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Bayesian Two-sample Prediction with Progressively Censored Data for Generalized Exponential Distribution Under Symmetric and Asymmetric Loss Functions

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Abstract. Statistical prediction analysis plays an important role in a wide range of fields. Examples include engineering systems, design of experiments, etc. In this paper, based on progressively Type-II right censored data, Bayesian two-sample point and interval predictors are developed under both informative and non-informative priors. By assuming a generalized exponential model, prediction bounds as well as Bayes point predictors are obtained under the squared error loss (SEL) and the Linear-Exponential (LINEX) loss functions for the order statistic in a future progressively Type-II censored sample with an arbitrary progressive censoring scheme. The derived results may be used for prediction of total time on test in lifetime experiments. In addition to numerical method, Gibbs sampling procedure (as Markov Chain Monte Carlo method) are used to assess approximate prediction bounds and Bayes point predictors under the SEL and LINEX loss functions. The performance of the proposed prediction procedures are also demonstrated via a Monte Carlo simulation study and an illustrative example, for each method.

Keywords. Bayesian prediction; generalized exponential model; gibbs sampling; LINEX loss function; Markov Chain Monte Carlo; progressive type-II censoring scheme; two-sample prediction.

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1 Introduction

Problem of prediction for a future random variable or future observations on the basis of the past and present information is valuable in the analysis of the lifetime data sets. Prediction of unobserved failure times or censored data is of interest in many fields such as medical sciences and reliability analysis. Generally, prediction problems are divided into two categories called one-sample and two-sample problems. In one-sample problem, the random variable which we wish to predict, say Y, and the available data, say X, are dependent. On the other hand, in a two-sample problem, X and Y are independent but the corresponding distribution functions have some shared parameters. For a greater details, see Aitchison and Dunsmore (1975).

Based on complete random sample, prediction problems have been discussed in the literature; see, Aitchison and Dunsmore (1975); Geisser (1993) and the references therein. Censoring arises when due to some restrictions such as cost and time, experiments often terminated before all units on the test have failed. There are many types of censoring such as Type-II right censoring, doubly Type-II censoring, random censoring and progressive Type-II right censoring.

On the basis of the progressively Type-II censored data, many authors developed statistical inference and prediction for future observations (failure times). For example, Cohen (1963) and Cohen and Norgaard (1977) studied statistical inference for several failure time distributions based on progressively Type-II censored data. Also, see Mann (1969, 1971), Thomas and Wilson (1972), Cacciari and Montanari (1987), Viveros and Balakrishnan (1994). Also, Balakrishnan et al. (2001) computed bounds for means and variances of progressively Type-II censored order statistics.

Fernandez (2000) considered Bayesian prediction for independent future sample arising from the Rayleigh distribution based on Type-II double censoring. Ali Mousa and Jaheen (2002) considered the two-parameter Burr Type-XII model for obtaining the Bayesian prediction in a two-sample problem on the basis of progressively censored data. In addition, Ali Mousa and AL-Sagheer (2005) derived Bayesian two-sample prediction bounds with progressively Type-II censored data coming from the Rayleigh model. Kundu

and Howlader (2010) presented Bayesian prediction for the inverse Weibull distribution under Type-II censoring scheme.

This paper deals with two-sample prediction problem on the basis of progressively Type-II censored data. Here, a description on progressive Type-II censoring due to Balakrishnan and Aggarwala (2000) is provided.

Suppose that we have n identical units for a lifetime test. For $1 \leqslant m < n$, $R_1, R_2, \ldots, R_m$ all are prefixed integers such that $R_1 + R_2 + \cdots + R_m + m = n$. At the first failure time $x_{1:m:n}$, we randomly withdraw R_1 items from the remaining $n-1$ units. Then immediately after the second observed failure time $x_{2:m:n}$, R_2 items are randomly selected and withdrawn from the remaining $n-2-R_1$ units at random, and so on. The experiment continues when the mth failure time $x_{m:m:n}$ occurs and the remaining items, $R_m = n-m-R_1-R_2-\cdots-R_{m-1}$, are removed. Thus, we have a progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ and m ordered observed failure times, denoted by $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$. Note that for $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n-m$, the progressively Type-II censored data reduces to the usual Type-II censored data.

1.1 The GE Distribution

The two-parameter generalized exponential distribution (GE) has been proposed by Gupta and Kundu (1999) as the most effective model (up to that time) to analyze many skewed lifetime data instead of some well-known lifetime models such as the gamma and Weibull distributions. The GE distribution is a flexible lifetime model that includes increasing hazard rate and decreasing hazard rate in addition to the constant hazard rate case corresponding to the exponential distribution.

Raqab and Ahsanullah (2001) considered the estimation of the location and scale parameters of the GE model based on the usual order statistics; i.e. a progressively Type-II censored data with $R_1 = R_2 = \cdots = R_{m-1} = R_m$ so that $m = n$. Raqab and Madi (2005) studied the Bayesian estimation and prediction for the GE model using importance sampling techniques. Madi and Raqab (2007) obtained Bayesian prediction of rainfall records based on the GE distribution. Also, Alamm et al. (2007) derived Bayesian prediction bounds for future order statistics obtained from a random sample with the one-parameter GE distribution on the basis of the observed ordered data. Madi and Raqab (2009) considered the problem of Bayesian estimation of the parameters as well as Bayesian prediction of the unobserved

failure times censored in multiple stages in the progressively censored sample from the GE distribution (Bayesian one-sample prediction) using the Gibbs and Metropolis samplers. Also, see Kundu and Gupta (2008); Mitra and Kundu (2008); Kundu and Pradhan (2009a,b); Pradhan and Kundu (2009) and the references therein.
Recently, Kim and Han (2015) considered the problem of maximum likelihood and Bayes estimation of the GE parameters as well as predicting an independent future order statistics from the GE distribution based on progressively first failure censored samples.

Notice that Ghafoori et al. (2011) considered Bayesian two-sample prediction bounds as well as Bayes predictive estimations for a future progressively Type-II censored sample in a general form of lifetime models, which proposed by AL-Hussaini (1999) including the Weibull and Pareto models, under the symmetric SEL function. Also, it is important to state that in the mentioned paper, the problem of Bayesian two-sample prediction carried out in AL-Hussaini family models under progressively Type-II censored samples. But this paper is devoted to Bayesian two-sample prediction, under SEL and LINEX loss functions, in the GE model which does not belong to the AL-Hussaini's family, based on progressive Type-II censoring schemes.

A random variable X is said to have the two-parameter GE distribution, denoted by GE(α, λ), if the probability distribution (pdf), the cumulative distribution (cdf) and the hazard functions (hf) are in the following forms, respectively:

$$\begin{aligned} f_{GE}(x|\alpha,\lambda) &= \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, \qquad x>0,\ \alpha,\lambda>0, \\ F_{GE}(x|\alpha,\lambda) &= (1-e^{-\lambda x})^{\alpha}, \qquad x>0,\ \alpha,\lambda>0, \end{aligned} \tag{1}$$

and

$$r_{GE}(x|\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}}{1-(1-e^{-\lambda x})^{\alpha}}, \qquad x>0,\ \alpha,\lambda>0, \tag{2}$$

where α and λ are the shape and the scale parameters, respectively. For $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$, the hf in the Equation (2) is increasing, constant and decreasing, respectively.

In this paper and in Section 2, we develop Bayesian prediction bounds as well as Bayes point predictors under the SEL and LINEX loss functions, for the two-parameter GE model based on a progressively Type-II censored data. Two methods including numerical and Markov Chain Monte Carlo

(MCMC) techniques under informative and non-informative priors are used in Sections 2 and 3, respectively. In addition, Section 4 is organized to assess the performance of the procedures obtained under informative and non-informative priors, by a Monte Carlo simulation study and an illustrative example for each method.

