s+8}+\frac{2(\psi(s+1)+\gamma_E)}{s+1}+\frac{4(\psi(s+2)+\gamma_E)}{s+1}-\frac{4(\psi(s+2)+\gamma_E)}{s+2}\\
&+\frac{4(\psi(s+3)+\gamma_E)}{s+3}-\frac{2(\pi^2+6(\psi(s+2)+\gamma_E)^2-6\psi'(s+2))}{6s+6}\\
&+\frac{12}{(s+1)^2}\left(\gamma_E+\frac{1}{s+1}+\psi(s+1)-(s+1)\psi'(s+2)\right)\\
&+\frac{4(\pi^2+6(\psi(s+3)+\gamma_E)^2-6\psi'(s+3))}{6s+12}\\
&-\frac{24}{(s+2)^2}\left(\gamma_E+\frac{1}{s+2}+\psi(s+2)-(s+2)\psi'(s+3)\right)\\
&-\frac{4(\pi^2+6(\psi(s+4)+\gamma_E)^2-6\psi'(s+4))}{6s+18}\\
&+\frac{24}{(s+3)^2}\left(\gamma_E+\frac{1}{s+3}+\psi(s+3)-(s+3)\psi'(s+4)\right)\Bigg)\,,
\end{aligned}
$$

$$\begin{aligned}
\Theta_g^{\mathrm{NLO}} = C_F^2\Bigg(&\frac{2}{(s+1)^3} + \frac{8}{(s+1)^2} - \frac{16.66}{s+1} - \frac{1}{(s+2)^3} - \frac{1}{2(s+2)^2} \\
&+ \frac{34.196}{s+2} - \frac{40.096}{s+3} + \frac{42.432}{s+4} - \frac{35.224}{s+5} \\
&+ \frac{17.392}{s+6} - \frac{4.4}{s+7} - \frac{2(\psi(s+3)+\gamma_E)}{s+2} \\
&+ \frac{1}{3s}\big(\pi^2 + 6(\psi(s+1)+\gamma_E)^2 - 6\psi'(s+1)\big) \\
&- \frac{8}{s^3}(1 + s\gamma_E + s(\psi(s) - s\psi'(s+1))) \\
&- \frac{2}{6s+6}\big(\pi^2 + 6(\psi(s+2)+\gamma_E)^2 - 6\psi'(s+2)\big) \\
&+ \frac{8}{(s+1)^2}\left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)\right) \\
&+ \frac{1}{6s+12}\big(\pi^2 + 6(\psi(s+3)+\gamma_E)^2 - 6\psi'(s+3)\big) \\
&- \frac{4}{(s+2)^2}\left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)\right)\Bigg) \\
&+ C_A C_F\Bigg(-\frac{4}{s^3} + \frac{6}{s^2} + \frac{17}{9s} - \frac{2\pi^2}{3s} - \frac{8}{(s+1)^2} + \frac{25.2}{s+1} \\
&- \frac{4}{(s+2)^3} - \frac{9}{(s+2)^2} - \frac{23.27}{s+2} - \frac{\pi^2}{3(s+2)} - \frac{8}{3(s+3)^2} + \frac{35.99}{s+3} \\
&- \frac{41.046}{s+4} + \frac{35.01}{s+5} - \frac{17.444}{s+6} + \frac{3.3}{s+7} + \frac{2(\psi(s+3)+\gamma_E)}{s+2} \\
&+ \frac{1}{s^2}\left(\ln(16) - 2\psi\left(\frac{s}{2}+1\right) + 2\psi\left(\frac{s+1}{2}\right) + s\psi'\left(\frac{s}{2}+1\right) - s\psi'\left(\frac{s+1}{2}\right)\right) \\
&- 2\left(\frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi\left(\frac{s}{2}+1\right)}{(s+1)^2} + \frac{\psi\left(\frac{s+1}{2}\right)}{(s+1)^2} + \frac{\psi'\left(\frac{s}{2}+1\right)}{2s+2} - \frac{\psi'\left(\frac{s+1}{2}\right)}{2s+2}\right) \\
&+ \frac{1}{2(s+2)^3}\Bigg(\frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) \\