2 Some Theoretical Results

As described in the introduction, suppose that we observed a progressively Type-II censored sample with scheme $(R_1, R_2, \ldots, R_m)$ and denoted by $\underline{x} = (x_1, x_2, \ldots, x_m)$. The joint likelihood function of the sample is (see, Balakrishnan and Aggarwala (2000), p. 8)

$$L(\alpha, \lambda|\underline{x}) \propto (\alpha\lambda)^m \exp\Big\{-\lambda \sum_{j=1}^{m} x_j\Big\} \times \prod_{j=1}^{m} \Big(1 - e^{-\lambda x_j}\Big)^{\alpha-1} \Big(1 - (1 - e^{-\lambda x_j})^{\alpha}\Big)^{R_j}. \tag{3}$$

Following Ali Mousa and AL-Sagheer (2005), assume that $Y_{1:M:N}^{(S_1,S_2,\ldots,S_M)}$, $Y_{2:M:N}^{(S_1,S_2,\ldots,S_M)}, \ldots, Y_{M:M:N}^{(S_1,S_2,\ldots,S_M)}$ is another (unobserved) independent progressively Type-II right censored ordered statistics of size M from a sample of size N with the progressive censoring scheme $(S_1, S_2, \ldots, S_M)$. The first sample is considered as "informative" (past) sample, whereas the second sample is the "future" sample. Briefly, we denote $Y_{s:M:N}^{(S_1,S_2,\ldots,S_M)}$ by Y_s which is representative of the sth order statistic (failure time) in the future sample $(1 \leqslant s \leqslant M)$. In the sequel, we consider the problem of predicting Y_s on the basis of the past sample $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$.

2.1 Predictive Density Function

Under the GE(α, λ) model, the pdf of Y_s, $1 \leqslant s \leqslant M$, is (see Balakrishnan and Aggarwala (2000), p. 26)

$$h(y_s|\alpha, \lambda) = C_{s-1} f_X(y_s|\alpha, \lambda) \sum_{i=1}^{s} a_i \Big(1 - F_X(y_s|\alpha, \lambda)\Big)^{\gamma_i - 1},$$

$$= C_{s-1}\ \alpha\lambda e^{-\lambda y_s}\Big(1-e^{-\lambda y_s}\Big)^{\alpha-1}\sum_{i=1}^{s} a_i\Big(1-(1-e^{-\lambda y_s})^{\alpha}\Big)^{\gamma_i-1}, \qquad (4)$$

where

$$\gamma_i = \sum_{j=i}^{M}(S_j+1) = N - \sum_{j=1}^{i-1}(S_j+1), \qquad C_{s-1} = \prod_{i=1}^{s}\gamma_i,$$

$$a_i = \prod_{j=1}^{s}\frac{1}{\gamma_j-\gamma_i}, \qquad \forall i \neq j, s > 1, \qquad (5)$$

and $a_1 = 1$ for $s = 1$. The Bayes predictive pdf is defined as (Aitchison and Dunsmore (1975), p. 17)

$$H(y_s|\underline{x}) = \int h(y_s|\underline{\theta})\pi(\underline{\theta}|\underline{x})\, d\underline{\theta}\,,$$

where $h(y_s|\underline{\theta})$ and $\pi(\underline{\theta}|\underline{x})$ are the pdf and posterior density function, respectively. Hereafter, both parameters α and λ are assumed to be unknown, we discuss the Bayesian point and interval prediction of Y_s separately in three cases of priors for α and λ:

Case I: α and λ have independent gamma prior distributions
Assume that α and λ are independent and follow the gamma (informative) prior densities (see, Kundu and Pradhan (2009b)) as

$$\pi_1(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}\ e^{-b\alpha}, \qquad \alpha > 0, \qquad (6)$$

and

$$\pi_2(\lambda) = \frac{d^c}{\Gamma(c)}\lambda^{c-1}\ e^{-d\lambda}, \qquad \lambda > 0, \qquad (7)$$

i.e. $\alpha \sim \Gamma(a,b)$ and $\lambda \sim \Gamma(c,d)$, where all the hyperparameters a, b, c and d are known and nonnegative constants. It may be mentioned that for known λ, $\pi_1(\alpha)$ in the Equation (6) is a conjugate prior on α. If both parameters are unknown, the joint conjugate priors do not exist. But because of flexibility of gamma distributions which includes the Jeffreys (non-informative) prior as a special case, we suggest the independent gamma priors on the scale and

shape parameters (see also, Kundu and Pradhan (2009b)). For Bayesian analysis of the gamma distribution, Son and Oh (2006) assumed the gamma prior on the scale parameter and independent non-informative prior on the shape parameter, which is a special case of the gamma distribution. So, from Equations (3), (6) and (7) the joint posterior density of the parameters α and λ becomes

$$\pi(\alpha,\lambda|\underline{x}) = M\alpha^{m+a-1}\lambda^{m+c-1}\exp\Big\{-(b\alpha+d\lambda+\lambda\sum_{j=1}^{m}x_j)\Big\} \\ \times\exp\Big\{(\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})+\sum_{j=1}^{m}R_j\ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}, \tag{8}$$

where M is the normalizing constant and given by

$$M^{-1} = \int_0^{+\infty}\int_0^{+\infty}\alpha^{m+a-1}\lambda^{m+c-1}\exp\Big\{-(b\alpha+d\lambda+\lambda\sum_{j=1}^{m}x_j)\Big\} \\ \times\exp\Big\{(\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})+\sum_{j=1}^{m}R_j\ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}\,d\alpha\,d\lambda.$$

By applying Equations (4) and (8), the Bayes predictive density function of Y_s, $1\leqslant s\leqslant M$, is

$$H(y_s|\underline{x}) = \int_0^{+\infty}\int_0^{+\infty}h(y_s|\alpha,\lambda)\pi(\alpha,\lambda|\underline{x})\,d\alpha\,d\lambda, \\ = MC_{s-1}\sum_{i=1}^{s}a_i\int_0^{+\infty}\int_0^{+\infty}\alpha^{m+a}\lambda^{m+c}\exp\Big\{-(b\alpha+d\lambda+\lambda\sum_{j=1}^{m}x_j+\lambda y_s)\Big\} \\ \times\exp\Big\{(\alpha-1)\ln(1-e^{-\lambda y_s})+(\gamma_i-1)\ln\Big(1-(1-e^{-\lambda y_s})^{\alpha}\Big)\Big\} \\ \times\exp\Big\{(\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})+\sum_{j=1}^{m}R_j\ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}\,d\alpha\,d\lambda, \tag{9}$$

where γ_i, C_{s-1}, a_i and M are given by Equations (7) and (10), respectively.

Case II: λ has informative prior and α has non-informative prior
Following Singh et al. (2008), we assume α and λ are independent as well as non-informative prior $\pi_1(\alpha) \propto \alpha^{-1}$, $0 < \alpha < \infty$, and the informative prior $\Gamma(c, d)$, for the scale parameter λ. Then, the prior density for the parameters α and λ is

$$\pi(\alpha, \lambda) = \frac{d^c}{\alpha\Gamma(c)}\lambda^{c-1}\, e^{-d\lambda}, \qquad 0 < \alpha < \infty, \quad \lambda > 0 \quad c, d > 0.$$

In this case, the joint posterior density of the parameters α and λ and the Bayes predictive density function of Y_s is similar to Case I with $a = b = 0$.

Case III: α and λ have non-informative priors
Here, we suppose that both parameters α and λ are independent and follow non-informative priors, i.e. $\pi_1(\alpha) \propto \alpha^{-1}$ and $\pi_2(\lambda) \propto \lambda^{-1}$. By considering $a = b = c = d = 0$ in Case I, the joint posterior density of the parameters and the Bayes predictive density function can be obtained.

2.2 Bayesian Prediction Bounds and Bayes Predictors

Bayesian prediction bounds for Y_s, $1 \leqslant s \leqslant M$, are obtained by evaluating $Pr(Y_s \geqslant \varepsilon|\underline{x})$, for some positive value of ε. It turns out from the Equation (9) that

$$\begin{aligned}
Pr(Y_s \geqslant \varepsilon|\underline{x}) &= \int_{\varepsilon}^{+\infty} H(y_s|\underline{x})\, dy_s, \\
&= MC_{s-1}\sum_{i=1}^{s} a_i \int_{\varepsilon}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} \alpha^{m+a}\lambda^{m+c} \\
&\times \exp\Big\{-(b\alpha + d\lambda + \lambda\sum_{j=1}^{m} x_j + \lambda y_s)\Big\} \\
&\times \exp\Big\{(\alpha-1)\ln(1-e^{-\lambda y_s}) + (\gamma_i - 1)\ln\Big(1-(1-e^{-\lambda y_s})^{\alpha}\Big)\Big\} \\
&\times \exp\Big\{(\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j}) \\
&+ \sum_{j=1}^{m} R_j \ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}\, d\alpha\, d\lambda\, dy_s. \qquad (10)
\end{aligned}$$

An equi.tail $\tau \times 100\%$ Bayesian prediction bounds for Y_s is obtained numerically by solving the following non-linear equations:
$Pr(Y_s \geqslant L_s(\underline{x})|\underline{x}) = \frac{1+\tau}{2}$ and $Pr(Y_s \geqslant U_s(\underline{x})|\underline{x}) = \frac{1-\tau}{2}$, where $L_s(\underline{x})$ and $U_s(\underline{x})$ are lower and upper Bayesian predictive bounds of Y_s, respectively.

Remark 1. The following lemma is useful for deriving Bayes point predictor of Y_s under the SEL function and simplifying integrals of related to it.

***Lemma* 1.** *We have*

$$\begin{aligned}\int_{\varepsilon}^{+\infty} h(y_s|\alpha,\lambda)\,dy_s &= \sum_{i=1}^{s} a_i C_{s-1} \int_{\varepsilon}^{+\infty} f_X(y_s|\alpha,\lambda)(1-F_X(y_s|\alpha,\lambda)\Big)^{\gamma_i-1}\,dy_s,\\ &= \sum_{i=1}^{s} a_i C_{s-1} \int_{F(\varepsilon)}^{1} (1-u)^{\gamma_i-1}\,du\\ &= \sum_{i=1}^{s} a_i C_{s-1} \frac{(1-F(\varepsilon))^{\gamma_i}}{\gamma_i}. \qquad (11)\end{aligned}$$

Proof. Note that by

$$F_X(y_s) = (1-e^{-\lambda y_s})^{\alpha} = u,$$

the desired result follows

$$\begin{aligned}\int_{\varepsilon}^{+\infty} h(y_s|\alpha,\lambda)\,dy_s &= \sum_{i=1}^{s} a_i C_{s-1} \int_{\varepsilon}^{+\infty} f_X(y_s|\alpha,\lambda)(1-F_X(y_s|\alpha,\lambda)\Big)^{\gamma_i-1}\,dy_s,\\ &= \sum_{i=1}^{s} a_i C_{s-1} \int_{\varepsilon}^{+\infty} \alpha\lambda e^{-\lambda y_s}\Big(1-e^{-\lambda y_s}\Big)^{\alpha-1}\\ &\quad\times \Big(1-(1-e^{-\lambda y_s})^{\alpha}\Big)^{\gamma_i-1}\,dy_s,\\ &= \sum_{i=1}^{s} a_i C_{s-1} \frac{(1-F(\varepsilon))^{\gamma_i}}{\gamma_i}.\end{aligned}$$

□

Now, from Equations (10) and (11) the Bayes point predictor of Y_s, $1 \leqslant s \leqslant M$, under the SEL function is

$$\widetilde{y}_s(SEL) = E(Y_s|\underline{x}) = \int_{0}^{+\infty} Pr(Y_s \geqslant \varepsilon|\underline{x})\,d\varepsilon,$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_\varepsilon^{+\infty} H(y_s|\underline{x})\,dy_s\,d\varepsilon \\
&= \int_0^{+\infty}\int_\varepsilon^{+\infty}\int_0^{+\infty}\int_0^{+\infty} h(y_s|\alpha,\lambda)\pi(\alpha,\lambda|\underline{x})\,d\alpha\,d\lambda\,dy_s\,d\varepsilon, \\
&= MC_{s-1}\sum_{i=1}^{s} a_i \int_0^{+\infty}\int_\varepsilon^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \alpha^{m+a}\lambda^{m+c} \\
&\times \exp\Big\{-(b\alpha + d\lambda + \lambda\sum_{j=1}^{m} x_j + \lambda y_s)\Big\} \\
&\times \exp\Big\{(\alpha-1)\ln(1-e^{-\lambda y_s}) + (\gamma_i - 1)\ln\Big(1-(1-e^{-\lambda y_s})^{\alpha}\Big)\Big\} \\
&\times \exp\Big\{(\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j}) \\
&+ \sum_{j=1}^{m} R_j \ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}\,d\alpha\,d\lambda\,dy_s\,d\varepsilon, \\
&= M\sum_{i=1}^{s} a_i C_{s-1}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{\alpha^{m+a-1}\lambda^{m+c-1}\Big(1-(1-e^{-\lambda\varepsilon})^{\alpha}\Big)^{\gamma_i-1}}{\gamma_i} \\
&\times \exp\Big\{-(b\alpha + d\lambda + \lambda\sum_{j=1}^{m} x_j)\Big\}\exp\Big\{(\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j}) \\
&+ \sum_{j=1}^{m} R_j \ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}\,d\varepsilon\,d\alpha\,d\lambda. \qquad (12)
\end{aligned}
$$

For Cases II and III in Subsection 2.1 by substituting $a = b = 0$ and $a = b = c = d = 0$, respectively, the lower and upper Bayesian predictive bounds and the Bayes point predictor of Y_s are derived.
The LINEX loss function with the shape and scale parameters τ and b_0 can be written as

$$L(\theta,\delta) = b_0[e^{\tau(\delta-\theta)} - \tau(\delta-\theta) - 1], \qquad \tau \neq 0,\ b_0 > 0, \qquad (13)$$

which introduced by Varian (1975). Properties of the LINEX loss function may be found in Zellner (1986). Without loss of generality, we here assume that $b_0 \equiv 1$. Let $M_{\theta|X}(t) = E_{\theta|X}(e^{\tau\theta})$ be the moment generating function of the posterior distribution of $\pi(\theta|X)$. Therefore, it is easy to see that the

unique Bayes estimate of θ under the LINEX loss function in the Equation (13) is

$$\delta_B(X) = -\frac{1}{\tau}\log(M_{\theta|X}(-t)), \tag{14}$$

provided that $M_{\theta|X}(\cdot)$ exists and is finite. Here, based on the Bayes predictive density function of Y_s, $1 \leqslant s \leqslant M$, in the Equation (9), the Bayes point predictor of Y_s under the LINEX loss function becomes

$$\begin{aligned}
\widetilde{y}_s(LINEX) &= -\frac{1}{\tau}\log E\big(e^{-\tau Y_s}|\underline{x}\big) = -\frac{1}{\tau}\log\Big[\int_0^{+\infty} e^{-\tau y_s} H(y_s|\underline{x})\,dy_s\Big],\\
&= -\frac{1}{\tau}\log\Big[MC_{s-1}\sum_{i=1}^{s} a_i \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \alpha^{m+a}\lambda^{m+c}\\
&\times \exp\Big\{-(b\alpha + d\lambda + \lambda\sum_{j=1}^{m} x_j + \lambda y_s - \tau y_s)\\
&+ (\alpha-1)\ln(1-e^{-\lambda y_s})\Big\}\\
&\times \exp\Big\{(\gamma_i - 1)\ln\Big(1-(1-e^{-\lambda y_s})^{\alpha}\Big)\\
&+ (\alpha-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\Big\}\\
&\times \exp\Big\{\sum_{j=1}^{m} R_j \ln\Big(1-(1-e^{-\lambda x_j})^{\alpha}\Big)\Big\}\,d\alpha\,d\lambda\,dy_s\Big].
\end{aligned} \tag{15}$$

It is worthwhile to mention that a numerical method could be applied to calculate the 95% Bayesian prediction bounds and Bayes point predictors under SEL and LINEX loss functions.