&- 2(s+2)\psi\left(\frac{s}{2}+1\right) + 2(s+2)\psi\left(\frac{s+1}{2}\right) \\
&+ (s+2)^2\psi'\left(\frac{s}{2}+1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{3s}\left(\pi^2 + 6(\psi(s+1)+\gamma_E)^2 - 6\psi'(s+1)\right)\\
&+ \frac{12}{s^3}\left(1 + s\gamma_E + s(\psi(s) - s\psi'(s+1))\right)\\
&+ \frac{2}{6s+6}\left(\pi^2 + 6(\psi(s+2)+\gamma_E)^2 - 6\psi'(s+2)\right)\\
&- \frac{12}{(s+1)^2}\left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)\right)\\
&- \frac{1}{6s+12}\left(\pi^2 + 6(\psi(s+3)+\gamma_E)^2 - 6\psi'(s+3)\right)\\
&+ \frac{6}{(s+2)^2}\left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)\right)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\Phi_g^{\mathrm{NLO}} = {}& C_F T_f\left(-\frac{16}{3s^2} + \frac{92}{9s} + \frac{4}{(s+1)^3} - \frac{10}{(s+1)^2} - \frac{4}{s+1}\right.\\
&\left.+ \frac{4}{(s+2)^3} - \frac{14}{(s+2)^2} + \frac{12}{s+2} - \frac{16}{3(s+3)^2} - \frac{164}{9(s+3)}\right)\\
&+ C_A T_f\left(\frac{8}{3s^2} - \frac{46}{9s} - \frac{4}{(s+1)^2} + \frac{58}{9(s+1)} + \frac{4}{(s+2)^2}\right.\\
&\left.- \frac{38}{9(s+2)} - \frac{8}{3(s+3)^2} + \frac{46}{9(s+3)} + \frac{8}{3}\psi'(s+1)\right)\\
&+ C_A^2\left(-\frac{8}{s^3} + \frac{22}{3s^2} + \frac{2}{(s+1)^3} + \frac{11}{(s+1)^2} + \frac{4.4407}{s+1} - \frac{17.9984}{(s+2)^3}\right.\\
&+ \frac{1}{(s+2)^2} - \frac{6.9024}{s+2} - \frac{\pi^2}{3(s+2)} + \frac{5.9702}{(s+3)^3} + \frac{22}{3(s+3)^2}\\
&- \frac{6.7917}{s+3} + \frac{\pi^2}{3(s+3)} - \frac{1.801}{(s+4)^3} - \frac{3.5389}{s+4} + \frac{1.3242}{(s+5)^3} + \frac{1.2736}{s+5}\\
&- \frac{0.6348}{(s+6)^3} - \frac{5.6479}{s+6} + \frac{0.1398}{(s+7)^3} + \frac{9.2228}{s+7} - \frac{7.6863}{s+8} + \frac{6.5284}{s+9}\\
&- \frac{3.1431}{s+10} + \frac{0.9985}{s+11} - \frac{0.1537}{s+12} - \frac{67(\psi(s+1)+\gamma_E)}{9}\\
&+ \frac{1}{3}\pi^2(\psi(s+1)+\gamma_E) - \frac{1}{s^2}\left(\ln(16) - 2\psi\left(\frac{s}{2}+1\right)\right.\\
&\left.\left. + 2\psi\left(\frac{s+1}{2}\right) + s\psi'\left(\frac{s}{2}+1\right) - s\psi'\left(\frac{s+1}{2}\right)\right)\right)
\end{aligned}
$$

$$+2\left(\frac{4}{(s+1)^3}-\frac{\ln(4)}{(s+1)^2}-\frac{\psi\left(\frac{s}{2}+1\right)}{(s+1)^2}+\frac{\psi\left(\frac{s+1}{2}\right)}{(s+1)^2}+\frac{\psi'\left(\frac{s}{2}+1\right)}{2s+2}-\frac{\psi'\left(\frac{s+1}{2}\right)}{2s+2}\right)$$