3 MCMC for Prediction Problems under Progressively Type-II Censored Data

In this section, we consider MCMC methods to generate samples from the posterior distributions and then, along with the importance sampling, compute Bayesian prediction bounds as well as Bayes point predictors under the SEL and LINEX loss functions based on a progressively Type-II censored data. A complete introduction and review of importance sampling as well as

MCMC methods including Gibbs sampling and Metropolis-Hastings (MH) and related algorithms can be found in Chen and Shao (1999); Kundu and Howlader (2010); Soliman et al. (2011a) and Soliman et al. (2012).

In addition, some useful tips and techniques for application of MCMC methods such as MH can be seen in Valiollahi et al. (2013) and Asgharzadeh et al. (2013). Also, some chapters of Rizzo (2008) focus on introduction and implementation of Monte Carlo and MCMC methods based on R language.

The joint posterior density of the parameters α and λ in the Equation (8) can be rewritten as

$$\begin{aligned}\pi(\alpha,\lambda|\underline{x}) \propto\ & \alpha^{m+a-1}\big(b-\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\big)^{(m+a)}\\ & \times\exp\Big\{-\alpha\big(b-\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\big)\Big\}\\ & \times\lambda^{m+c-1}\exp\big\{-\lambda\big(d+\sum_{j=1}^{m}x_j\big)\big\}\exp\Big\{-\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\Big\}\\ & \times\frac{\exp\Big\{\sum_{j=1}^{m}R_j\ln\big(1-(1-e^{-\lambda x_j})^{\alpha}\big)\Big\}}{\big(b-\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\big)^{(m+a)}},\end{aligned}\tag{16}$$

or equivalently $\pi(\alpha,\lambda|\underline{x}) \propto g_1(\alpha|\lambda,\underline{x})g_2(\lambda|\underline{x})h_1(\alpha,\lambda|\underline{x})$.
Here, $g_1(\alpha|\lambda,\underline{x})$ is the gamma pdf with the shape and scale parameters as $(m+a)$ and $\big(b-\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\big)^{-1}$, respectively. Also, $g_2(\lambda|\underline{x})$ has the gamma pdf with the shape and scale parameters as $(m+c)$ and $(d+\sum_{j=1}^{m}x_j)^{-1}$, respectively. $h_1(\alpha,\lambda)=\frac{\exp\left\{\sum_{j=1}^{m}R_j\ln\left(1-(1-e^{-\lambda x_j})^{\alpha}\right)\right\}}{\left(b-\sum_{j=1}^{m}\ln(1-e^{-\lambda x_j})\right)^{(m+a)}}$, is related to importance sampling.

Since, it is so hard to compute Equations (10), (12) and (15) analytically, we suggest approximating them using the Gibbs and importance sampling techniques (Soliman et al. (2011b)).

$g_1(\alpha|\lambda,\underline{x})$ and $g_2(\lambda|\underline{x})$ are the gamma pdfs and therefore, generating samples of α and λ is easy. In order to obtain Bayesian prediction bounds as well as Bayes point predictors under the SEL and LINEX loss functions based on Gibbs and importance sampling, we carry out the following algorithm based on Soliman et al. (2011a):

Gibbs algorithm

(1) Set $t = 1$.

(2) Generate $\lambda^{(t)}$ from $g_2(\cdot|data)$.

(3) Generate $\alpha^{(t)}$ from $g_1(\cdot|\lambda^{(t)}, data)$.

(4) Obtain $\lambda^{(t)}$ and $\alpha^{(t)}$.

(5) Set $t = t + 1$.

(6) Repeat (2)-(6) $N1$ times and obtain $((\lambda_1, \alpha_1), (\lambda_2, \alpha_2), \ldots, (\lambda_{N1}, \alpha_{N1}))$. On the other hand, the corresponding probability and survival function of Y_s, $1 \leqslant s \leqslant M$, are

$$F(y_s|\alpha, \lambda) = 1 - C_{s-1} \sum_{i=1}^{s} \frac{a_i}{\gamma_i} \Big(1 - (1 - e^{-\lambda y_s})^{\alpha}\Big)^{\gamma_i - 1}, \tag{17}$$

and

$$G(y_s|\alpha, \lambda) = C_{s-1} \sum_{i=1}^{s} \frac{a_i}{\gamma_i} \Big(1 - (1 - e^{-\lambda y_s})^{\alpha}\Big)^{\gamma_i - 1}, \tag{18}$$

respectively. Suppose $((\lambda_1, \alpha_1), (\lambda_2, \alpha_2), \ldots, (\lambda_{N1}, \alpha_{N1}))$ are MCMC samples obtained from $\pi(\alpha, \lambda|\underline{x})$ using the Gibbs sampling technique. Then, the corresponding Bayes predictive survival function of Y_s, $1 \leqslant s \leqslant M$, is

$$G^*(y_s|\underline{x}) = \int_0^{+\infty} \int_0^{+\infty} G(y_s|\alpha, \lambda)\pi(\alpha, \lambda|\underline{x})\, d\alpha\, d\lambda, \tag{19}$$

and the simulation consistent estimator of $G^*(y_s|\underline{x})$ is given by

$$\widehat{G^*}(y_s|\underline{x}) = \sum_{j=1}^{N_1} G(y_s|\alpha_j, \lambda_j) w_j, \tag{20}$$

where $w_i = \frac{h_1(y_s|\alpha_i, \lambda_i)}{\sum_{j=1}^{N_1} h_1(y_s|\alpha_i, \lambda_i)}$.

(7) An approximate equi-tail $\tau \times 100\%$ Bayesian prediction bounds (L_s, U_s) for Y_s are obtained numerically by evaluating $\widehat{G^*}(U_s|\underline{x}) = \frac{1+\tau}{2}$ and $\widehat{G^*}(L_s|\underline{x}) = \frac{1-\tau}{2}$.

(8) Also, an approximate Bayes point predictor of Y_s under the SEL function may be in following form

$$\begin{aligned}\widehat{\widetilde{y_s}(SEL)} &= \widehat{E(Y_s|\underline{x})} \\ &= \sum_{j=1}^{N_1} C_{s-1} \sum_{i=1}^{s} a_i \Big(\int_0^{+\infty} \frac{1}{\gamma_i} \big(1 - (1 - e^{-\lambda_j \varepsilon})^{\alpha_j}\big)^{\gamma_i} \, d\varepsilon \Big) w_j. \end{aligned} \tag{21}$$

(9) Finally, an approximate Bayes point predictor of Y_s under the LINEX loss function becomes

$$\begin{aligned}\widehat{\widetilde{y_s}(LINEX)} &= -\frac{1}{\tau} \log \widehat{E\big(e^{-\tau Y_s}|\underline{x}\big)} \\ &= -\frac{1}{\tau} \log \left\{ \int_0^{+\infty} e^{-\tau y_s} \sum_{j=1}^{N_1} h(y_s|\alpha_j, \lambda_j) w_j \, dy_s \right\}, \\ &= -\frac{1}{\tau} \log \left\{ \sum_{j=1}^{N_1} w_j \alpha_j \lambda_j C_{s-1} \sum_{i=1}^{s} a_i \Big(\int_0^{+\infty} e^{-(\tau+\lambda_j) y_s} \right. \\ &\left. \times (1 - e^{-\lambda_j y_s})^{\alpha_j - 1} \Big(1 - (1 - e^{-\lambda_j y_s})^{\alpha_j}\Big)^{\gamma_i - 1} \, dy_s \right\}. \end{aligned} \tag{22}$$

4 Empirical Results

In this section, the performances of the proposed procedures in Sections 2 and 3 are investigated by a Monte Carlo simulation study and an illustrative example for each method.

4.1 Simulation Study in Numerical Solution

In this subsection, the performances of the obtained Bayesian prediction bounds as well as Bayes point predictors proposed in Section 2 are carried

out. The 95% Bayesian prediction bounds for Y_s and Bayes point predictors of Y_s under the SEL and LINEX loss functions are computed as follows:

(1) For given values of the parameters and the prior parameters (hyperparameters) a, b, c and d, according to an algorithm proposed by Balakrishnan and Sandhu (1995), a progressively Type-II censored sample is generated.

(2) The 95% Bayesian prediction bounds for Y_s and Bayes point predictors of Y_s are obtained from Equations (10), (12) and (15) respectively, for different informative sample sizes, $m = 15, 15, 15, 20$,
and $s = 2, 6, 12$ coming from the GE distribution.

(3) For 10^3 simulated independent future progressively Type-II samples of size $N = 24$ from the GE pdf, simulated coverage probabilities of prediction bounds for Y_s are derived by using the statistical package R version 3.2.2.

(4) Steps 1-3 are repeated 10^3 times and reported the average Bayesian prediction bounds, the average Bayes point predictors, average widths (AWs), average of simulated Bayesian coverage probabilities (CPs) and a new criterion, we called PL, which is defined as $\frac{CP}{AW}$ (Salehi and Ahmadi, 2014), for prediction intervals.

Remark 2. After simplifying the integrals of Equations (10) and (12) by the Equation (11), remained integrals can be computed by "integrate" and "area" functions in the statistical package R version 3.2.2, to provide the 95% Bayesian prediction bounds and Bayes point predictors.

Also, we performed the above algorithm for informative and non-informative priors described earlier. Table 1 displays three different cases of R_i and S_i's for $n = 30, 36$, $N = 24$, $m = 15, 15, 15, 20$ and $M = 12$.

In Case I, we consider $a = 8$, $b = 2$, $c = 8$ and $d = 4$ for the two-parameter GE model. Table 2 summarizes the results.

In Case II, we choose $a = 0$, $b = 0$, $c = 8$ and $d = 4$ for the two-parameter GE model. The results are given in Table 3.

Finally, the results for Case III, where both parameters have non-informative priors, are presented in Table 4.

One can see from Tables 2-4 that the simulated Bayesian coverage probabilities of Y_s are close to the nominal level 95%. Also, choosing informative

Table 1. Various censoring scheme $R_i, i = 1, 2, \ldots, m$ and $S_i, i = 1, 2, \ldots, M$.

	$R_i, i = 1, 2, \ldots, m$	$S_i, i = 1, 2, \ldots, M$
1	15,0, ...,0	12,0,...,0
2	$\overbrace{0,\ldots,0}^{7}$,15,$\overbrace{0\ldots,0}^{7}$	$\overbrace{0,\ldots,0}^{6}$,12,$\overbrace{0,\ldots,0}^{5}$
3	0,...,0,15	0,...,0,12
4	$\overbrace{0,\ldots,0}^{12}$,$\overbrace{2,\ldots,2}^{8}$	12,0,...,0

Table 2. Bayes point predictors for Y_s based on the SEL and LINEX loss functions, the 95% Bayesian prediction bounds, AWs, simulated Bayesian CPs and PL criterion, for the two-parameter GE, with $\alpha = 4$, $\lambda = 2$ and $\tau = 2$.

		Gamma priors for α and λ					
		Point prediction		Interval Prediction			
Scheme	Y_s	SEL	LINEX	(Lower, Upper)	AW	CP	PL
1	Y_2	0.374	0.377	(0.140, 0.713)	0.572	0.959	1.677
	Y_6	0.839	0.775	(0.450, 1.388)	0.937	0.974	1.039
	Y_{12}	2.417	2.087	(1.232, 4.494)	3.261	0.962	0.294
2	Y_2	0.317	0.306	(0.128, 0.569)	0.440	0.955	2.167
	Y_6	0.571	0.553	(0.328, 0.891)	0.563	0.972	1.726
	Y_{12}	2.198	2.981	(1.065, 4.249)	3.183	0.962	0.302
3	Y_2	0.315	0.303	(0.125, 0.564)	0.438	0.952	2.172
	Y_6	0.575	0.557	(0.331, 0.893)	0.561	0.974	1.735
	Y_{12}	0.917	1.000	(0.585, 1.360)	0.775	0.977	1.261
4	Y_2	0.385	0.365	(0.155, 0.713)	0.558	0.964	1.728
	Y_6	0.858	0.882	(0.480, 1.382)	0.902	0.975	1.081
	Y_{12}	2.451	3.966	(1.287, 4.476)	3.189	0.956	0.299

Table 3. Bayes point predictors for Y_s based on the SEL and LINEX loss functions, the 95% Bayesian prediction bounds, AWs, simulated Bayesian CPs and PL criterion, for the two-parameter GE, with $\alpha = 4$, $\lambda = 2$ and $\tau = 2$.

		Non-informative prior for α and Gamma prior for λ					
		Point prediction		Interval Prediction			
Scheme	Y_s	SEL	LINEX	(Lower, Upper)	AW	CP	PL
1	Y_2	0.347	0.350	(0.102, 0.693)	0.590	0.949	1.607
	Y_6	0.816	0.759	(0.405, 1.389)	0.983	0.979	0.995
	Y_{12}	2.473	2.124	(1.226, 4.781)	3.554	0.969	0.272
2	Y_2	0.293	0.279	(0.088, 0.555)	0.467	0.944	2.021
	Y_6	0.550	0.529	(0.280, 0.884)	0.604	0.970	1.606
	Y_{12}	2.270	2.945	(1.061, 4.596)	3.534	0.966	0.273
3	Y_2	0.287	0.273	(0.082, 0.549)	0.466	0.941	2.016
	Y_6	0.549	0.528	(0.276, 0.883)	0.606	0.973	1.604
	Y_{12}	0.908	0.977	(0.550, 1.386)	0.835	0.987	1.180
4	Y_2	0.364	0.342	(0.120, 0.698)	0.577	0.959	1.662
	Y_6	0.852	0.870	(0.452, 1.404)	0.952	0.982	1.031
	Y_{12}	2.529	3.959	(1.297, 4.814)	3.517	0.957	0.272

Table 4. Bayes point predictors for Y_s based on the SEL and LINEX loss functions, the 95% Bayesian prediction bounds, AWs, simulated Bayesian CPs and PL criterion, for the two-parameter GE, with $\alpha = 4$, $\lambda = 2$ and $\tau = 2$.