$$-\frac{1.9992}{(s+2)^3}\left(\frac{16}{(s+1)^2}+\frac{12s}{(s+1)^2}+(s+2)\ln(16)-2(s+2)\psi\left(\frac{s}{2}+1\right)\right.$$

$$\left.+2(s+2)\psi\left(\frac{s+1}{2}\right)+(s+2)^2\psi'\left(\frac{s}{2}+1\right)-(s+2)^2\psi'\left(\frac{s+1}{2}\right)\right)$$

$$+\frac{0.0149}{(s+1)^3}\left(\frac{164}{(s+1)^2(s+2)^2}+\frac{284s}{(s+1)^2(s+2)^2}+\frac{188s^2}{(s+1)^2(s+2)^2}\right.$$

$$+\frac{60s^3}{(s+1)^2(s+2)^2}+\frac{8s^4}{(s+1)^2(s+2)^2}-4(s+3)\ln(2)$$

$$-2(s+3)\psi\left(\frac{s}{2}+1\right)+2(s+3)\psi\left(\frac{s+1}{2}\right)$$

$$\left.+(s+3)^2\psi'\left(\frac{s}{2}+1\right)-(s+3)^2\psi'\left(\frac{s+1}{2}\right)\right)$$

$$-\frac{0.9005}{(s+4)^3}\left(\frac{2176}{(s+1)^2(s+2)^2(s+3)^2}+\frac{4392s}{(s+1)^2(s+2)^2(s+3)^2}\right.$$

$$+\frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2}+\frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2}$$

$$+\frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2}+\frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2}$$

$$+4(s+4)\ln(2)-2(s+4)\psi\left(\frac{s}{2}+1\right)+2(s+4)\psi\left(\frac{s+1}{2}\right)$$

$$\left.+(s+4)^2\psi'\left(\frac{s}{2}+1\right)-(s+4)^2\psi'\left(\frac{s+1}{2}\right)\right)$$

$$-\frac{0.6621}{(s+5)^3}\left(\frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}\right.$$

$$+\frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}+\frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$

$$+\frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}+\frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$

$$+\frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}+\frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$

$$+\frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}+\frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2}$$

$$
\begin{aligned}
&- 4(s+5)\ln(2) - 2(s+5)\psi\left(\frac{s}{2}+1\right) + 2(s+5)\psi\left(\frac{s+1}{2}\right) \\
&+ (s+5)^2\psi'\left(\frac{s}{2}+1\right) - (s+5)^2\psi'\left(\frac{s+1}{2}\right)\Bigg) \\
&+ \frac{4}{s^3}(1+\gamma_E s + s(\psi(s) - s\psi'(s+1))) \\
&- \frac{8}{(s+1)^2}\left(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)\right) \\
&+ \frac{4}{(s+2)^2}\left(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)\right) \\
&- \frac{4}{(s+3)^2}\left(\gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4)\right) \\
&- \frac{0.3174}{(s+6)^2}\Bigg(\ln(16) - 2\psi\left(\frac{s}{2}+4\right) + 2\psi\left(\frac{s+7}{2}\right) \\
&+ (s+6)\psi'\left(\frac{s}{2}+4\right) - (s+6)\psi'\left(\frac{s+7}{2}\right)\Bigg) \\
&+ \frac{0.0699}{(s+7)^2}\Bigg(\ln(16) + 2\psi\left(\frac{s}{2}+4\right) - 2\psi\left(\frac{s+9}{2}\right) \\
&- (s+7)\psi'\left(\frac{s}{2}+4\right) + (s+7)\psi'\left(\frac{s+9}{2}\right)\Bigg) \\
&+ 4\left(\left((\psi(s+1)+\gamma_E)\psi'(s+1) - \frac{1}{2}\psi''(s+1)\right) - \frac{22}{3}\psi'(s+1) + 3\psi''(s+1)\right).
\end{aligned}
$$

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