		Non-informative priors for α and λ					
		Point prediction		Interval Prediction			
Scheme	Y_s	SEL	LINEX	(Lower, Upper)	AW	CP	PL
1	$Y2$	0.343	0.322	(0.092, 0.718)	0.626	0.959	1.531
	Y_6	0.861	0.792	(0.411, 1.535)	1.123	0.986	0.877
	Y_{12}	2.680	2.245	(1.273, 5.605)	4.332	0.957	0.220
2	Y_2	0.280	0.263	(0.071, 0.564)	0.492	0.947	1.923
	Y_6	0.561	0.537	(0.269, 0.948)	0.678	0.983	1.448
	Y_{12}	2.538	3.105	(1.114, 5.672)	4.558	0.953	0.209
3	Y_2	0.276	0.260	(0.068, 0.562)	0.493	0.947	1.920
	Y_6	0.561	0.536	(0.265, 0.954)	0.688	0.988	1.435
	Y_{12}	0.982	1.069	(0.563, 1.626)	1.062	0.991	0.933
4	Y_2	0.356	0.333	(0.102, 0.717)	0.615	0.967	1.572
	Y_6	0.890	0.907	(0.452, 1.535)	1.082	0.987	0.911
	Y_{12}	2.751	4.105	(1.348, 5.692)	4.343	0.943	0.217

priors for the scale and shape parameters (Table 2) decreases AWs and increases the PL criterion in comparison to Table 3, slightly. Also, based on AW and PL, results of Table 3, where only the shape parameter has non-informative prior, is better than results of Table 4.
In each scheme, we observe that the larger s has the longer AW and smaller PL criterion. It seems that this result is due to the positive skewness of GE distribution for $\alpha > 1$.
In general, we came to conclusion that scheme 3 (Type-II censored data), in comparison to other schemes, has smaller AW and larger PL.

4.2 Simulation Study in MCMC Solution

This subsection devoted to showing how the MCMC sampling methods are useful for analyzing data sets. In this subsection, the performances of the approximate Bayesian prediction bounds and Bayes point predictors proposed in Section 3 via Gibbs sampler and importance sampling are carried out. The approximate 95% Bayesian prediction bounds for Y_s and Bayes point predictors of Y_s under the SEL and the LINEX loss functions are derived by the statistical package R version 3.2.2.
Steps 1-9 of the algorithm described in Section 3 are repeated 10^3 times and reported the average Bayesian prediction bounds, the average Bayes point predictors, AWs, CPs and PLs for prediction intervals.
In addition, here $N_1 = 500$ as Gibbs replications. We performed the Gibbs algorithm for informative and non-informative priors and three Cases that described before.

Again, in the most schemes and for the most s, the simulated Bayesian coverage probabilities of Y_s are approximately close to the nominal level 95%. Unfortunately, under a few schemes and s, we could not reach the nominal level 95%.
In addition, the rest of results are similar to the numerical method (Tables 2-4).

In general, by comparing peer to peer Tables 2-7, we can conclude the numerical method has slightly better results, in termes of AW and PL criteria. But, we suggest MCMC method because of their ease of calculation.

4.3 An Illustrative Example

Here, a real data set is used for illustrating the proposed prediction bounds and the Bayes predictive predictor for the three cases of the priors in Subsec-

Table 5. Bayes point predictors for Y_s based on the SEL and LINEX loss functions, the 95% Bayesian prediction bounds, AWs, simulated Bayesian CPs and PL criterion, for the two-parameter GE under MCMC, with $\alpha = 4$, $\lambda = 2$ and $\tau = 2$.

		Gamma priors for α and λ					
		Point prediction		Interval Prediction			
Scheme	Y_s	SEL	LINEX	(Lower, Upper)	AW	CP	PL
1	Y_2	0.510	0.402	(0.088, 1.103)	1.014	0.997	0.983
	Y_6	1.532	1.253	(0.625, 2.667)	2.041	0.842	0.412
	Y_{21}	5.753	3.716	(2.649, 10.891)	8.241	0.247	0.030
2	Y_2	0.356	0.286	(0.065, 0.687)	0.621	0.979	1.575
	Y_6	0.771	0.668	(0.318, 1.239)	0.920	0.980	1.065
	Y_{12}	4.277	2.841	(1.815, 8.498)	6.683	0.551	0.082
3	Y_2	0.330	0.269	(0.068, 0.598)	0.529	0.945	1.784
	Y_6	0.676	0.590	(0.292, 1.045)	0.752	0.980	1.302
	Y_{12}	1.216	1.097	(0.687, 1.742)	1.054	0.879	0.833
4	Y_2	0.398	0.321	(0.079, 0.796)	0.717	0.978	1.363
	Y_6	1.128	0.965	(0.488, 1.887)	1.399	0.970	0.693
	Y_{12}	4.066	2.914	(1.928, 7.572)	5.644	0.627	0.111

Table 6. Bayes point predictors for Y_s based on the SEL and LINEX loss functions, the 95% Bayesian prediction bounds, AWs, simulated Bayesian CPs and PL criterion, for the two-parameter GE under MCMC, with $\alpha = 4$, $\lambda = 2$ and $\tau = 2$.

		Non-informative prior for α and Gamma prior for λ					
		Point prediction		Interval Prediction			
Scheme	Y_s	SEL	LINEX	(Lower, Upper)	AW	CP	PL
1	Y_2	0.396	0.293	(0.044, 0.913)	0.868	0.990	1.140
	Y_6	1.265	1.022	(0.440, 2.313)	1.872	0.978	0.522
	Y_{12}	5.191	3.312	(2.250, 10.124)	7.874	0.436	0.055
2	Y_2	0.283	0.206	(0.035, 0.569)	0.533	0.915	1.714
	Y_2	0.633	0.539	(0.218, 1.057)	0.838	0.989	1.179
	Y_{12}	3.868	2.556	(1.553, 7.890)	6.337	0.728	0.114
3	Y_2	0.272	0.204	(0.041, 0.508)	0.467	0.842	1.801
	Y_6	0.568	0.487	(0.213, 0.902)	0.689	0.958	1.390
	Y_{12}	1.058	0.947	(0.555, 1.544)	0.989	0.970	0.981
4	Y_2	0.337	0.262	(0.052, 0.697)	0.644	0.937	1.454
	Y_6	0.986	0.836	(0.388, 1.696)	1.308	0.992	0.758
	Y_{12}	3.812	2.720	(1.750, 7.210)	5.460	0.743	0.136

Table 7. Bayes point predictors for Y_s based on the SEL and LINEX loss functions, the 95% Bayesian prediction bounds, AWs, simulated Bayesian CPs and PL criterion, for the two-parameter GE under MCMC, with $\alpha = 4$, $\lambda = 2$ and $\tau = 2$.

		Non-informative priors for α and λ					
		Point prediction		Interval Prediction			
Scheme	Y_s	SEL	LINEX	(Lower, Upper)	AW	CP	PL
1	Y_2	0.345	0.190	(0.021, 0.926)	0.905	0.985	1.088
	Y_6	1.406	1.052	(0.393, 2.847)	2.453	0.976	0.398
	Y_{12}	7.142	3.859	(2.827, 14.546)	11.718	0.206	0.017
2	Y_2	0.239	0.098	(0.017, 0.538)	0.521	0.858	1.646
	Y_6	0.601	0.496	(0.163, 1.117)	0.954	0.987	1.034
	Y_{12}	5.068	2.886	(1.802, 10.914)	9.111	0.564	0.061
3	Y_2	0.232	0.119	(0.022, 0.474)	0.451	0.758	1.679
	Y_6	0.543	0.455	(0.170, 0.937)	0.767	0.954	1.243
	Y_{12}	1.133	0.991	(0.537, 1.759)	1.222	0.963	0.788
4	Y_2	0.307	0.216	(0.034, 0.689)	0.655	0.918	1.401
	Y_6	1.029	0.848	(0.358, 1.884)	1.526	0.993	0.651
	Y_{12}	4.499	2.985	(1.949, 8.769)	6.820	0.615	0.090

tions 2 and 3. The following data set is due to Lawless (1982) (p. 228) and arose in tests on endurance of deep groove ball bearings. The data are the numbers of million revolution before failure for each of the 23 ball bearings in a life test:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Gupta and Kundu (2001) observed that the two-parameter GE distribution is an adequate model for this data set; see, Kumar Dey and Kundu (2009) and Kundu and Gupta (2008). In order to apply MCMC, data set divided by 30. For $m = 11$, $R = (\overbrace{1,\ldots,1}^{5},2,\overbrace{1,\ldots,1}^{5})$, $M = 20$, $S = (20,0,\ldots,0)$ and $N = 40$, in Case I, we supposed $a = 12$, $b = 10$, $c = 15$, $d = 14$, $\tau = 2$ and $s = 18$.

(1) Numerical method: By considering informative priors, the 95%

Bayesian prediction bounds and the Bayes point predictors under the SEL and the LINEX loss functions for Y_{18} are obtained from Equations (10), (12) and (15) as (2.664, 7.876) (length=5.212), 3.641 and 4.045, respectively.

Also, let non-informative prior $\pi(\alpha) \equiv \alpha^{-1}$, for the shape parameter α and informative prior $\Gamma(15, 14)$ for the scale parameter λ, the 95% Bayesian prediction bounds and the Bayes point predictors under the SEL and the LINEX loss functions for Y_{18} are derived as (2.731, 8.132) (length=5.401), 3.733 and 4.110, respectively.
Finally, if if both parameters have non-informative priors ($a = b = 0$, $c = d = 0$), the 95% Bayesian prediction bounds and the Bayes point predictors under the SEL and the LINEX loss functions for Y_{18} are (2.711, 8.044) (length=5.332), 3.701 and 4.086, respectively.
We observe that informative priors provide minor influence on the length of the prediction bounds and decrease it.

(2) MCMC method: Under MCMC, assuming informative priors, the approximate 95% Bayesian prediction bounds and Bayes point predictors under the SEL and the LINEX loss functions for Y_{18} by $N_1 = 500$ Gibbs replications, based on Equations (20), (21) and (22) are (3.257, 9.530) (length=6.272), 5.870 and 4.308, respectively. Also, using non-informative prior $\pi(\alpha) \equiv \alpha^{-1}$, for the shape parameter α and informative prior $\Gamma(15, 14)$ for the scale parameter λ, the approximate 95% Bayesian prediction bounds and the Bayes point predictors under the SEL and the LINEX loss functions for Y_{18} are obtained as (4.295, 12.529) (length=8.234), 7.874 and 5.179, respectively.
Finally, let non-informative priors ($a = b = c = d = 0$) for both parameters, the approximate 95% Bayesian prediction bounds and the Bayes point predictors under the SEL and the LINEX loss functions for Y_{18} are given by (4.702, 14.573) (length=9.871), 8.894 and 5.522, respectively. Here, we observe that informative priors have an influence on the decreasing of prediction lengths.

5 Conclusions

In this paper, we have obtained the prediction bounds and Bayes point predictors for the sth order statistic, denoted by Y_s, with a known arbitrary progressive censoring scheme from the second independent future sample, under the two-parameter GE model. The Bayes point predictors were derived under the SEL and the LINEX loss functions by an numerical method

and MCMC techniques.

Empirical evidences from the Monte Carlo simulation show that the performance of the prediction method for some various censoring schemes under the informative and the non-informative priors for both methods. Also, in the Monte Carlo simulation study, we considered various values of the hyperparameters. The results of three cases of informative and non-informative priors, for each method, were compared in terms of AW and PL.
Also, in most schemes, we have similar results, in terms of Aw and PL, between the numerical method and MCMC technique. In conclusion, we would prefer to use MCMC methods rather than apply the numerical method because of easy calculations.

The proposed procedures for the prediction problem may be considered for other censoring schemes and for some other lifetime distributions such as the Generalized Pareto distribution. Also, the results of this paper may be extended to other asymmetric loss functions involving the reflected normal loss function introduced by Spiring (1993).

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References

Aitchison, J. and Dunsmore, I.R. (1975). *Statistical Prediction Analysis.* Cambridge University Press, London.

Alamm, A.A., Raqab, M.Z. and Madi, M.T. (2007). Bayesian Prediction Intervals for Future Order Statistics from the generalized exponential distribution. *JIRSS*, **6**, 17-30.

AL-Hussaini, E.K. (1999). Predicting Observable from a General Class of Distributions. *J. Statist. Plann. Inference*, **79**, 79-91.

Ali Mousa, M.A.M. and AL-Sagheer, S.A. (2005). Bayesian Prediction for Progressively Type-II Censored Data from the Rayleigh Model. *Comm. Statist. Theory Methods*, **34**, 2353-2361.

Ali Mousa, M.A.M. and Jaheen, Z.F. (2002). Bayesian Prediction for Progressively Censored Data from the Burr Model. *Statist. Papers*, **43**, 587-593.

Asgharzadeh, A., Valiollahi R. and Raqab, M.Z. (2013). Estimation of the Stress Strength Reliability for the Generalized Logistic Distribution. *Stat. Methodol.*, **15**, 73-94.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications*. Birkhauser, Boston.

Balakrishnan, N., Cramer, E. and Kamps, U. (2001). Bounds for Means and Variances of Progressive Type-II Censored Order Statistics. *Statist. Probab. Lett.*, **54**, 301-315.

Balakrishnan, N. and Sandhu, R.A. (1995). A Simple Simulation Algorithm for Generating Progressive Type-II Censored Samples. *Amer. Statist.*, **49**, 229-230.

Cacciari, M. and Motanari, G.C. (1987). A Method to Estimate the Weibull Parameters for Progressively Censored tests. *IEEE Trans. Reliab.*, **R-36**, 87-93.

Chen, M.H. and Shao, Q.M. (1999). Monte Carlo Estimation of Bayesian Credible and HPD Intervals. *J. Comput. Graph. Stat.*, **8**, 69-92.

Cohen, A.C. (1963). Progressively Censored Samples in the Life Testing. *Technometrics*, **5**, 327-339.

Cohen, A.C. and Norgaard, N.J. (1977). Progressively Censored Sampling in the Three-Parameter Gamma Distribution. *Technometrics*, **19**, 333-340.

Fernandez, A.J. (2000). Bayesian Inference from Type-II Doubly Censored Rayleigh Data. *Statist. Probab. Lett.*, **48**, 393-399.

Ghafoori, S., Habibi Rad, A. and Doostparast, M. (2011). Bayesian Two-Sample Prediction with Progressively Type-II Censored Data for Some Lifetime Models. *JIRSS*, **10**, 63-86.

Geisser, S. (1993). *Predictive Inference: An Introduction*. Chapman & Hall, New York.

Gupta, R.D. and Kundu, D. (1999). Generalized Exponential Distribution. *Aust. N. Z. J. Stat.*, **41**, 173-188.

Gupta, R.D. and Kundu, D. (2001). Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions. *Biom. J.*, **43**, 117-130.

Kim, C. and Han, K. (2015). Bayesian Estimation of Generalized Exponential Distribution under Progressive First Failure Censored Sample. *Appl. Math. Sci.*, **9**, 2037-2047.

Kumar Dey, A. and Kundu, D. (2009). Discriminating Among the Log-Normal, Weibull, and Generalized Exponential Distributions. *IEEE Trans. Reliab.*, **58**, 416-424.

Kundu, D. and Gupta, R.D. (2008). Generalized Exponential Distribution: Bayesian Estimations. *Comput. Statist. Data Anal.*, **52**, 1873-1883.

Kundu, D. and Howlader, H. A. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. *Comput. Statist. Data Anal.*, **54**, 1547-1558.

Kundu, D. and Pradhan, B. (2009a). Estimating the Parameters of the Generalized Exponential Distribution in Presence of Hybrid Censoring. *Comm. Statist. Theory Methods*, **38**, 2030-2041.

Kundu, D. and Pradhan, B. (2009b). Bayesian Inference and Life Testing Plans for Generalized Exponential Distribution. *Sci. China Ser. A*, **52**, 1373-1388.

Lawless, J.F. (1982). *Statistical Models and Methods for Lifetime Data*. Wiley, NewYork.

Madi, M.T. and Raqab, M.Z. (2007). Bayesian Prediction of Rainfall Records using the Generalized Exponential Distribution. *Environmetrics*, **18**, 541-549.

Madi, M.T. and Raqab, M.Z. (2009). Bayesian Inference for the Generalized Exponential Distribution based on Progressively Censored Data. *Comm. Statist. Theory Methods*, **38**, 2016-2029.

Mann, N.R. (1969). Exact Three-Order-Statistic Confidence Bounds on Reliable Life for a Weibull Model with Progressive Censoring. *J. Amer. Statist. Assoc.*, **64**, 306-315.

Mann, N.R. (1971). Best Linear Invariant Estimation for Weibull Parameters under Progressive Censoring. *Technometrics*, **13**, 521-534.

Mitra, S. and Kundu, D. (2008). Analysis of the Left Censored Data from the Generalized Exponential Distribution. *J. Stat. Comput. Simul.*, **78**, 669-679.

Pradhan, K. and Kundu, D. (2009). On Progressively Censored Generalized Exponential. *Test*, **18**, 497-515.

Raqab, M.Z. and Ahsanullah, M. (2001). Estimation of the Location and Scale Parameters of Generalized Exponential Distribution Based on Order Statistics. *J. Stat. Comput. Simul.*, **69**, 109-124.

Raqab, M.Z. and Madi, M.T. (2005). Bayesian Inference for the Generalized Exponential Distribution. *J. Stat. Comput. Simul.*, **75**, 841-852.

Rizzo, M.L. (2008). *Statistical Computing with R*. Chapman & Hall/CRC, Boca Raton, FL.

Salehi, M. and Ahmadi, J. (2014). Record Ranked set Sampling Scheme. *Metron*, **72**, 351-365.

Singh, R., Singh, S.K., Singh, U. and Singh, G.P. (2008). Bayes Estimator of Generalized Exponential Parameters under LINEX Loss Function using Lindley's Approximation. *Data Sci. J.*, **7**, 65-75.

Soliman, A.A., Abd Ellah, A.H., Abou-Elheggag, N.A. and Modhesh, A.A. (2011a). Bayesian

Inference and Prediction of Burr Type-XII Distribution for Progressive First Failure Censored Sampling. *Intelligent Information Management*, **3**, 175-185.

Soliman, A.A., Abd Ellah, A.H., Abou-Elheggag, N.A. and Abd-Elmougod, G.A. (2011b). A Simulation-based Approach to the Study of Coefficient of Variation of Gompertz Distribution under Progressive First-failure Censoring. *Indian J. Pure Appl. Math.*, **42**, 335-356.

Soliman, A. A., Abd Ellah, A. H., Abou-Elheggag, N. A. and Ahmed, E. A. (2012). Modified Weibull model: A Bayes study using MCMC Approach based on Progressive Censoring Data. *Reliab. eng. Syst. Safe.*, **100**, 48-57.

Son Y.S. and Oh, M. (2006). Bayes Estimation of the Two-Parameter Gamma Distribution. *Comm. Statist. Theory Methods*, **35**, 285-293.

Spiring, F.A. (1993). The Reflected Normal Loss Lunction. *Canad. J. Statist.*, **21**, 321-330.

Thomas, D.R. and Wilson, W.M. (1972). Linear Order Statistic Estimation for the Two-Parameter Weibull and Extreme-Value Distributions from Type-II Progressively Censored Samples. *Technometrics*, **14**, 679-691.

Valiollahi R., Asgharzadeh, A. and Raqab, M.Z. (2013). Estimation of $P(Y < X)$ for Weibull Distribution under Progressive Type-II Censoring. *Comm. Statist. Theory Methods*, **42**, 4476-4498.

Varian, H.R. (1975). A Bayesian Approach to Real Estate Assessment, In: Stephen, E.F., Zellner, A. (Eds.). *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, North-Holland, Amsterdam, 195-208.

Viveros, R. and Balakrishnan, N. (1994). Interval Estimation of Life Characteristics from Progressively Censored Data. *Technometrics*, **36**, 84-91.

Zellner, A. (1986). Bayesian Estimation and Prediction using Asymmetric Loss Function. *J. Amer. Statist.*, **81**, 446-451.

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مجله‌ی پژوهش‌های آماری ایران
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پیش‌بینی دونمونه‌ای بیزی در داده‌های سانسورشده‌ی فزاینده از توزیع نمایی تعمیم‌یافته، تحت تابع‌های زیان متقارن و نامتقارن

غفوری، آرزو حبیبی راد و مهدی دوست‌پرست

دانشگاه فردوسی مشهد

چکیده. در بسیاری تحقیقات آماری پیش‌بینی نقش مهمی دارد. مثال‌هایی در این زمینه شامل سیستم‌های مهندسی، طراحی آزمایش‌ها و غیره می‌باشند. در این مقاله، بر اساس داده‌های سانسور فزاینده‌ی نوع دوم در الگوی نمایی تعمیم‌یافته، پیش‌بینی‌کننده‌های بیزی نقطه‌ای و فاصله‌ای تحت توزیع‌های پیشین آگاهی‌بخش و ناآگاهی‌بخش مورد بررسی قرار می‌گیرند. همچنین کران‌های پیش‌بینی و پیش‌بینی‌کننده‌های نقطه‌ای بیزی را تحت دو تابع زیان توان دوم خطا و خطی- نمایی، برای آماره‌ی مرتب در یک نمونه‌ی سانسور فزاینده‌ی نوع دوم آینده با طرح سانسور دلخواه، به دست می‌آوریم. نتایج مستخرج ممکن است در آزمایش‌های طول عمر برای پیش‌بینی زمان کل آزمایش مورد استفاده قرار گیرند. علاوه بر روش عددی، روش نمونه‌گیری گیبس (به‌عنوان روشی از زنجیر مارکوف مونت‌کارلو) برای ارزیابی کران‌های پیش‌بینی و پیش‌بینی‌کننده‌های نقطه‌ای بیزی تقریبی تحت توابع زیان توان دوم خطا و خطی- نمایی مورد استفاده قرار گرفته است. عملکرد روش‌های پیش‌بینی پیش‌نهادی از طریق یک مطالعه‌ی شبیه‌سازی مونت کارلو و یک مثال عددی (واقعی) برای هر روش نشان داده شده است.

واژگان کلیدی. الگوی نمایی تعمیم‌یافته؛ پیش‌بینی بیزی؛ پیش‌بینی دونمونه‌ای؛ تابع زیان خطی- نمایی؛ طرح سانسور فزاینده‌ی نوع دوم؛ زنجیر مارکوف مونت‌کارلو؛ نمونه‌گیری گیبس.

نسخه‌ی کامل این مقاله، به زبان انگلیسی در صفحه‌های ۱۲۹ تا ۱۵۴ آمده است.